

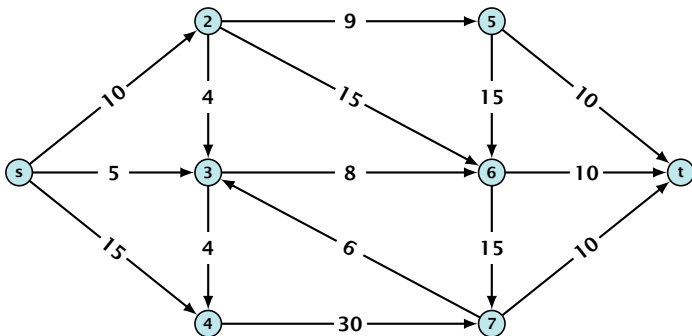
Part IV

Flows and Cuts

11 Introduction

Flow Network

- ▶ directed graph $G = (V, E)$; edge capacities $c(e)$
- ▶ two special nodes: source s ; target t ;
- ▶ no edges entering s or leaving t ;
- ▶ at least for now: no parallel edges;



Cuts

Definition 41

An (s, t) -cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

Definition 42

The **capacity** of a cut A is defined as

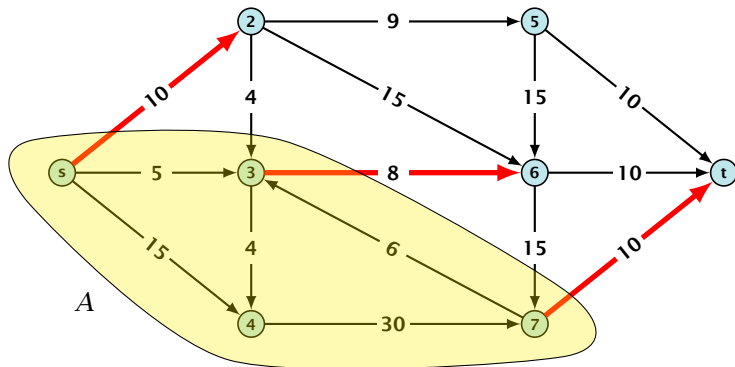
$$\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e) ,$$

where $\text{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

Minimum Cut Problem: Find an (s, t) -cut with minimum capacity.

Cuts

Example 43



The capacity of the cut is $\text{cap}(A, V \setminus A) = 28$.

Flows

Definition 44

An (s, t) -flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) .$$

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) .$$

(flow conservation constraints)

Flows

Definition 45

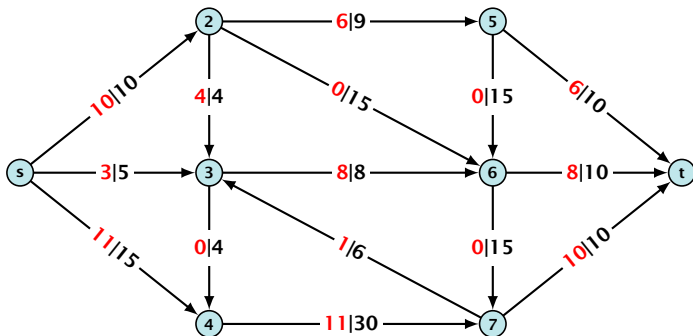
The value of an (s, t) -flow f is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) .$$

Maximum Flow Problem: Find an (s, t) -flow with maximum value.

Flows

Example 46



The value of the flow is $\text{val}(f) = 24$.

Lemma 47 (Flow value lemma)

Let f a flow, and let $A \subseteq V$ be an (s, t) -cut. Then the *net-flow* across the cut is equal to the amount of flow leaving s , i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) .$$

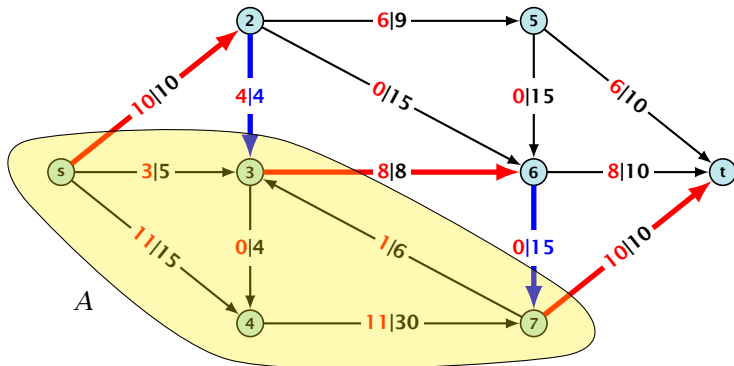
Proof.

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) && = 0 \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right) \\ &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)\end{aligned}$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A .



Example 48



Corollary 49

Let f be an (s, t) -flow and let A be an (s, t) -cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

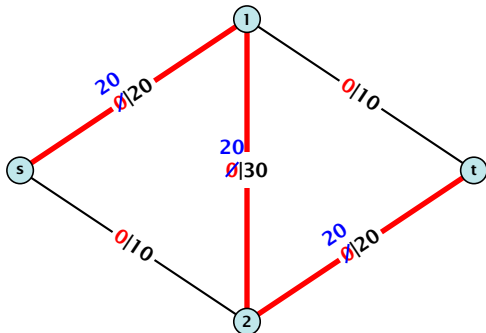
$$\begin{aligned} \text{cap}(A, V \setminus A) &< \text{val}(f') \\ &= \sum_{e \in \text{out}(A)} f'(e) - \sum_{e \in \text{into}(A)} f'(e) \\ &\leq \sum_{e \in \text{out}(A)} f'(e) \\ &\leq \text{cap}(A, V \setminus A) \end{aligned}$$



12 Augmenting Path Algorithms

Greedy-algorithm:

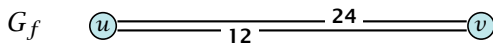
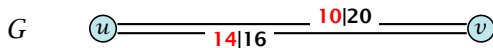
- ▶ start with $f(e) = 0$ everywhere
- ▶ find an s - t path with $f(e) < c(e)$ on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible



The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

- ▶ Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v .
- ▶ G_f has edge e'_1 with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e'_2 with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.



Augmenting Path Algorithm

Definition 50

An **augmenting path** with respect to flow f , is a path in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 45 FordFulkerson($G = (V, E, c)$)

- 1: Initialize $f(e) \leftarrow 0$ for all edges.
- 2: **while** \exists augmenting path p in G_f **do**
- 3: augment as much flow along p as possible.

Augmenting Path Algorithm

Theorem 51

A flow f is a maximum flow **iff** there are no augmenting paths.

Theorem 52

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A, B such that $\text{val}(f) = \text{cap}(A, B)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

1. \Rightarrow 2.

This we already showed.

2. \Rightarrow 3.

If there were an augmenting path, we could improve the flow.
Contradiction.

3. \Rightarrow 1.

- ▶ Let f be a flow with no augmenting paths.
- ▶ Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \\ &= \sum_{e \in \text{out}(A)} c(e) \\ &= \text{cap}(A, V \setminus A)\end{aligned}$$

This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A .

Analysis

Assumption:

All capacities are integers between 1 and C .

Invariant:

Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral throughout the algorithm.

Lemma 53

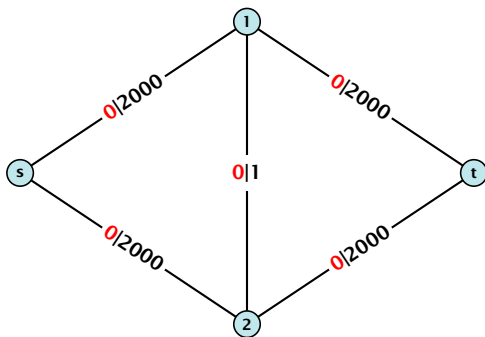
The algorithm terminates in at most $\text{val}(f^) \leq nC$ iterations, where f^* denotes the maximum flow. Each iteration can be implemented in time $\mathcal{O}(m)$. This gives a total running time of $\mathcal{O}(nmC)$.*

Theorem 54

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.

A bad input

Problem: The running time may not be polynomial.

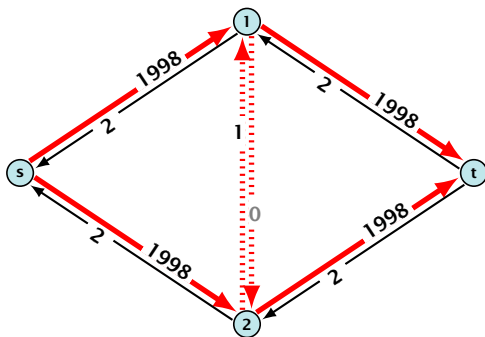


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A bad input

Problem: The running time may not be polynomial.

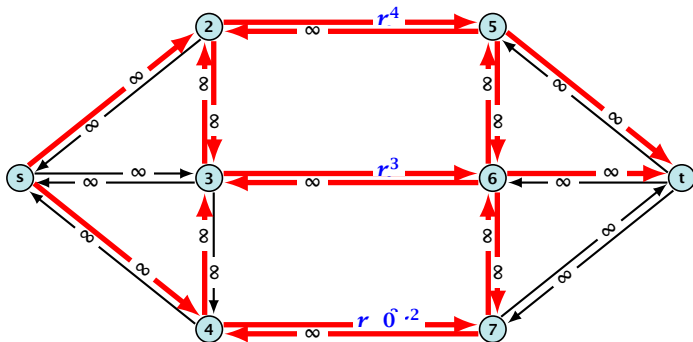


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



Running time may be infinite!!!

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

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Overview: Shortest Augmenting Paths

Lemma 55

The length of the shortest augmenting path never decreases.

Lemma 56

After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 57

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.

Proof.

- ▶ We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- ▶ $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.

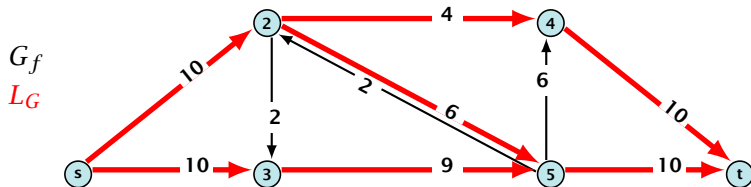


Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s - v path in G_f .

Let L_G denote the **subgraph** of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s - t path in G_f if it is a an s - t path in L_G .

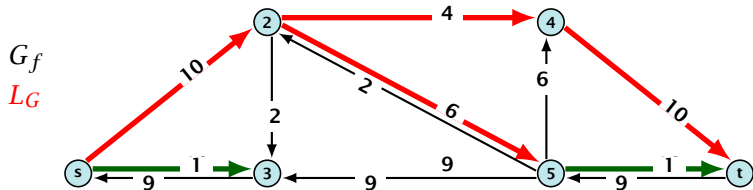


Shortest Augmenting Path

First Lemma: The length of the shortest augmenting path never decreases.

- ▶ After an augmentation the following changes are done in G_f .
- ▶ Some edges of the chosen path may be deleted (bottleneck edges).
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t .



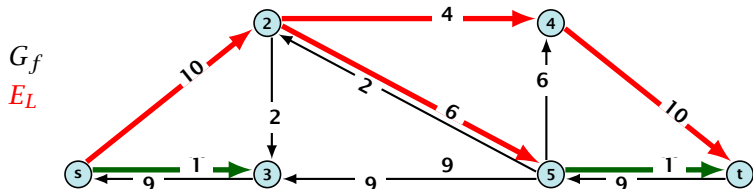
Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let E_L denote the set of edges in graph L_G **at the beginning** of a **round** when the distance between s and t is k .

An s - t path in G_f that does use edges not in E_L has length larger than k , even when considering edges added to G_f during the round.

In each augmentation one edge is deleted from E_L .



Shortest Augmenting Paths

Theorem 58

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 59 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:

There always exists a set of m augmentations that gives a maximum flow.

Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

Shortest Augmenting Paths

We maintain a subset E_L of the edges of G_f with the guarantee that a shortest s - t path using only edges from E_L is a shortest augmenting path.

With each augmentation some edges are deleted from E_L .

When E_L does not contain an s - t path anymore the distance between s and t strictly increases.

Note that E_L is not the set of edges of the level graph but a subset of level-graph edges.

Suppose that the initial distance between s and t in G_f is k .

E_L is initialized as the level graph L_G .

Perform a DFS search to find a path from s to t using edges from E_L .

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from E_L .

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation **during** a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

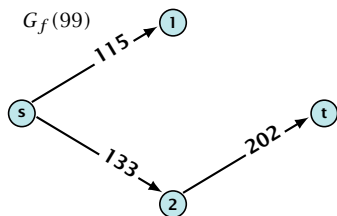
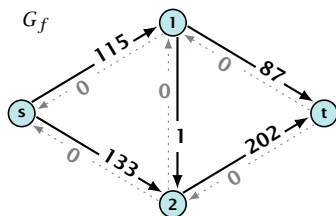
Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter Δ .
- ▶ $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ .



Capacity Scaling

Algorithm 46 maxflow(G, s, t, c)

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
```

Capacity Scaling

Assumption:

All capacities are integers between 1 and C .

Invariant:

All flows and capacities are/remain integral throughout the algorithm.

Correctness:

The algorithm computes a maxflow:

- ▶ because of integrality we have $G_f(1) = G_f$
- ▶ therefore after the last phase there are no augmenting paths anymore
- ▶ this means we have a maximum flow.

Capacity Scaling

Lemma 60

There are $\lceil \log C \rceil$ iterations over Δ .

Proof: obvious.

Lemma 61

Let f be the flow at the end of a Δ -phase. Then the maximum flow is smaller than $\text{val}(f) + 2m\Delta$.

Proof: less obvious, but simple:

- ▶ An s - t cut in $G_f(\Delta)$ gives me an upper bound on the amount of flow that my algorithm can still add to f .
- ▶ The edges that currently have capacity at most Δ in G_f form an s - t cut with capacity at most $2m\Delta$.

Capacity Scaling

Lemma 62

There are at most $2m$ augmentations per scaling-phase.

Proof:

- ▶ Let f be the flow at the end of the previous phase.
- ▶ $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$
- ▶ each augmentation increases flow by Δ .

Theorem 63

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

Preflows

Definition 64

An (s, t) -preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) .$$

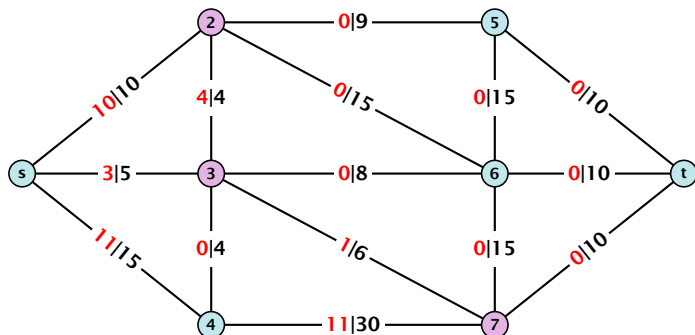
(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) \leq \sum_{e \in \text{into}(v)} f(e) .$$

Preflows

Example 65



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an **active node**.

Preflows

Definition:

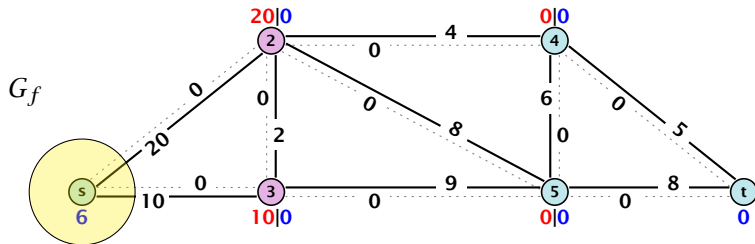
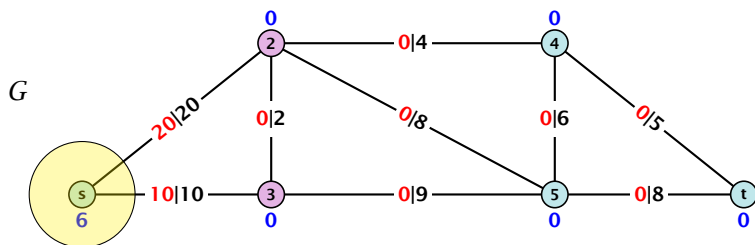
A **labelling** is a function $\ell : V \rightarrow \mathbb{N}$. It is **valid** for preflow f if

- ▶ $\ell(u) \leq \ell(v) + 1$ for all edges in the residual graph G_f (only non-zero capacity edges!!!)
- ▶ $\ell(s) = n$
- ▶ $\ell(t) = 0$

Intuition:

The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows



Preflows

Lemma 66

A *preflow* that has a valid labelling saturates a cut.

Proof:

- ▶ There are n nodes but $n + 1$ different labels from $0, \dots, n$.
- ▶ There must exist a label $d \in \{0, \dots, n\}$ such that none of the nodes carries this label.
- ▶ Let $A = \{v \in V \mid \ell(v) > d\}$ and $B = \{v \in V \mid \ell(v) < d\}$.
- ▶ We have $s \in A$ and $t \in B$ and there is no edge from A to B in the residual graph G_f ; this means that (A, B) is a saturated cut.

Lemma 67

A *flow* that has a valid labelling is a maximum flow.

Push Relabel Algorithms

Idea:

- ▶ start with some preflow and some valid labelling
- ▶ successively change the preflow while maintaining a valid labelling
- ▶ stop when you have a flow (i.e., no more active nodes)

Note that this is somewhat dual to an augmenting path algorithm. The former maintains the property that it has a feasible flow. It successively changes this flow until it saturates some cut in which case we conclude that the flow is maximum. A preflow push algorithm maintains the property that it has a saturated cut. The preflow is changed iteratively until it fulfills conservation constraints in which case we can conclude that we have a maximum flow.

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is **admissible** if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation

Consider an active node u with **excess flow**

$f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissible arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

- ▶ **saturating push**: $\min\{f(u), c_f(e)\} = c_f(e)$
the arc e is deleted from the residual graph
- ▶ **non-saturating push**: $\min\{f(u), c_f(e)\} = f(u)$
the node u becomes inactive

Push Relabel Algorithms

The relabel operation

Consider an active node u that does not have an outgoing admissible arc.

Increasing the label of u by 1 results in a valid labelling.

- ▶ Edges (w, u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u) + 1$.
- ▶ An outgoing edge (u, w) had $\ell(u) < \ell(w) + 1$ before since it was not admissible. Now: $\ell(u) \leq \ell(w) + 1$.

Push Relabel Algorithms

Intuition:

We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0 . If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water than this would be natural).

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Push Relabel Algorithms

Algorithm 47 $\text{maxflow}(G, s, t, c)$

```
1: find initial preflow  $f$ 
2: while there is active node  $u$  do
3:     if there is admiss. arc  $e$  out of  $u$  then
4:          $\text{push}(G, e, f, c)$ 
5:     else
6:          $\text{relabel}(u)$ 
7: return  $f$ 
```

In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push Algorithm

relabel push push push relabel 6

times non-saturated push relabel 6

push relabel 6 times non-saturated

push relabel 6 times non-saturated

relabel push relabel push relabel 6

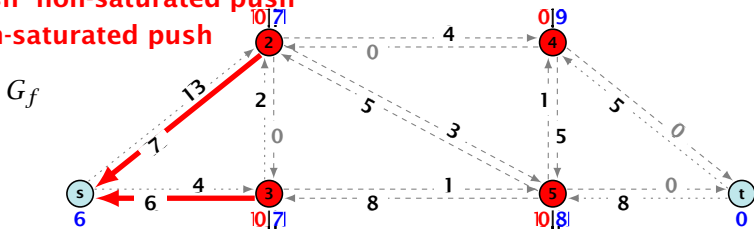
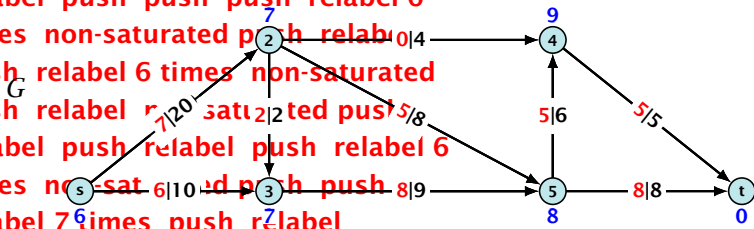
times non-saturated push push

relabel 7 times push relabel

non-saturated push non-saturated

push non-saturated push

non-saturated push



Analysis

Lemma 68

An active node has a path to s in the residual graph.

Proof.

- ▶ Let A denote the set of nodes that can reach s , and let B denote the remaining nodes. Note that $s \in A$.
- ▶ In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.
- ▶ In the residual graph there are no edges into A , and, hence, no edges leaving A /entering B can carry any flow.
- ▶ Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B .

Let $f : E \rightarrow \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$\begin{aligned} f(B) &= \sum_{b \in B} f(b) \\ &= \sum_{b \in B} \left(\sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \\ &= \sum_{b \in B} \left(\sum_{v \in A} f(v, b) + \sum_{v \in B} f(v, b) - \sum_{v \in A} f(b, v) - \sum_{v \in B} f(b, v) \right) \\ &= \underbrace{\sum_{b \in B} \sum_{v \in A} f(v, b)}_{= 0} - \underbrace{\sum_{b \in B} \sum_{v \in A} f(b, v)}_{\geq 0} + \underbrace{\sum_{b \in B} \sum_{v \in B} f(v, b) - \sum_{b \in B} \sum_{v \in B} f(b, v)}_{= 0} \\ &\leq 0 \quad = 0 \quad = 0 \end{aligned}$$

Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

Analysis

Lemma 69

The label of a node cannot become larger than $2n - 1$.

Proof.

- ▶ When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n .

Analysis

Lemma 70

There are only $\mathcal{O}(n^3)$ calls to discharge when using the relabel-to-front heuristic.

Proof.

- ▶ When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n .

Lemma 71

The number of *saturating pushes* performed is at most $\mathcal{O}(mn)$.

Proof.

- ▶ Suppose that we just made a saturating push along (u, v) .
- ▶ Hence, the edge (u, v) is deleted from the residual graph.
- ▶ For the edge to appear again, a push from v to u is required.
- ▶ Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissible edges.
- ▶ For a push from v to u the edge (v, u) must become admissible. The label of v must increase by at least 2.
- ▶ Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v) .

Lemma 72

The number of *non-saturating pushes* performed is at most $\mathcal{O}(n^2m)$.

Proof.

- ▶ Define a potential function $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$
- ▶ A saturating push increases Φ by at most $2n$.
- ▶ A relabel increases Φ by at most 1.
- ▶ A non-saturating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.
- ▶ Hence,

$$\begin{aligned} \# \text{non-saturating_pushes} &\leq \# \text{relabels} + 2n \cdot \# \text{saturating_pushes} \\ &\leq \mathcal{O}(n^2m) . \end{aligned}$$

Analysis

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2m)$.

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- ▶ check whether edge (v, u) needs to be added to G_f
- ▶ check whether (u, v) needs to be deleted (saturating push)
- ▶ check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissible
- ▶ check for all incoming edges if they become non-admissible

13.2 Relabel to front

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 48 discharge(u)

```
1: while  $u$  is active do
2:    $v \leftarrow u.current\text{-neighbour}$ 
3:   if  $v = \text{null}$  then
4:     relabel( $u$ )
5:      $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list-head}$ 
6:   else
7:     if  $(u, v)$  admissable then push( $u, v$ )
8:     else  $u.current\text{-neighbour} \leftarrow v.next\text{-in-list}$ 
```

13.2 Relabel to front

Lemma 73

If $v = \text{null}$ in line 3, then there is no outgoing admissible edge from u .

The lemma holds because push- and relabel-operations on nodes different from u cannot make edges outgoing from u admissible.

This shows that $\text{discharge}(u)$ is correct, and that we can perform a relabel in line 4.

13.2 Relabel to front

Algorithm 49 relabel-to-front(G, s, t)

```
1: initialize preflow
2: initialize node list  $L$  containing  $V \setminus \{s, t\}$  in any order
3: foreach  $u \in V \setminus \{s, t\}$  do
4:    $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list}\text{-head}$ 
5:  $u \leftarrow L.head$ 
6: while  $u \neq \text{null}$  do
7:    $old\text{-height} \leftarrow \ell(u)$ 
8:    $discharge(u)$ 
9:   if  $\ell(u) > old\text{-height}$  then
10:     move  $u$  to the front of  $L$ 
11:    $u \leftarrow u.next$ 
```

13.2 Relabel to front

Lemma 74 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

- 1. The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L .*
- 2. No node before u in the list L is active.*

Proof:

► Initialization:

1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
2. We start with u being the head of the list; hence no node before u can be active

► Maintenance:

1.
 - Pushes do not create any new admissible edges. Therefore, not relabeling u leaves L topologically sorted.
 - After relabeling, u cannot have admissible incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

13.2 Relabel to front

Proof:

► Maintenance:

2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u ; the $\text{discharge}(u)$ operation only terminates when u is not active anymore.

For the case that we do a relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arcs point to successors of u .

Note that the invariant for $u = \text{null}$ means that we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to front

Lemma 75

There are at most $\mathcal{O}(n^3)$ calls to $\text{discharge}(u)$.

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#\text{relabels} + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to front

Lemma 76

The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting *u.current-neighbour*). In total we have $\mathcal{O}(n^2)$ relabel-operations.

13.2 Relabel to front

Note that by definition a saturating push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can at the same time be a non-saturating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 77

*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.current-neighbour$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $degree(u) + 1$ many entries (+1 for null-entry).

13.2 Relabel to front

Lemma 78

The cost for all non-saturating push-operations is only $\mathcal{O}(n^3)$.

A non-saturating push-operation takes constant time and ends the current call to `discharge()`. Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 79

The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

13.3 Highest label

Algorithm 50 highest-label(G, s, t)

- 1: initialize preflow
- 2: **foreach** $u \in V \setminus \{s, t\}$ **do**
- 3: $u.current-neighbour \leftarrow u.neighbour-list-head$
- 4: **while** \exists active node u **do**
- 5: select active node u with highest label
- 6: discharge(u)

13.3 Highest label

Lemma 80

When using highest label the number of non-saturating pushes is only $\mathcal{O}(n^3)$.

After a non-saturating push from u a relabel is required to make a currently non-active node x , with $\ell(x) \geq \ell(u)$ active again (note that this includes u).

Hence, after n non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.3 Highest label

Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of relabel-to-front.

Question:

How do we find the next node for a discharge operation?

13.3 Highest label

Maintain lists L_i , $i \in \{0, \dots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k , traverse the lists $k - 1, \dots, 0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to s or t the list $k - 1$ must be non-empty (i.e., the search takes constant time).

13.3 Highest label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#non-saturating-pushes-to-s-or-t)$$

Lemma 81

The number of non-saturating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 82

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

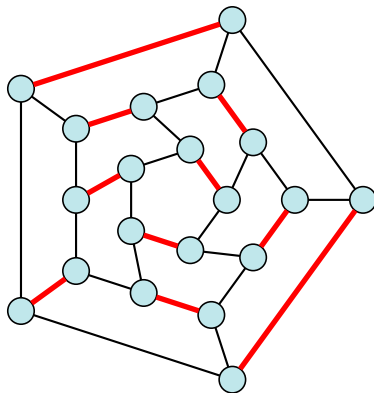
13.3 Highest label

Proof of the Lemma.

- ▶ We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- ▶ After a node v (which must have $\ell(v) = n + 1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.
- ▶ This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- ▶ As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

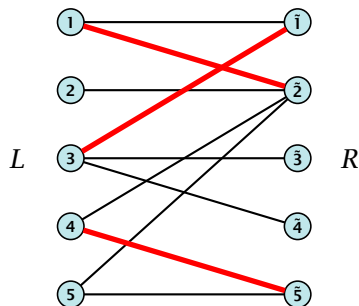
Matching

- ▶ Input: undirected graph $G = (V, E)$.
- ▶ $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- ▶ Maximum Matching: find a matching of maximum cardinality



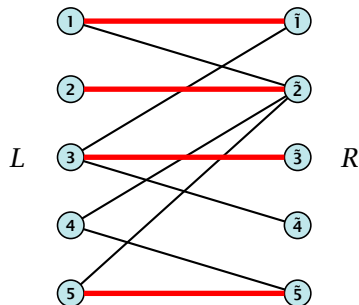
Bipartite Matching

- ▶ Input: undirected, **bipartite** graph $G = (L \uplus R, E)$.
- ▶ $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- ▶ Maximum Matching: find a matching of maximum cardinality



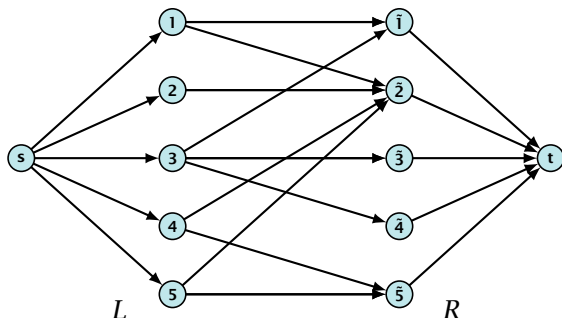
Bipartite Matching

- ▶ Input: undirected, **bipartite** graph $G = (L \uplus R, E)$.
- ▶ $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- ▶ Maximum Matching: find a matching of maximum cardinality



Maxflow Formulation

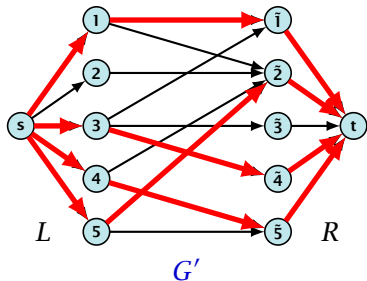
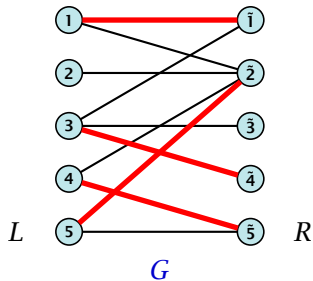
- ▶ Input: undirected, **bipartite** graph $G = (L \uplus R \uplus \{s, t\}, E')$.
- ▶ Direct all edges from L to R .
- ▶ Add source s and connect it to all nodes on the left.
- ▶ Add t and connect all nodes on the right to t .
- ▶ All edges have unit capacity.



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

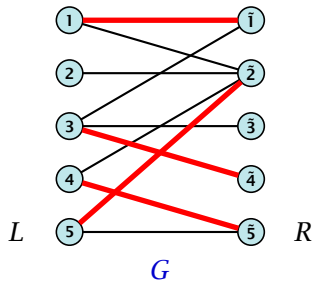
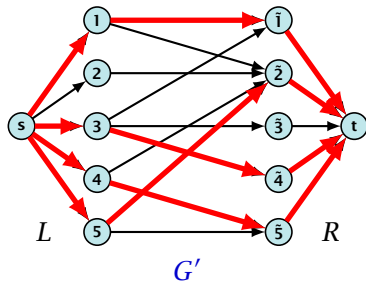
- ▶ Given a maximum matching M of cardinality k .
- ▶ Consider flow f that sends one unit along each of k paths.
- ▶ f is a flow and has cardinality k .



Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

- ▶ Let f be a maxflow in G' of value k
- ▶ Integrality theorem $\Rightarrow k$ integral; we can assume f is 0/1.
- ▶ Consider $M =$ set of edges from L to R with $f(e) = 1$.
- ▶ Each node in L and R participates in at most one edge in M .
- ▶ $|M| = k$, as the flow must use at least k middle edges.



14.1 Matching

Which flow algorithm to use?

- ▶ Generic augmenting path: $\mathcal{O}(m \text{val}(f^*)) = \mathcal{O}(mn)$.
- ▶ Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

Baseball Elimination

team i	wins w_i	losses ℓ_i	remaining games			
			Atl	Phi	NY	Mon
Atlanta	83	71	-	1	6	1
Philadelphia	80	79	1	-	0	2
New York	78	78	6	0	-	0
Montreal	77	82	1	2	0	-

Which team can end the season with most wins?

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- ▶ But also Philadelphia is eliminated. Why?

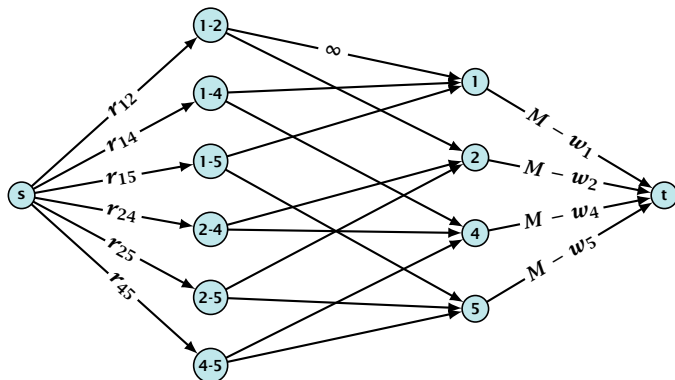
Baseball Elimination

Formal definition of the problem:

- ▶ Given a set S of teams, and one specific team $z \in S$.
- ▶ Team x has already won w_x games.
- ▶ Team x still has to play team y , r_{xy} times.
- ▶ Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow networks for $z = 3$. M is number of wins Team 3 can still obtain.

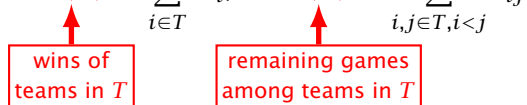


Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \quad r(T) := \sum_{i, j \in T, i < j} r_{ij}$$



If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 83

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{i,j \in S \setminus \{z\}, i < j} r_{ij}$.

Proof (\Leftarrow)

- ▶ Consider the mincut A in the flow network. Let T be the set of **team-nodes** in A .
- ▶ If for a node x - y not both team nodes x and y are in T , then x - $y \notin A$ as otw. the cut would cut an infinite capacity edge.
- ▶ We don't find a flow that saturates all source edges:

$$\begin{aligned}r(S \setminus \{z\}) &> \text{cap}(S, V \setminus S) \\ &\geq \sum_{i < j: i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i) \\ &\geq r(S \setminus \{z\}) - r(T) + |T|M - w(T)\end{aligned}$$

- ▶ This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (\Rightarrow)

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is **integral**.
- ▶ For every pairing x - y it defines how many games team x and team y should win.
- ▶ The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- ▶ This is less than $M - w_x$ because of capacity constraints.
- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- ▶ Hence, team z is not eliminated.

Project Selection

Project selection problem:

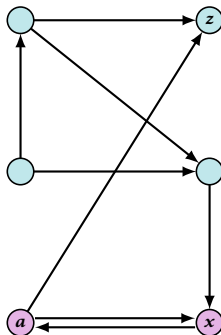
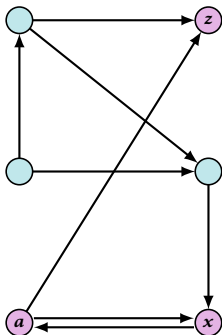
- ▶ Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- ▶ Some projects have requirements (taking course EA2 requires course EA1).
- ▶ Dependencies are modelled in a graph. Edge (u, v) means “can’t do project u without also doing project v .”
- ▶ A subset A of projects is **feasible** if the prerequisites of every project in A also belong to A .

Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

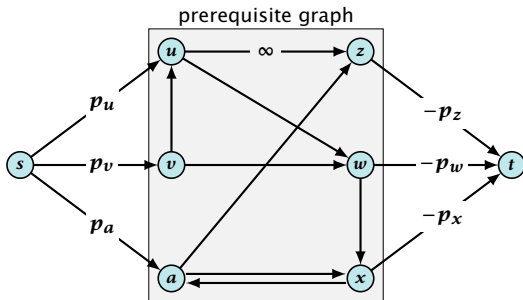
- ▶ $\{x, a, z\}$ is a feasible subset.
- ▶ $\{x, a\}$ is infeasible.



Project Selection

Mincut formulation:

- ▶ Edges in the prerequisite graph get infinite capacity.
- ▶ Add edge (s, v) with capacity p_v for nodes v with positive profit.
- ▶ Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



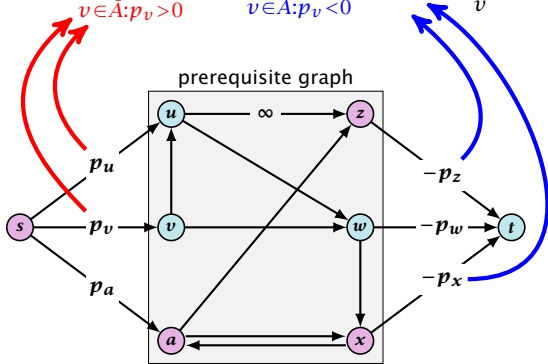
Theorem 84

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

▶ A is feasible because of capacity infinity edges.

▶ $\text{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v) = \sum_v p_v - \sum_{v \in A} p_v$



For the formula we define $p_s := 0$. Note that minimizing the capacity of the cut $(A, V \setminus A)$ corresponds to maximizing profits of projects in A .

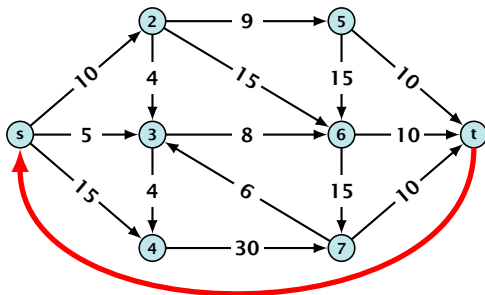
Mincost Flow

Consider the following problem:

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: 0 \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

- ▶ $G = (V, E)$ is an **oriented graph**.
- ▶ $u : E \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- ▶ $c : E \rightarrow \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).
- ▶ $b : V \rightarrow \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Solve Maxflow Using Mincost Flow



- ▶ Given a flow network for a standard maxflow problem.
- ▶ Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- ▶ Add an edge from t to s with infinite capacity and cost -1 .
- ▶ Then, $\text{val}(f^*) = -\text{cost}(f_{\min})$, where f^* is a maxflow, and f_{\min} is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- ▶ Given a flow network for a standard maxflow problem, and a value k .
- ▶ Set $b(v) = 0$ for every node apart from s or t . Set $b(s) = -k$ and $b(t) = k$.
- ▶ Set edge-costs to zero, and keep the capacities.
- ▶ There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

Generalization

Our model:

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: 0 \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

where $b : V \rightarrow \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \rightarrow \mathbb{R}$;

A more general model?

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: a(v) \leq f(v) \leq b(v) \end{aligned}$$

where $a : V \rightarrow \mathbb{R}$, $b : V \rightarrow \mathbb{R}$; $\ell : E \rightarrow \mathbb{R} \cup \{-\infty\}$, $u : E \rightarrow \mathbb{R} \cup \{\infty\}$
 $c : E \rightarrow \mathbb{R}$;

Reduction I

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: a(v) \leq f(v) \leq b(v) \end{aligned}$$

We can assume that $a(v) = b(v)$:

Add new node r .

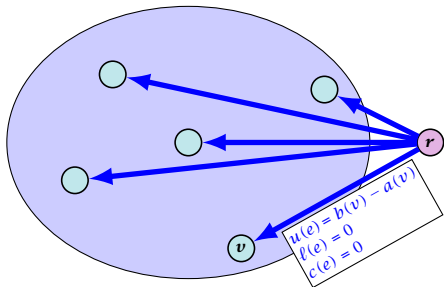
Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v) .

Set $a(v) = b(v)$ for all $v \in V$.

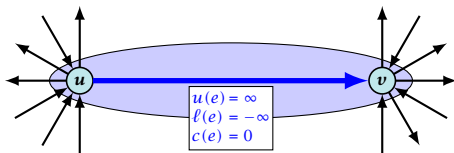
Set $b(r) = \sum_{v \in V} b(v)$.



Reduction II

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:

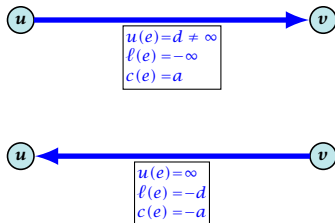


If $c(e) = 0$ we can simply contract the edge/identify nodes u and v

Reduction III

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

We can assume that $\ell(e) \neq -\infty$:

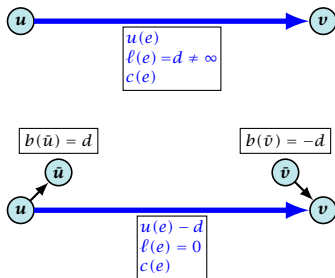


Replace the edge by an edge in opposite direction.

Reduction IV

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

We can assume that $\ell(e) = 0$:



The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- ▶ She needs to supply r_i napkins on N successive days.
- ▶ She can buy new napkins at p cents each.
- ▶ She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- ▶ She can use a slow laundry that takes $k > m$ days and costs s cents each.
- ▶ At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- ▶ Minimize cost.

Residual Graph

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

15 Mincost Flow

A **circulation** in a graph $G = (V, E)$ is a function $f : E \rightarrow \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$ (G may be a directed graph instead of just an oriented graph).

A circulation is **feasible** if it fulfills capacity constraints, i.e., $f(e) \leq u(e)$ for every edge of G .

15 Mincost Flow

$g = f^* - f$ is obtained by computing $\Delta(e) = f^*(e) - f(e)$ for every edge $e = (u, v)$. If the result is positive set $g((u, v)) = \Delta(e)$ and $g((v, u)) = 0$; otw. set $g((u, v)) = 0$ and $g((v, u)) = -\Delta(e)$.

Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph.

15 Mincost Flow

Lemma 86

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \rightarrow \mathbb{R}$.

Proof.

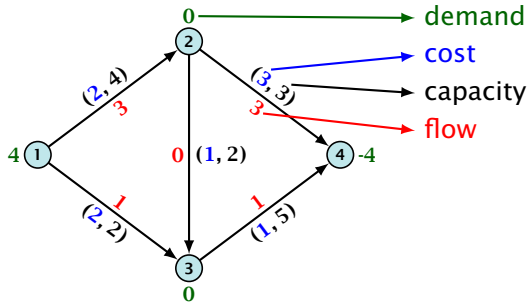
- ▶ Suppose that we have a negative cost circulation.
- ▶ Find directed path only using edges that have non-zero flow.
- ▶ If this path has negative cost you are done.
- ▶ Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- ▶ You still have a circulation with negative cost.
- ▶ Repeat.

15 Mincost Flow

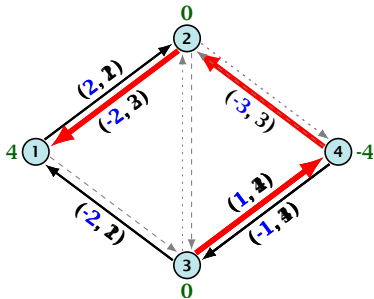
Algorithm 51 CycleCanceling($G = (V, E), c, u, b$)

- 1: establish a feasible flow f in G
- 2: **while** G_f contains negative cycle **do**
- 3: use Bellman-Ford to find a negative circuit Z
- 4: $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
- 5: augment δ units along Z and update G_f

15 Mincost Flow



15 Mincost Flow



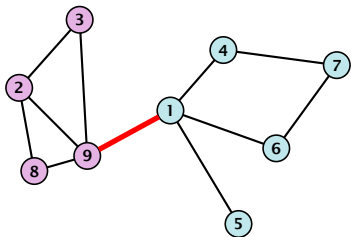
15 Mincost Flow

Lemma 87

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e , $|c(e)| \leq C$ and $|u(e)| \leq U$.

16 Global Mincut

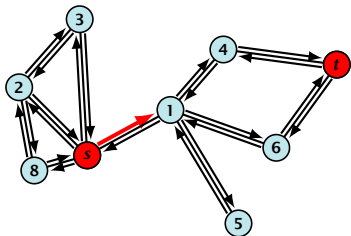
Given an **undirected, capacitated graph** $G = (V, E, c)$ find a partition of V into two non-empty sets $S, V \setminus S$ s.t. the capacity of edges between both sets is minimized.



16 Global Mincut

We can solve this problem using standard maxflow/mincut.

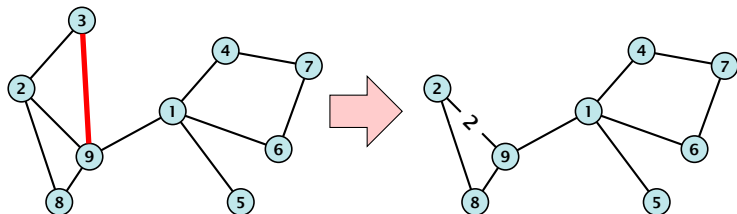
- ▶ Construct a directed graph $G' = (V, E')$ that has edges (u, v) and (v, u) for every edge $\{u, v\} \in E$.
- ▶ Fix an arbitrary node $s \in V$ as source. Compute a minimum s - t cut for all possible choices $t \in V, t \neq s$. (Time: $\mathcal{O}(n^4)$)
- ▶ Let $(S, V \setminus S)$ be a minimum global mincut. The above algorithm will output a cut of capacity $\text{cap}(S, V \setminus S)$ whenever $|\{s, t\} \cap S| = 1$.



Edge Contractions

- ▶ Given a graph $G = (V, E)$ and an edge $e = \{u, v\}$.
- ▶ The graph G/e is obtained by “identifying” u and v to form a new node.
- ▶ Resulting parallel edges are replaced by a single edge, whose capacity equals the sum of capacities of the parallel edges.

Example 88



- ▶ Edge-contractions do not decrease the size of the mincut.

Edge Contractions

We can perform an edge-contraction in time $\mathcal{O}(n)$.

Randomized Mincut Algorithm

Algorithm 52 KargerMincut($G = (V, E, c)$)

- 1: **for** $i = 1 \rightarrow n - 2$ **do**
- 2: choose $e \in E$ randomly with probability $c(e)/C(E)$
- 3: $G \leftarrow G/e$
- 4: **return** only cut in G

- ▶ Let G_t denote the graph after the $(n - t)$ -th iteration, when t nodes are left.
- ▶ Note that the final graph G_2 only contains a single edge.
- ▶ The cut in G_2 corresponds to a cut in the original graph G with the same capacity.
- ▶ What is the probability that this algorithm returns a mincut?

Example: Randomized Mincut Algorithm



What is the probability that this algorithm returns a mincut?

What is the probability that a given mincut A is still possible after round i ?

- ▶ It is still possible to obtain cut A in the end if so far **no** edge in $(A, V \setminus A)$ has been contracted.

Analysis

What is the probability that we select an edge from A in iteration i ?

- ▶ Let $\min = \text{cap}(A, V \setminus A)$ denote the capacity of a mincut.
- ▶ Let $\text{cap}(v)$ be capacity of edges incident to vertex $v \in V_{n-i+1}$.
- ▶ Clearly, $\text{cap}(v) \geq \min$.
- ▶ Summing $\text{cap}(v)$ over all edges gives

$$2c(E) = 2 \sum_{e \in E} c(e) = \sum_{v \in V} \text{cap}(v) \geq (n - i + 1) \cdot \min$$

- ▶ Hence, the probability of choosing an edge from the cut is at most $\min / c(E) \leq 2 / (n - i + 1)$.

$n - i + 1$ is the number of nodes in graph $G_{n-i+1} = (V_{n-i+1}, E_{n-i+1})$, the graph at the start of iteration i .

Analysis

The probability that we do **not** choose an edge from the cut in iteration i is

$$1 - \frac{2}{n - i + 1} = \frac{n - i - 1}{n - i + 1} .$$

The probability that the cut is alive after iteration $n - t$ (after which t nodes are left) is

$$\prod_{i=1}^{n-t} \frac{n - i - 1}{n - i + 1} = \frac{t(t - 1)}{n(n - 1)} .$$

Choosing $t = 2$ gives that with probability $1/\binom{n}{2}$ the algorithm computes a mincut.

Analysis

Repeating the algorithm $c \ln n \binom{n}{2}$ times gives that the probability that we are never successful is

$$\left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2} c \ln n} \leq \left(e^{-1/\binom{n}{2}}\right)^{\binom{n}{2} c \ln n} \leq n^{-c},$$

where we used $1 - x \leq e^{-x}$.

Theorem 89

The randomized mincut algorithm computes an optimal cut with high probability. The total running time is $\mathcal{O}(n^4 \log n)$.

Improved Algorithm

Algorithm 53 RecursiveMincut($G = (V, E, c)$)

```
1: for  $i = 1 \rightarrow n - n/\sqrt{2}$  do
2:   choose  $e \in E$  randomly with probability  $c(e)/C(E)$ 
3:    $G \leftarrow G/e$ 
4: if  $|V| = 2$  return cut-value;
5:  $cuta \leftarrow$  RecursiveMincut( $G$ );
6:  $cutb \leftarrow$  RecursiveMincut( $G$ );
7: return  $\min\{cuta, cutb\}$ 
```

Running time:

- ▶ $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + \mathcal{O}(n^2)$
- ▶ This gives $T(n) = \mathcal{O}(n^2 \log n)$.

Note that the above implementation only works for very special values of n .

Probability of Success

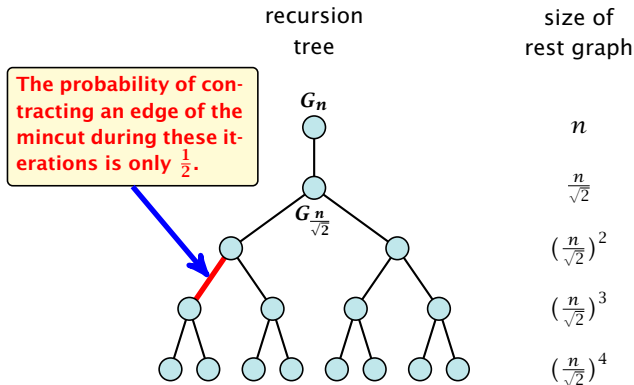
The probability of contracting an edge from the mincut during one iteration through the for-loop is only

$$\frac{t(t-1)}{n(n-1)} \approx \frac{t^2}{n^2} = \frac{1}{2},$$

as $t = \frac{n}{\sqrt{2}}$.

For the following analysis we ignore the slight error and assume that this probability is at most $\frac{1}{2}$.

Probability of Success



We can estimate the success probability by using the following game on the recursion tree. Delete every edge with probability $\frac{1}{2}$. If in the end you have a path from the root to **at least one** leaf node you are successful.

Probability of Success

Let for an edge e in the recursion tree, $h(e)$ denote the height (distance to leaf level) of the parent-node of e (end-point that is higher up in the tree). Let h denote the height of the root node.

Call an edge e **alive** if there exists a path from the parent-node of e to a descendant leaf, after we randomly deleted edges. Note that an edge can only be alive if it hasn't been deleted.

Lemma 90

The probability that an edge e is alive is at least $\frac{1}{h(e)+1}$.

Probability of Success

Proof.

- ▶ An edge e with $h(e) = 1$ is alive if and only if it is not deleted. Hence, it is alive with probability at least $\frac{1}{2}$.
- ▶ Let p_d be the probability that an edge e with $h(e) = d$ is alive. For $d > 1$ this happens for edge $e = \{c, p\}$ if it is not deleted **and** if one of the child-edges connecting to c is alive.
- ▶ This happens with probability

$$p_d = \frac{1}{2} (2p_{d-1} - p_{d-1}^2) \quad \boxed{\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]}$$

$$= p_{d-1} - \frac{p_{d-1}^2}{2}$$

$$\geq \frac{1}{d} - \frac{1}{2d^2} \geq \frac{1}{d} - \frac{1}{d(d+1)} = \frac{1}{d+1} .$$

$x - x^2/2$ is monotonically increasing for $x \in [0, 1]$

16 Global Mincut

Lemma 91

One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ and has a success probability of $\Omega(\frac{1}{\log n})$.

Doing $\Theta(\log^2 n)$ runs gives that the algorithm succeeds with high probability. The total running time is $\mathcal{O}(n^2 \log^3 n)$.

17 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a **cut-tree** $T = (V, F, w)$ is a tree with edge-set F and capacities w that fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$, $f(s, t)$ in G is equal to $f_T(s, t)$.
2. **Cut Property:** A minimum s - t cut in T is also a minimum cut in G .

Here, $f(s, t)$ is the value of a maximum s - t flow in G , and $f_T(s, t)$ is the corresponding value in T .

Overview of the Algorithm

The algorithm maintains a partition of V , (sets S_1, \dots, S_t), and a spanning tree T on the vertex set $\{S_1, \dots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

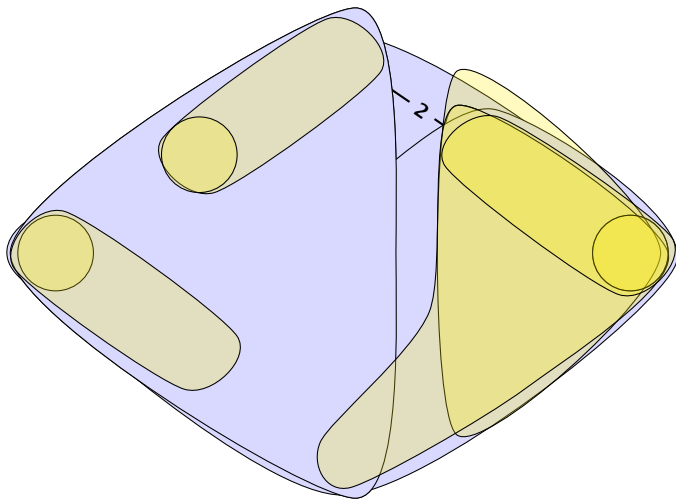
- ▶ In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y .
- ▶ S_i is then removed from T and replaced by X and Y .
- ▶ X and Y are connected by an edge, and the edges that before the split were incident to S_i are attached to either X or Y .

In the end this gives a tree on the vertex set V .

Details of the Split-operation

- ▶ Select S_i that contains at least two nodes a and b .
- ▶ Compute the connected components of the forest obtained from the current tree T after deleting S_i . Each of these components corresponds to a set of vertices from V .
- ▶ Consider the graph H obtained from G by contracting these connected components into single nodes.
- ▶ Compute a minimum a - b cut in H . Let A , and B denote the two sides of this cut.
- ▶ Split S_i in T into two sets/nodes $S_i^a := S_i \cap A$ and $S_i^b := S_i \cap B$ and add edge $\{S_i^a, S_i^b\}$ with capacity $f_H(a, b)$.
- ▶ Replace an edge $\{S_i, S_x\}$ by $\{S_i^a, S_x\}$ if $S_x \subset A$ and by $\{S_i^b, S_x\}$ if $S_x \subset B$.

Example: Gomory-Hu Construction



Lemma 92

For nodes $s, t, x \in V$ we have $f(s, t) \geq \min\{f(s, x), f(x, t)\}$

Lemma 93

For nodes $s, t, x_1, \dots, x_k \in V$ we have

$$f(s, t) \geq \min\{f(s, x_1), f(x_1, x_2), \dots, f(x_{k-1}, x_k), f(x_k, t)\}$$

Lemma 94

Let S be some minimum r - s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v - w -cut T with $T \subset S$.

Proof: Let X be a minimum v - w cut with $X \cap S \neq \emptyset$ and $X \cap (V \setminus S) \neq \emptyset$. Note that $S \setminus X$ and $S \cap X$ are v - w cuts inside S .

We may assume w.l.o.g. $s \in X$.

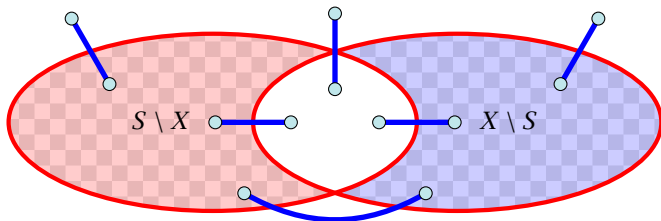
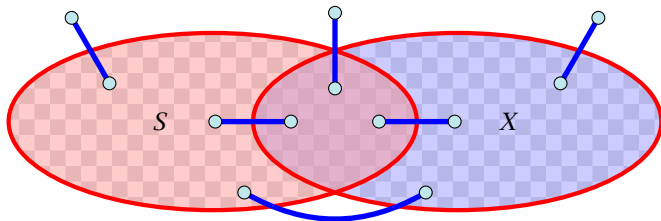
First case $r \in X$.

- ▶ $\text{cap}(X \setminus S) + \text{cap}(S \setminus X) \leq \text{cap}(S) + \text{cap}(X)$.
- ▶ $\text{cap}(X \setminus S) \geq \text{cap}(S)$ because $X \setminus S$ is an r - s cut.
- ▶ This gives $\text{cap}(S \setminus X) \leq \text{cap}(X)$.

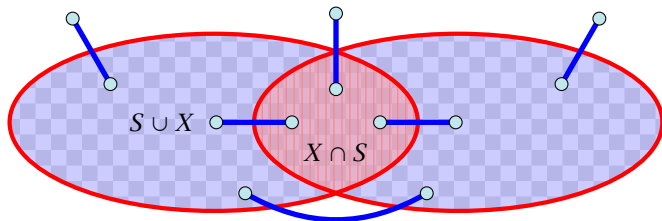
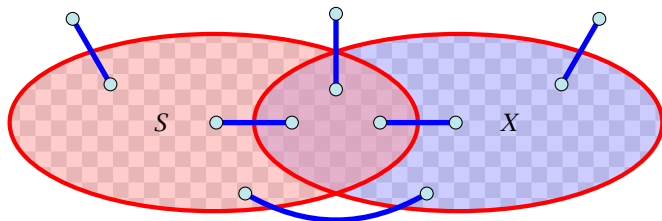
Second case $r \notin X$.

- ▶ $\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$.
- ▶ $\text{cap}(X \cup S) \geq \text{cap}(S)$ because $X \cup S$ is an r - s cut.
- ▶ This gives $\text{cap}(S \cap X) \leq \text{cap}(X)$.

$$\text{cap}(S \setminus X) + \text{cap}(X \setminus S) \leq \text{cap}(S) + \text{cap}(X)$$



$$\text{cap}(X \cup S) + \text{cap}(S \cap X) \leq \text{cap}(S) + \text{cap}(X)$$



Analysis

Lemma 94 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s, t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s, t) = f(s, t)$, where $f_H(s, t)$ is the value of a minimum s - t mincut in graph H .

Invariant [existence of representatives]:

For any edge $\{S_i, S_j\}$ in T , there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i, S_j) = f(a, b)$ and the cut defined by edge $\{S_i, S_j\}$ is a minimum a - b cut in G .

Analysis

We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

- ▶ Let $s = x_0, x_1, \dots, x_{k-1}, x_k = t$ be the unique simple path from s to t in the final tree T . From the invariant we get that $f(x_i, x_{i+1}) = w(x_i, x_{i+1})$ for all j .
- ▶ Then

$$\begin{aligned} f_T(s, t) &= \min_{i \in \{0, \dots, k-1\}} \{w(x_i, x_{i+1})\} \\ &= \min_{i \in \{0, \dots, k-1\}} \{f(x_i, x_{i+1})\} \leq f(s, t) . \end{aligned}$$

- ▶ Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.
- ▶ Since by the invariant this edge induces an s - t cut with capacity $f(x_j, x_{j+1})$ we get $f(s, t) \leq f(x_j, x_{j+1}) = f_T(s, t)$.

Analysis

- ▶ Hence, $f_T(s, t) = f(s, t)$ (flow equivalence).
- ▶ The edge $\{x_j, x_{j+1}\}$ is a mincut between s and t in T .
- ▶ By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in G (which separates s and t).
- ▶ Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. s and t , this is an s - t mincut (cut property).

Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Let S_i denote our selected cluster with nodes a and b . Because of the invariant all edges leaving $\{S_i\}$ in T correspond to some mincuts.

Therefore, contracting the connected components does not change the mincut btw. a and b due to Lemma 94.

After the split we have to choose representatives for all edges. For the new edge $\{S_i^a, S_i^b\}$ with capacity $w(S_i^a, S_i^b) = f_H(a, b)$ we can simply choose a and b as representatives.

Proof of Invariant

For edges that are not incident to S_i we do not need to change representatives as the neighbouring sets do not change.

Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S_i^a\}$ in the new tree (the case when it is replaced by $\{X, S_i^b\}$ is analogous).

If $s \in S_i^a$ we can keep x and s as representatives.

Otherwise, we choose x and a as representatives. We need to show that $f(x, a) = f(x, s)$.

Proof of Invariant

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in G of capacity $f(x, s)$. Since, x and a are on opposite sides of this cut, we know that $f(x, a) \leq f(x, s)$.

The set B forms a mincut separating a from b . Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B . Because of Lemma 94 we know that $f'(x, a) = f(x, a)$ as $x, a \notin B$.

We further have $f'(x, a) \geq \min\{f'(x, v_B), f'(v_B, a)\}$.

Since $s \in B$ we have $f'(v_B, x) \geq f(s, x)$.

Also, $f'(a, v_B) \geq f(a, b) \geq f(x, s)$ since the a - b cut that splits S_i into S_i^a and S_i^b also separates s and x .

Analysis

