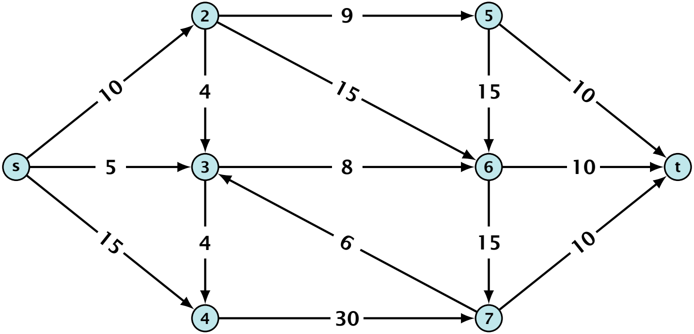
Part IV

Flows and Cuts

11 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source s; target t;
- no edges entering s or leaving t;
- at least for now: no parallel edges;



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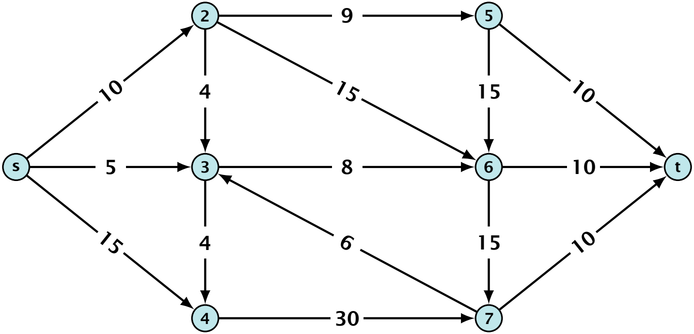

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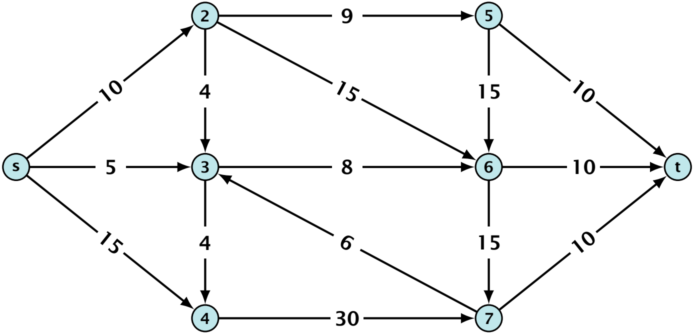

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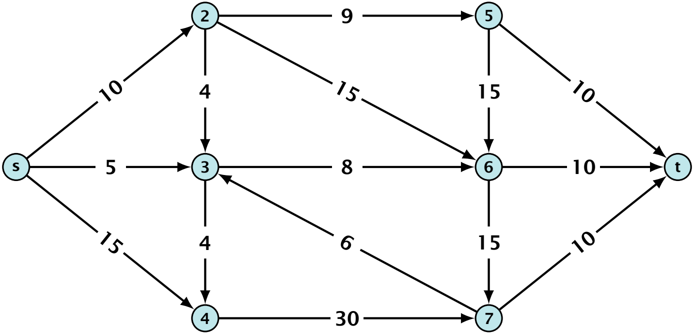

Cuts

Definition 41

An (s,t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

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The capacity of a cut A is defined as

$$\operatorname{cap}(A, V \setminus A) := \sum_{e \in \operatorname{out}(A)} c(e) \enspace ,$$

where $\operatorname{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

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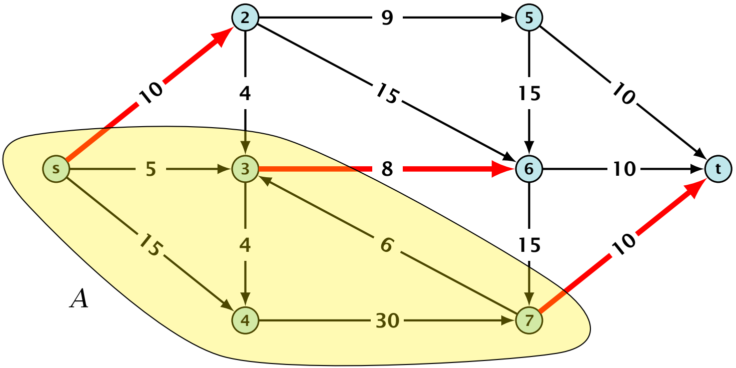
Minimum Cut Problem: Find an (s,t)-cut with minimum capacity.

Cuts

Example 43



The capacity of the cut is $\mathrm{cap}(A, V \setminus A) = 28$.

Flows

Definition 44

An (s,t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e
$$0 \le f(e) \le c(e) \enspace .$$
(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$
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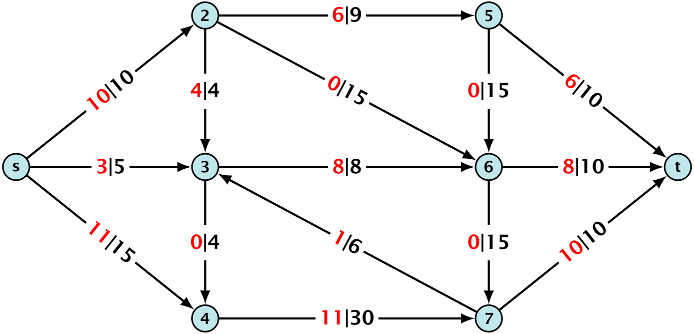

The value of the flow is $\mathrm{val}(f) = 24$.

Lemma 47 (Flow value lemma)

Let f a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \enspace .$$

Proof.

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$$\begin{aligned}\operatorname{val}(f) &= \sum_{e\in\operatorname{out}(s)} f(e)\\ &= \sum_{e\in\operatorname{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}}\Bigg(\sum_{e\in\operatorname{out}(v)} f(e) - \sum_{e\in\operatorname{in}(v)} f(e)\Bigg)\end{aligned}$$

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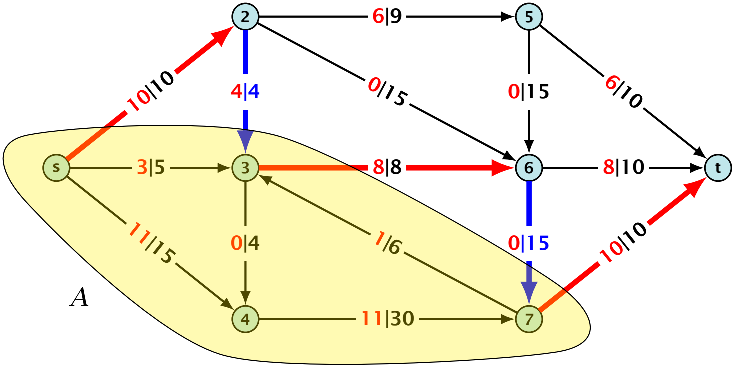
&= \sum_{e\in\mathrm{out}(A)} f(e) - \sum_{e\in\mathrm{into}(A)} f(e)
\end{aligned}$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A.

□

Example 48



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3|5
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8|10
11/15
0|4
1/6
0|15
10|10
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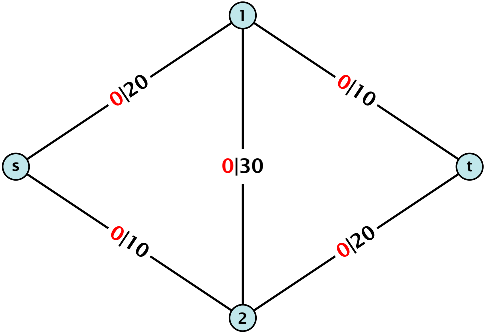
12 Augmenting Path Algorithms

Greedy-algorithm:

- start with $f(e) = 0$ everywhere
- find an s-t path with $f(e) < c(e)$ on every edge
- augment flow along the path
- repeat as long as possible



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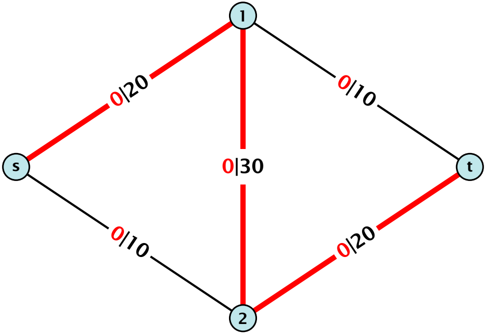

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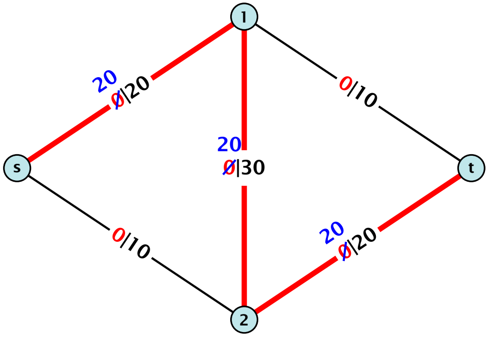

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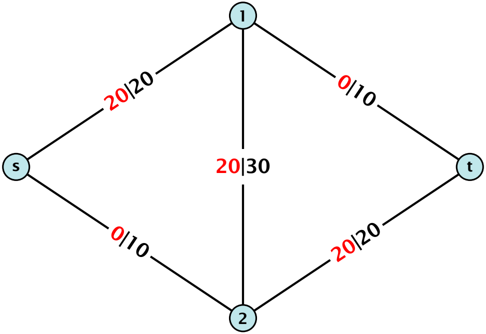

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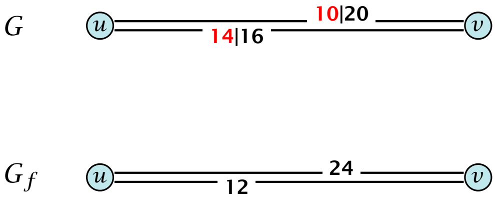

Augmenting Path Algorithm

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An augmenting path with respect to flow f, is a path in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 45 FordFulkerson($G = (V, E, c)$)

1: Initialize $f(e) \leftarrow 0$ for all edges.
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Theorem 51

*A flow f is a maximum flow **iff** there are no augmenting paths.*

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The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

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This we already showed.

2\. $\Rightarrow$ 3.
If there were an augmenting path, we could improve the flow. Contradiction.

3\. $\Rightarrow$ 1.

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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The algorithm terminates in at most $\mathrm{val}(f^*) \le nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$*. This gives a total running time of* $\mathcal{O}(nmC)$*.*

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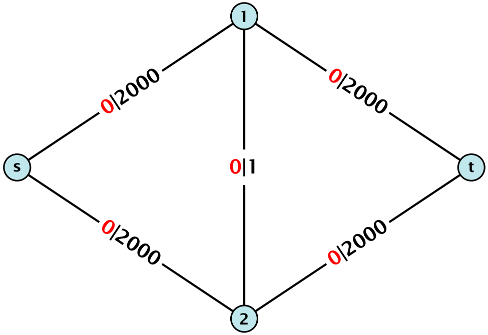
A bad input

Problem: The running time may not be polynomial.



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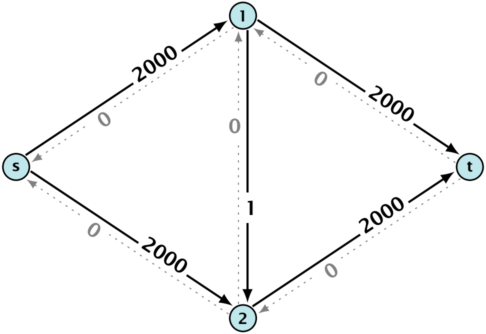

Question:
Can we tweak the algorithm so that the running time is polynomial in the input length?

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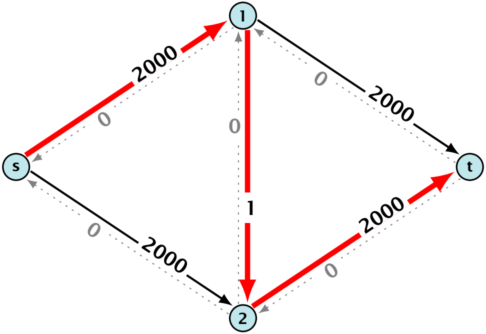

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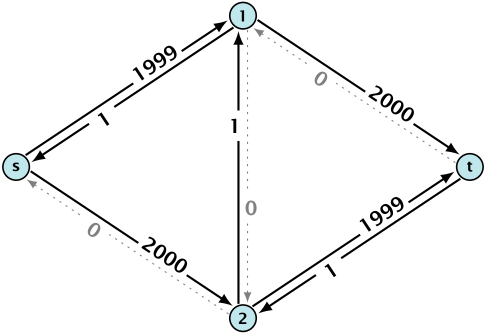

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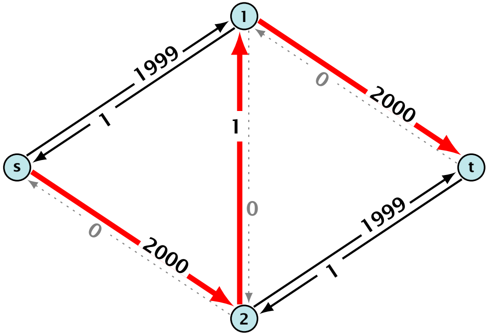

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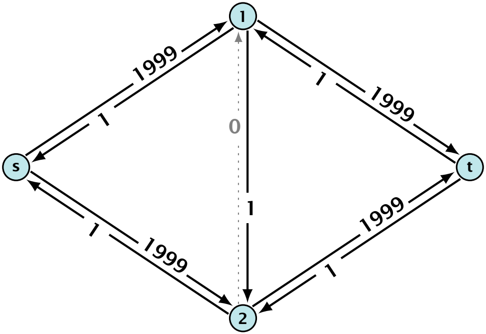

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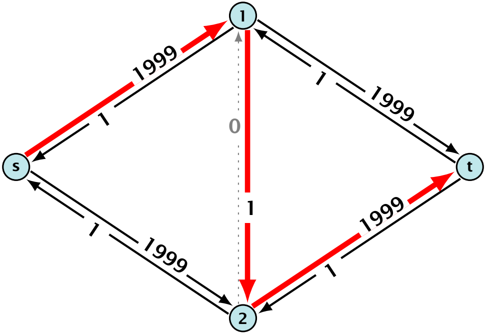

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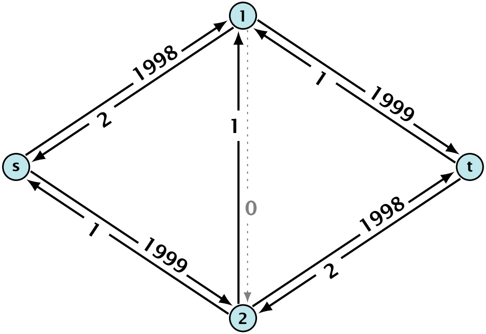

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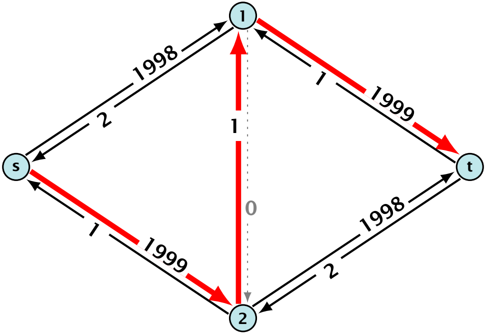

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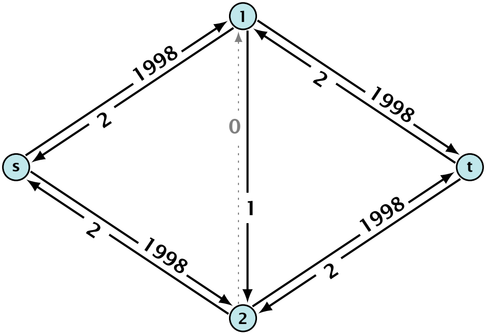

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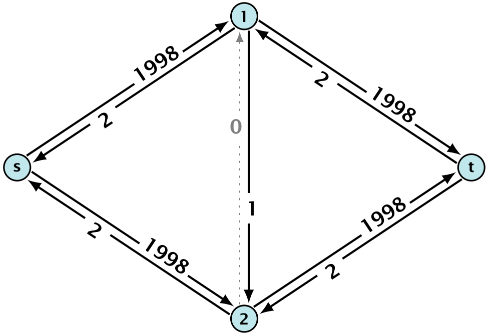

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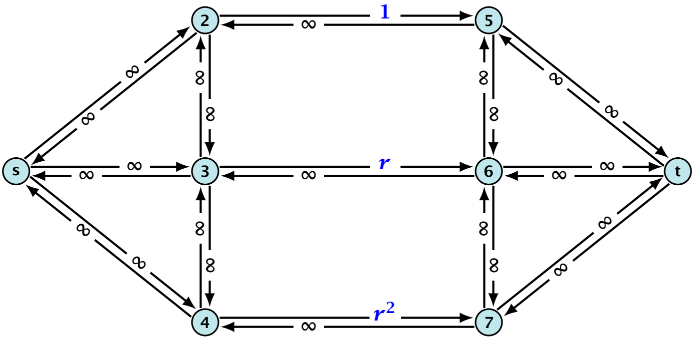
Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



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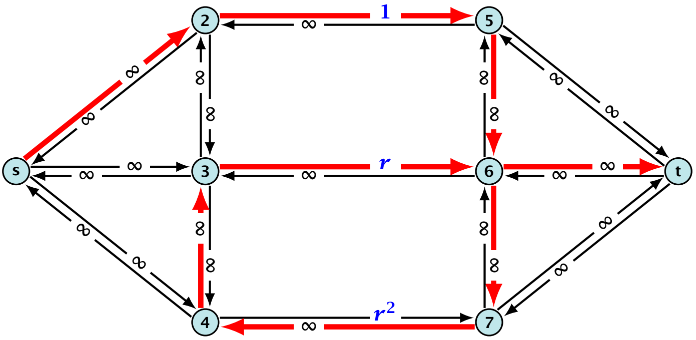

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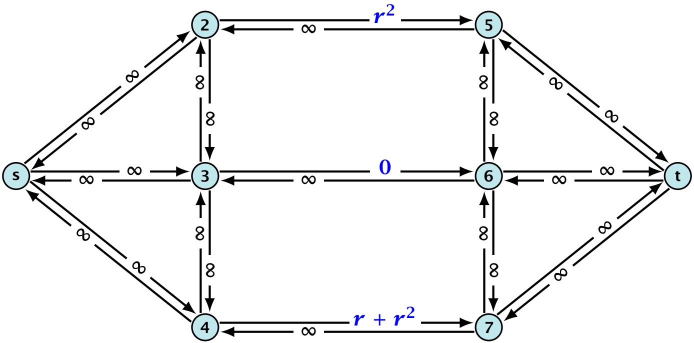

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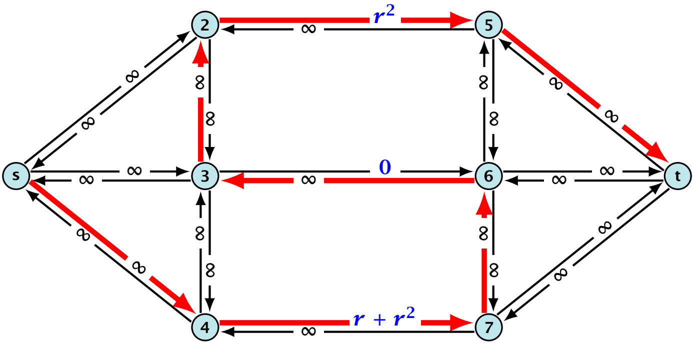

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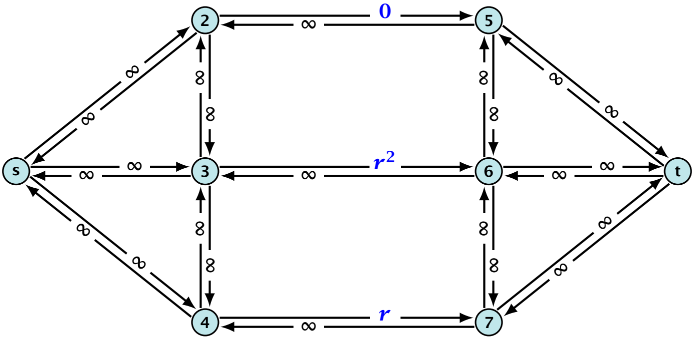

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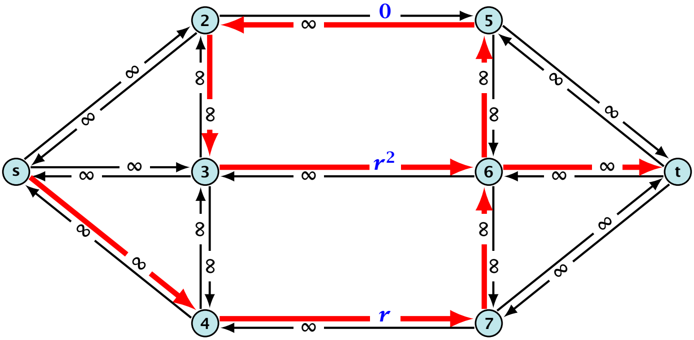

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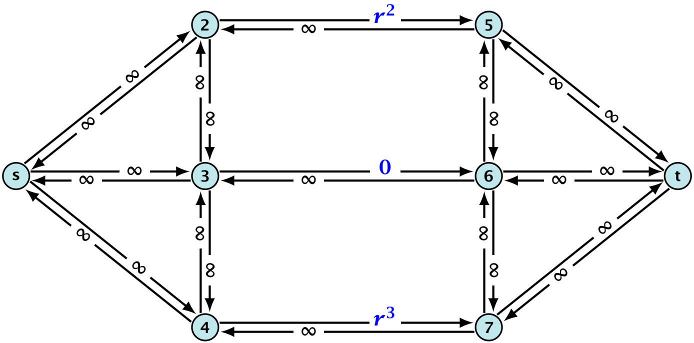

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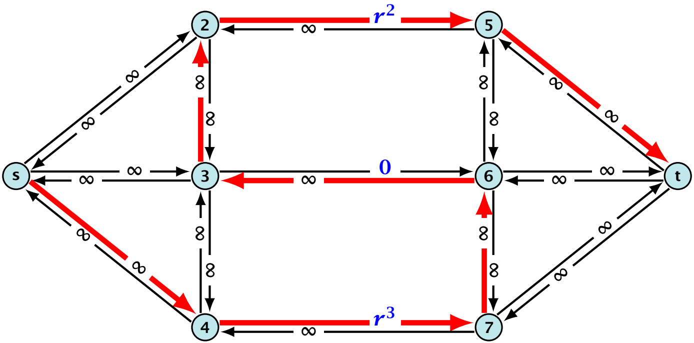

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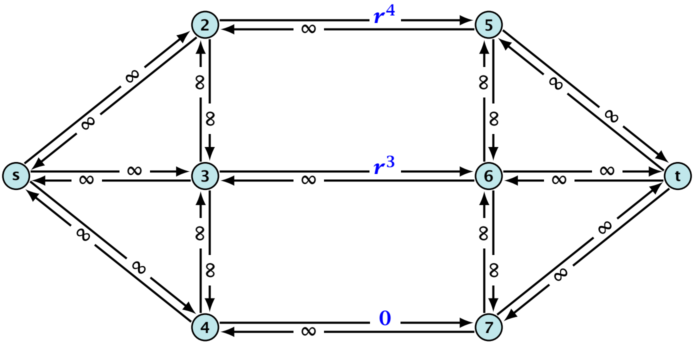

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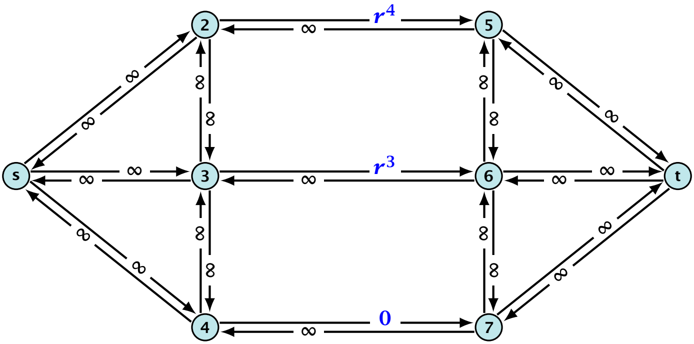

Running time may be infinite!!!

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Overview: Shortest Augmenting Paths

Lemma 55

The length of the shortest augmenting path never decreases.

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- We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
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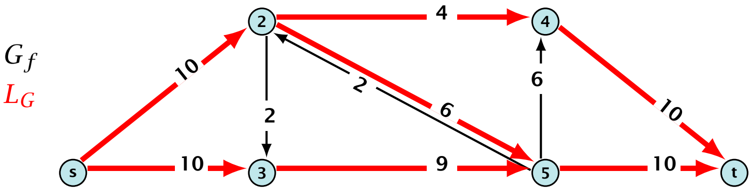

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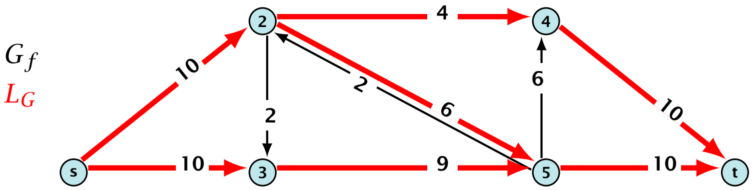

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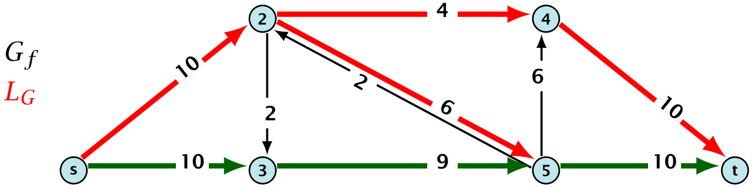

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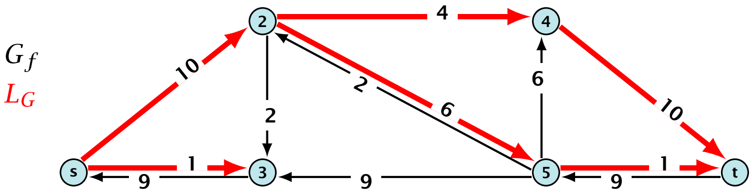

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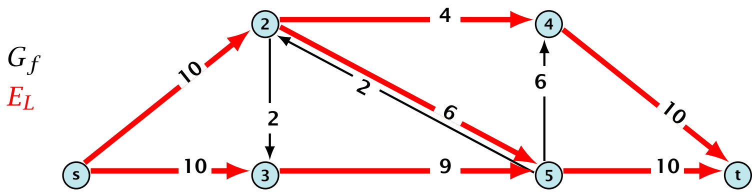

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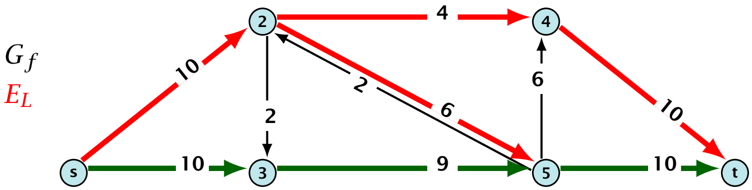

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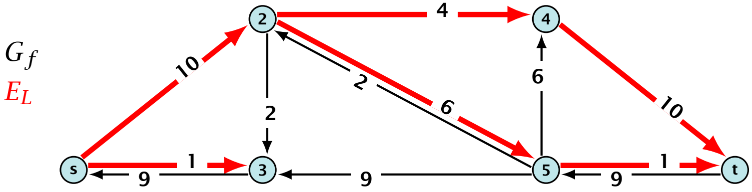

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Theorem 58

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 59 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

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There always exists a set of m augmentations that gives a maximum flow.

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We maintain a subset E_L of the edges of G_f with the guarantee that a shortest s-t path using only edges from E_L is a shortest augmenting path.

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Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

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- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter Δ.
- $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.

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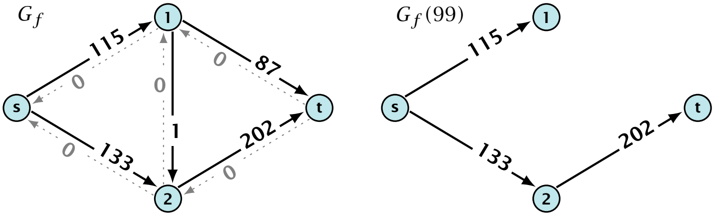

Capacity Scaling

Algorithm 46 maxflow(G, s, t, c)

foreach $e \in E$ **do** $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
while $\Delta \geq 1$ **do**
    $G_f(\Delta) \leftarrow \Delta$-residual graph
    while there is augmenting path P in $G_f(\Delta)$ **do**
        $f \leftarrow$ augment(f, c, P)
        update$(G_f(\Delta))$
    $\Delta \leftarrow \Delta/2$
return f

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All capacities are integers between 1 and C.

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- this means we have a maximum flow.

Capacity Scaling

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Lemma 60

There are $\lceil \log C \rceil$ *iterations over* Δ.

Proof: obvious.

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Let f *be the flow at the end of a* Δ*-phase. Then the maximum flow is smaller than* $\mathrm{val}(f) + 2m\Delta$*.*

Proof: less obvious, but simple:

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- An s-t cut in $G_f(\Delta)$ gives me an upper bound on the amount of flow that my algorithm can still add to f.

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Proof: less obvious, but simple:

- An s-t cut in $G_f(\Delta)$ gives me an upper bound on the amount of flow that my algorithm can still add to f.
- The edges that currently have capacity at most Δ in G_f form an s-t cut with capacity at most $2m\Delta$.

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Lemma 62

There are at most $2m$ augmentations per scaling-phase.

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- Let f be the flow at the end of the previous phase.
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Theorem 63

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

Preflows

Definition 64
An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \le f(e) \le c(e) .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_{e \in \text{out}(v)} f(e) \le \sum_{e \in \text{into}(v)} f(e) .$$

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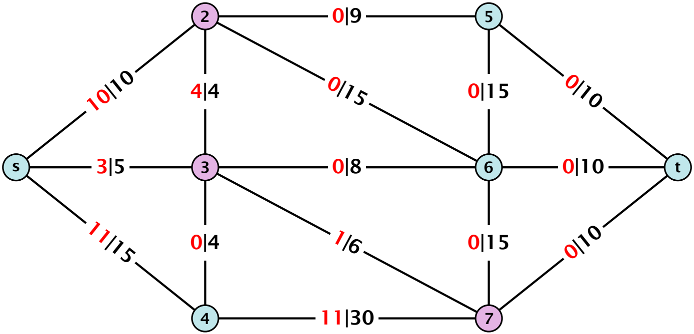
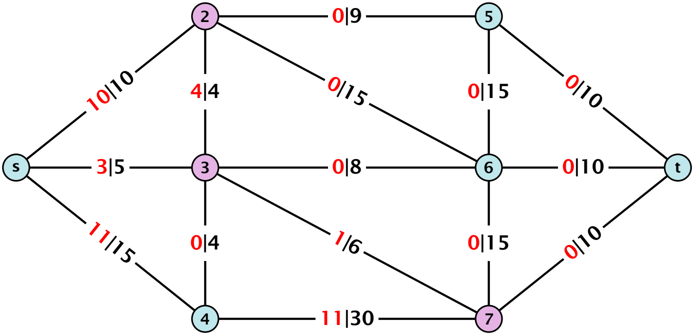


Preflows

Example 65



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

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Definition:
A labelling is a function $\ell : V \rightarrow \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \leq \ell(v) + 1$ for all edges in the residual graph G_f (only non-zero capacity edges!!!)

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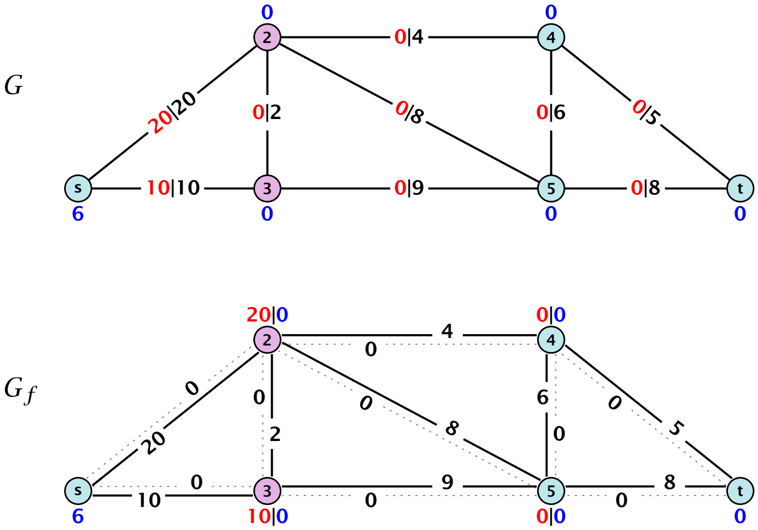
Intuition:

The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows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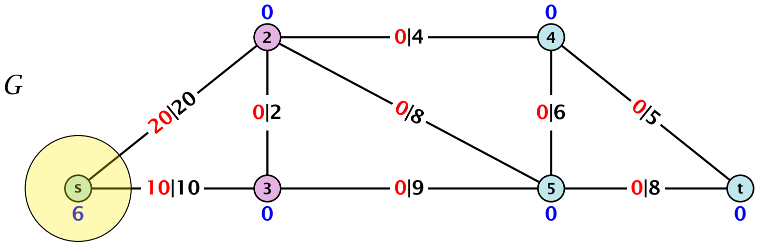
Preflows



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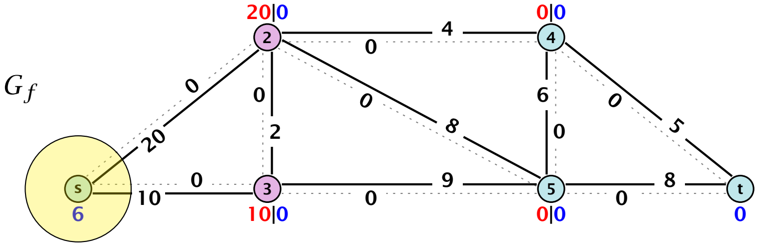

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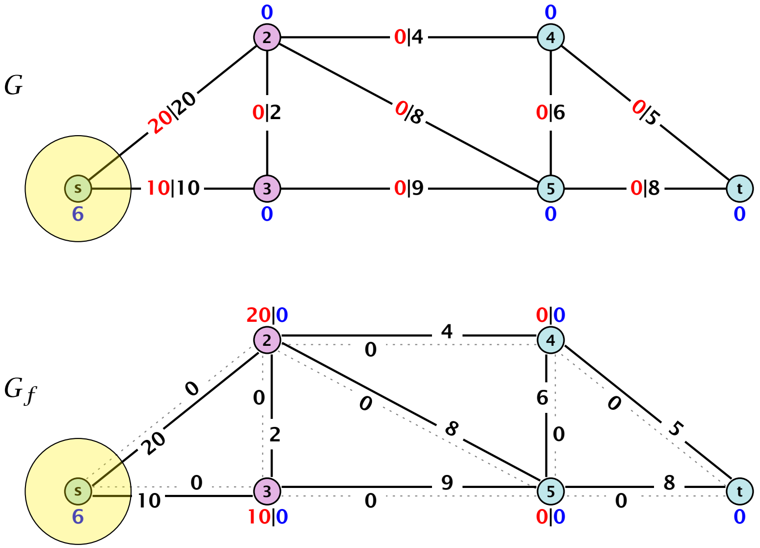
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8

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A preflow that has a valid labelling saturates a cut.

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- Let $A = \{v \in V \mid \ell(v) > d\}$ and $B = \{v \in V \mid \ell(v) < d\}$.
- We have $s \in A$ and $t \in B$ and there is no edge from A to B in the residual graph G_f; this means that (A, B) is a saturated cut.

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Lemma 67

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

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Idea:

- start with some preflow and some valid labelling

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- start with some preflow and some valid labelling
- successively change the preflow while maintaining a valid labelling
- stop when you have a flow (i.e., no more active nodes)

Changing a Preflow

An arc (u,v) with $c_f(u,v) > 0$ in the residual graph is admissable if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation

Consider an active node u with excess flow $f(u) = \sum_{e\in\text{into}(u)} f(e) - \sum_{e\in\text{out}(u)} f(e)$ and suppose $e = (u,v)$ is an admissable arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

- saturating push: $\min\{f(u), c_f(e)\} = c_f(e)$
  the arc e is deleted from the residual graph
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Push Relabel Algorithms

The relabel operation
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Increasing the label of u by 1 results in a valid labelling.

- Edges (w,u) incoming to u still fulfill their constraint $\ell(w) \le \ell(u)+1$.
- An outgoing edge (u,w) had $\ell(u) < \ell(w)+1$ before since it was not admissable. Now: $\ell(u) \le \ell(w)+1$.

Push Relabel Algorithms

Intuition:
We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water than this would be natural).

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Push Relabel Algorithms

Algorithm 47 maxflow(G, s, t, c)

```
find initial preflow f
while there is active node u do
    if there is admiss. arc e out of u then
        push(G, e, f, c)
    else
        relabel(u)
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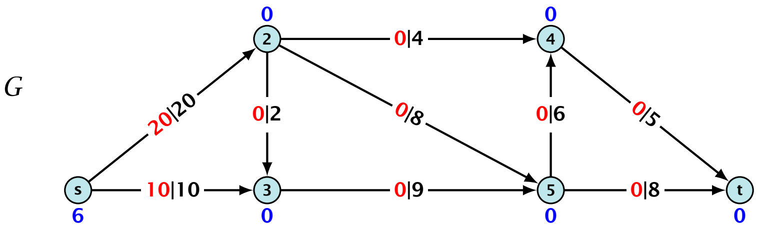
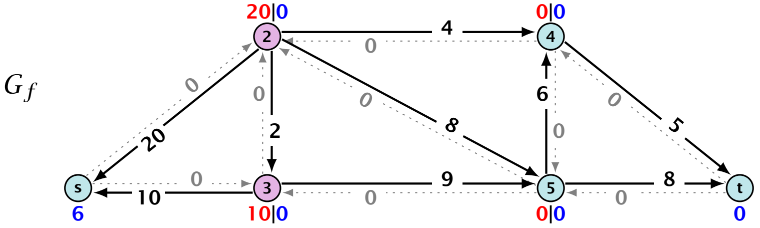
In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push Algorithm





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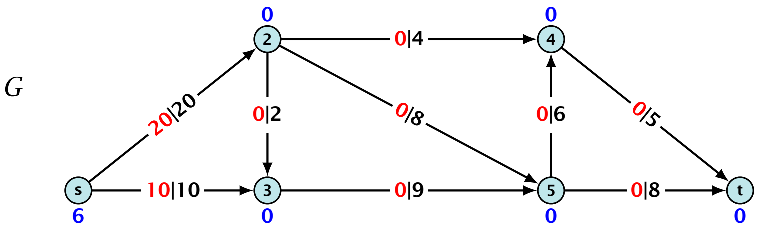
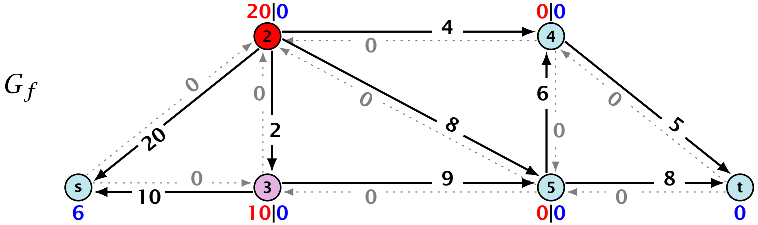
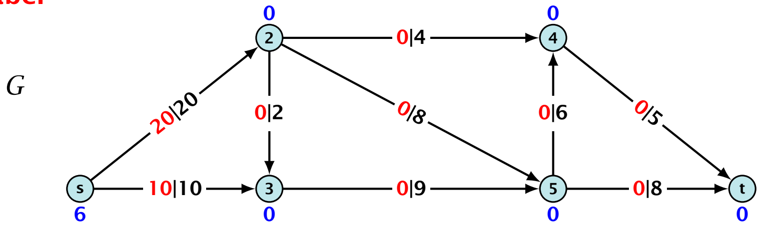
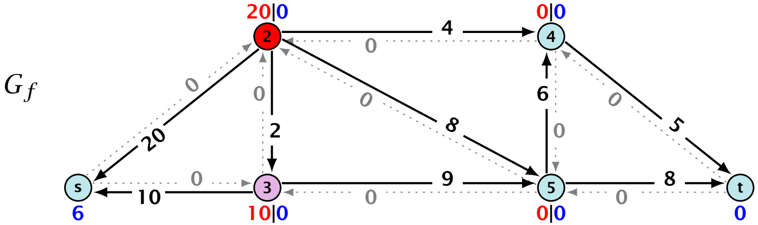




Preflow Push Algorithm

relabel





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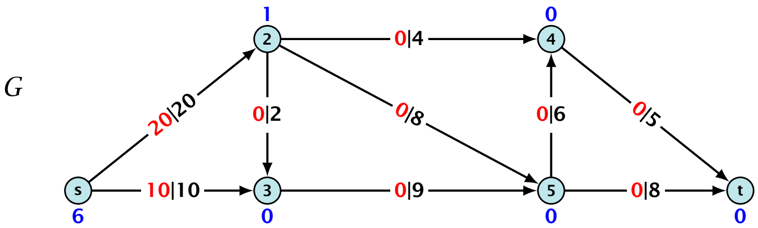
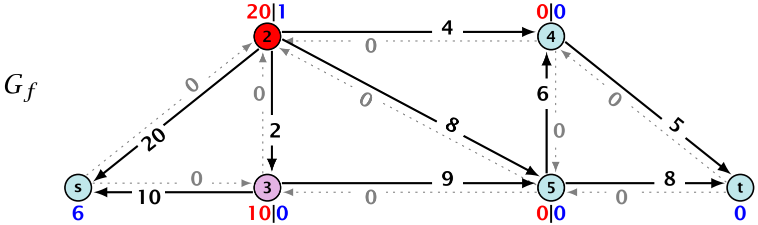
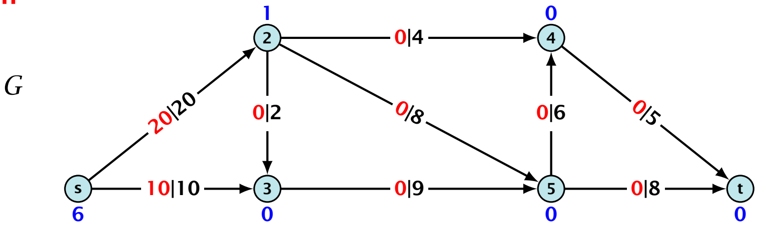
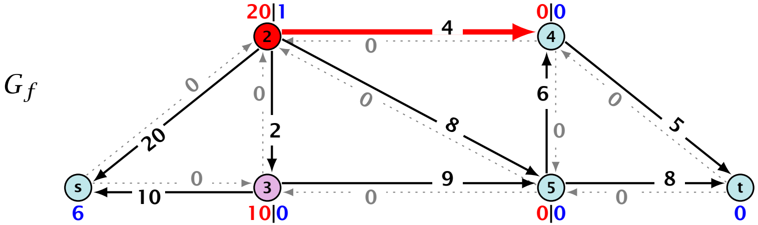




Preflow Push Algorithm

push





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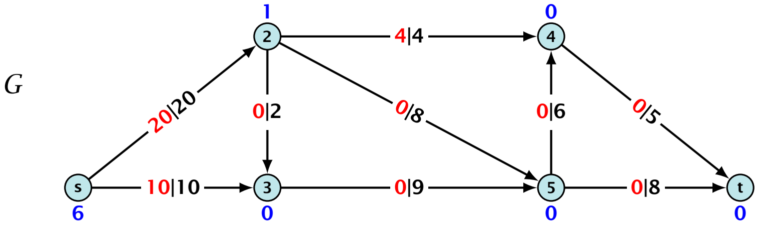
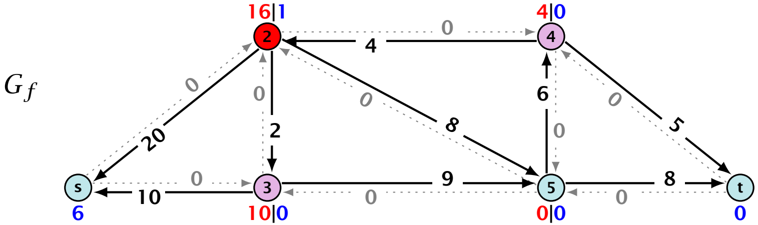
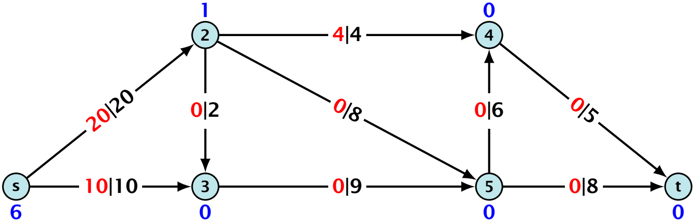




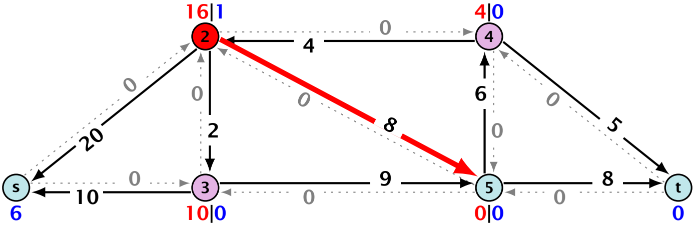
Preflow Push Algorithm

push

G



G_f



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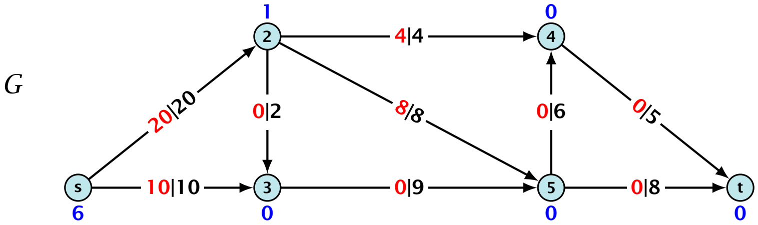

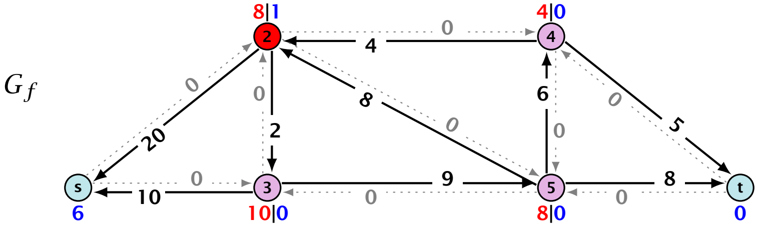

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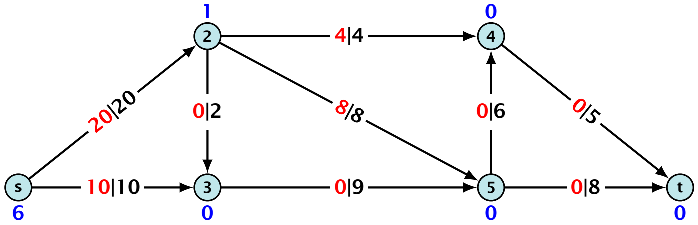
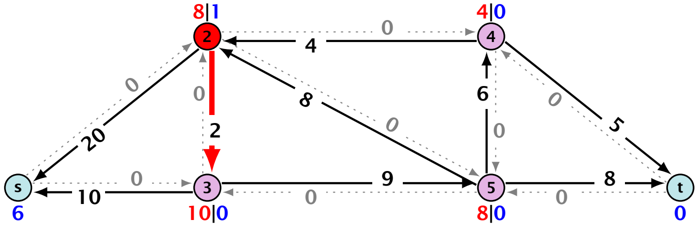
push

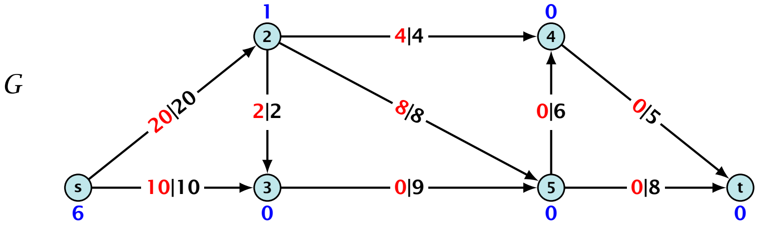
G

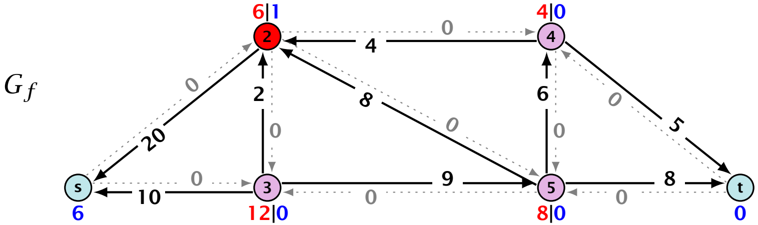


G_f



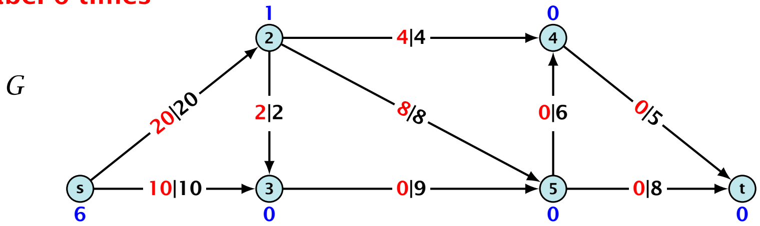
Preflow Push Algorithm

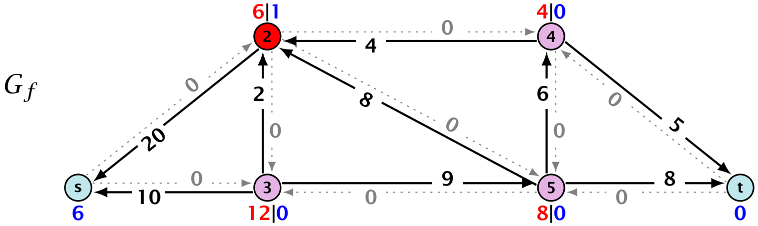




Preflow Push Algorithm

relabel 6 times





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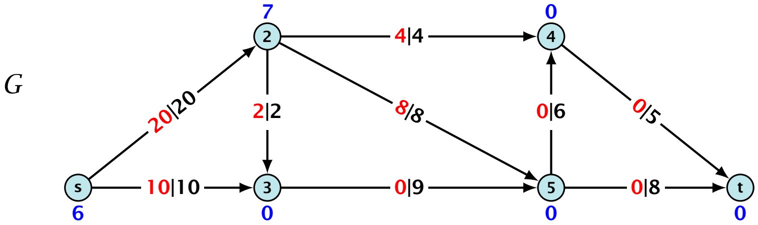
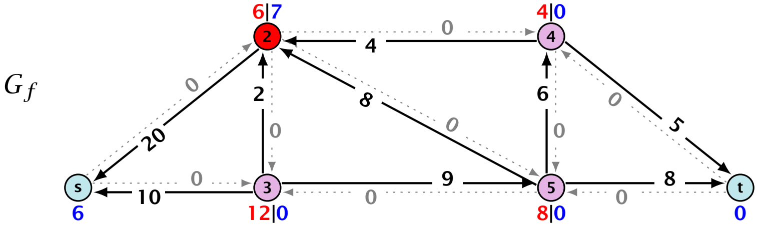
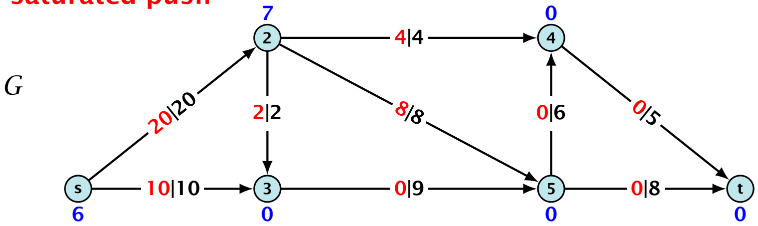
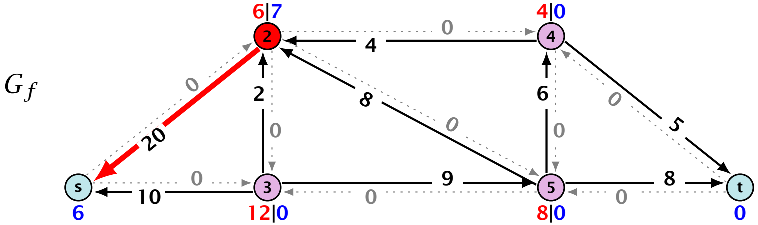




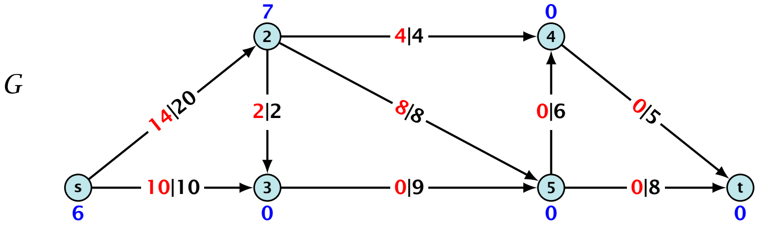
Preflow Push Algorithm

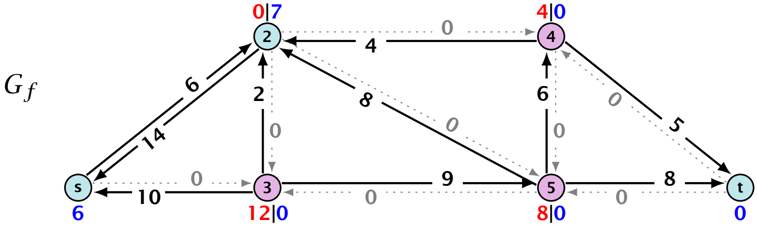
non-saturated push





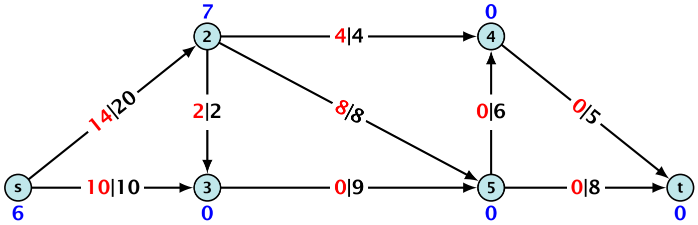
Preflow Push Algorithm





Preflow Push Algorithm

G



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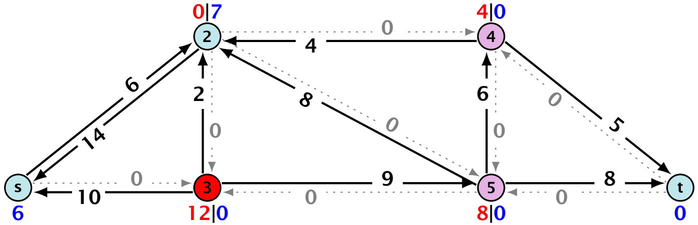
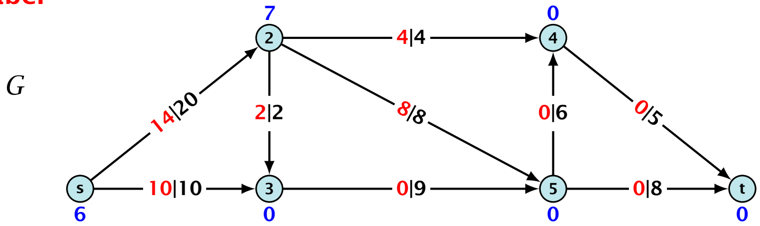
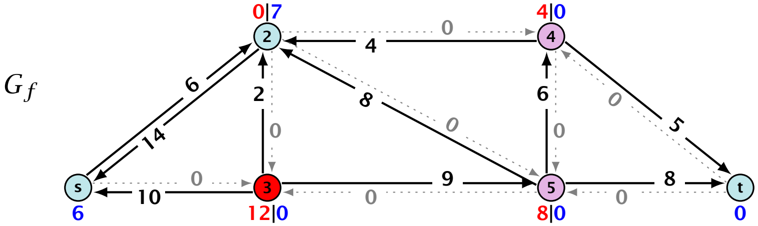


Preflow Push Algorithm

relabel





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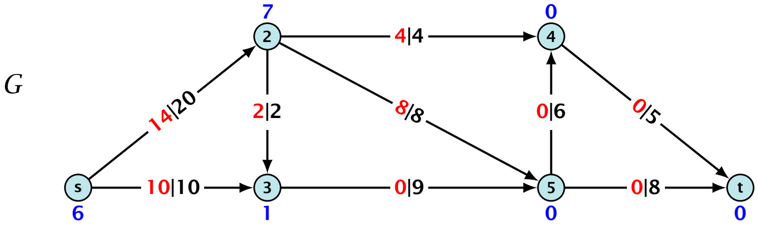
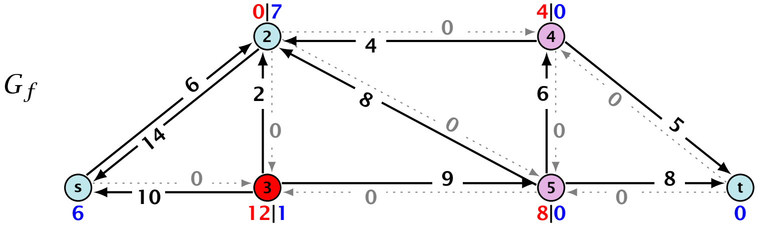
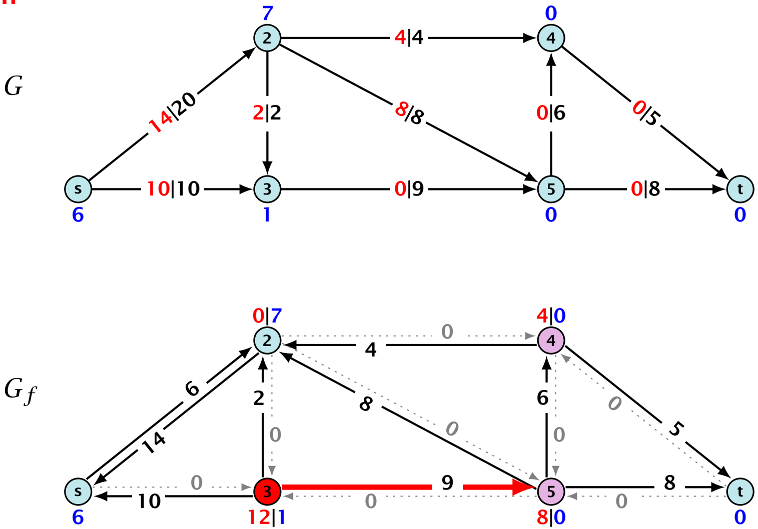




Preflow Push Algorithm

push



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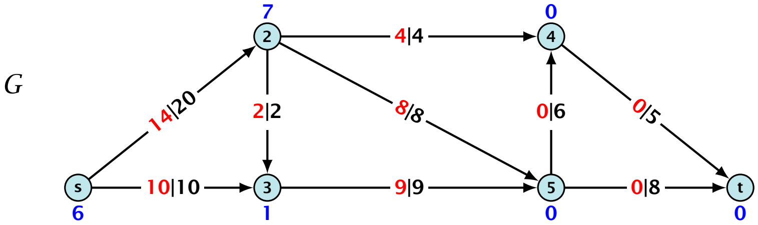
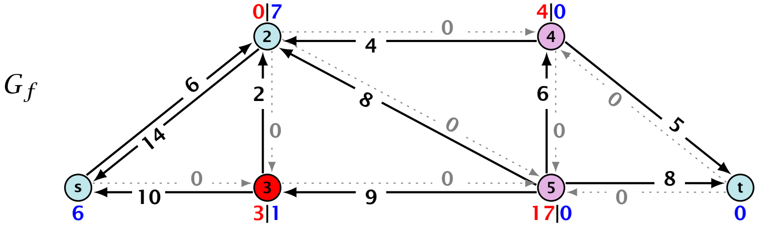
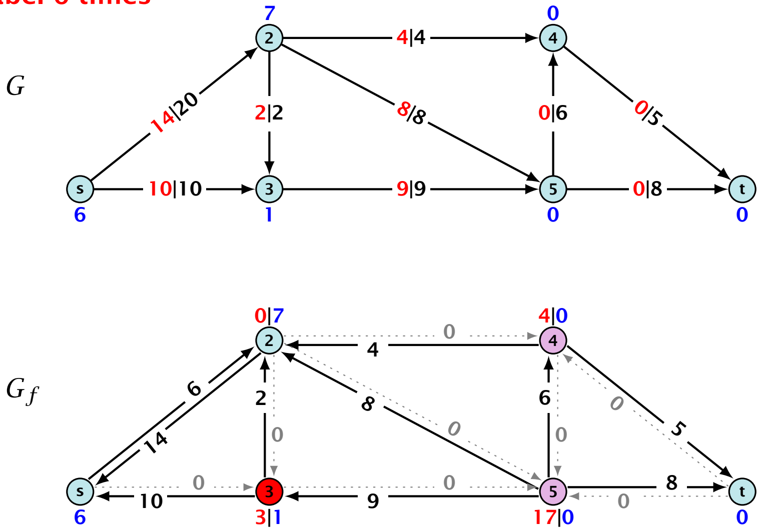




Preflow Push Algorithm

relabel 6 times



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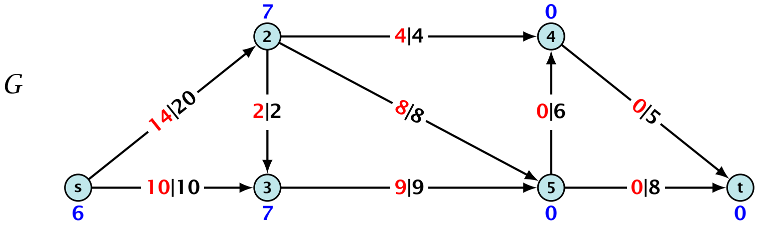
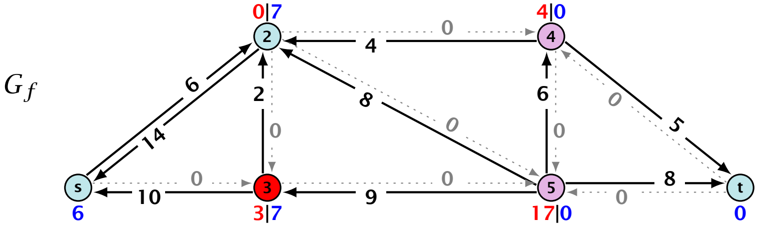
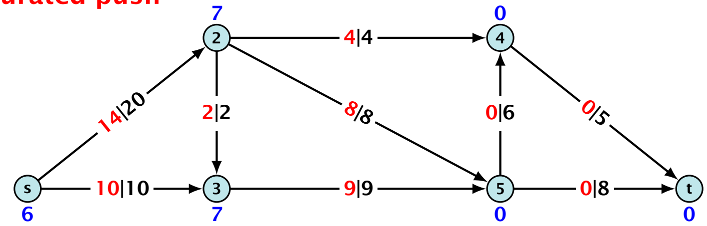




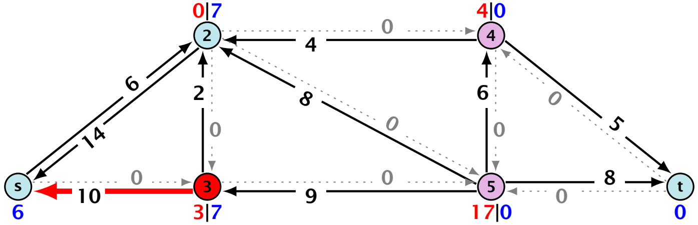
Preflow Push Algorithm

non-saturated push

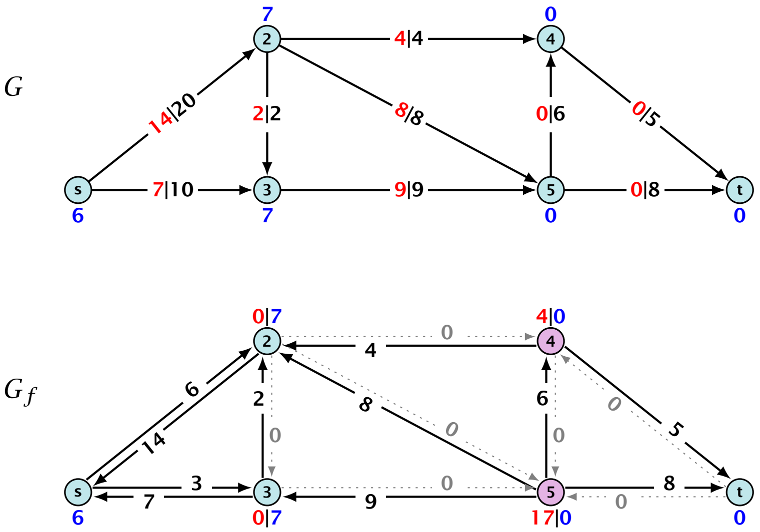
G



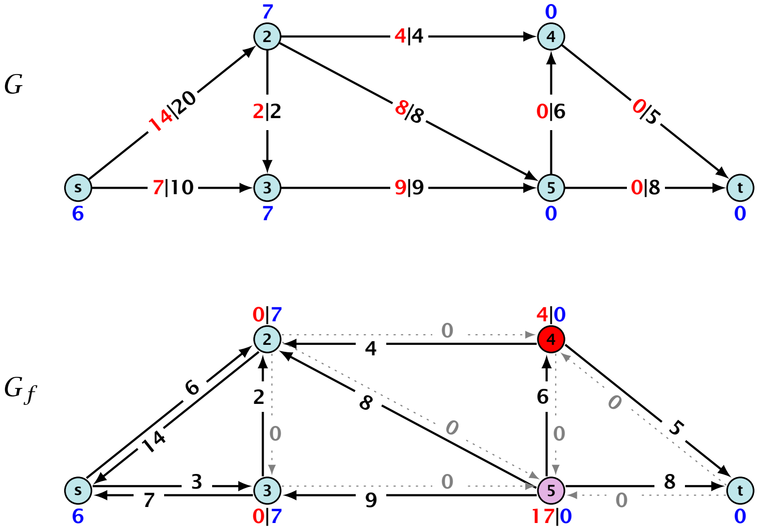
G_f



Preflow Push Algorithm



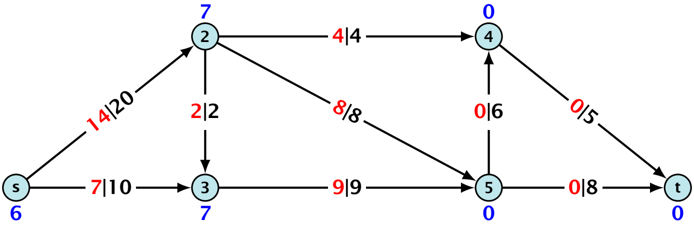
Preflow Push Algorithm



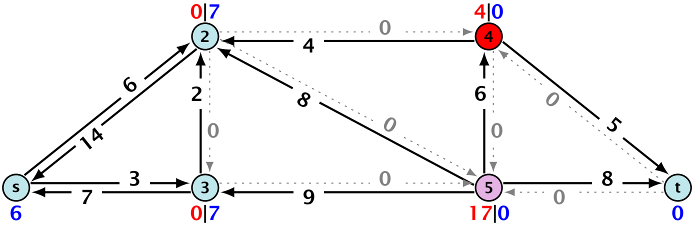
Preflow Push Algorithm

relabel

G



G_f



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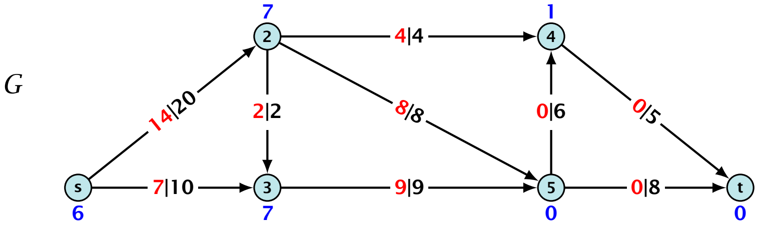
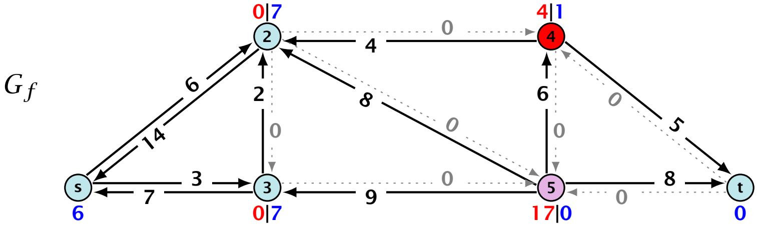
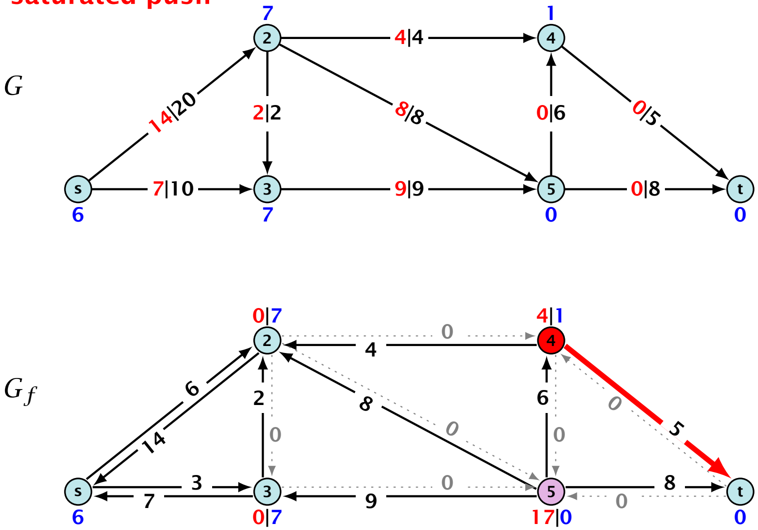


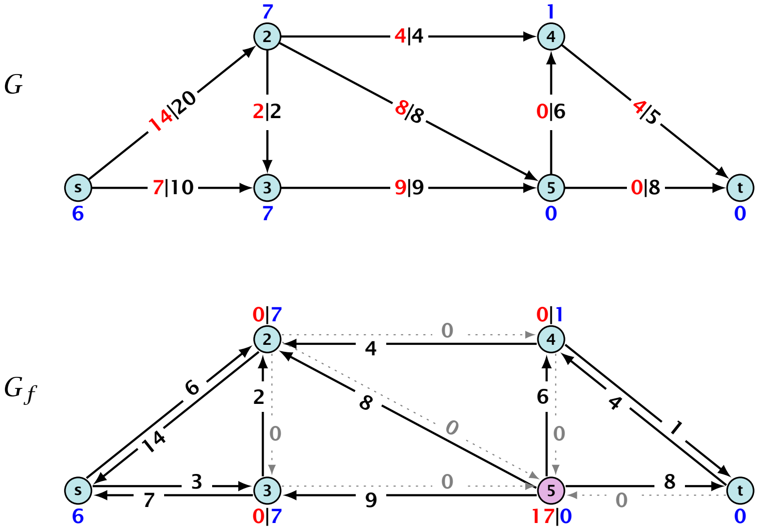


Preflow Push Algorithm

non-saturated push



Preflow Push Algorithm



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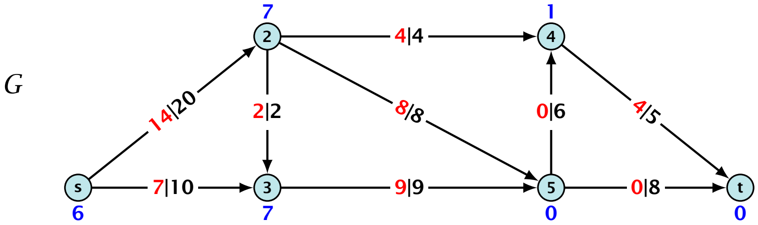
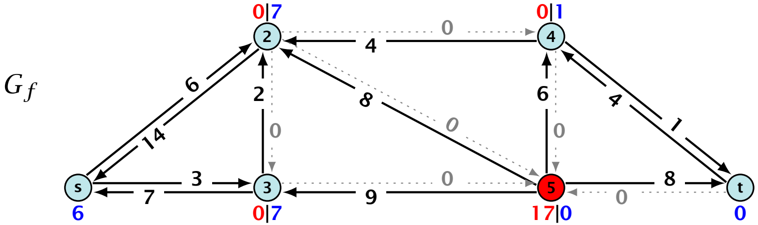
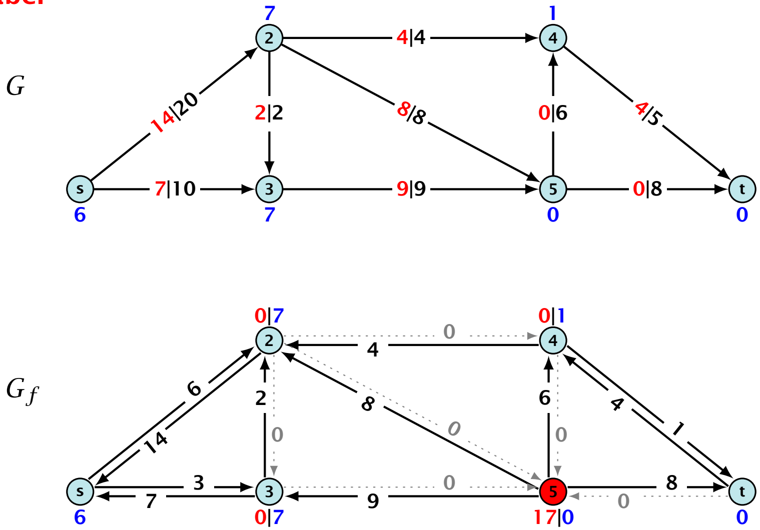




Preflow Push Algorithm

relabel



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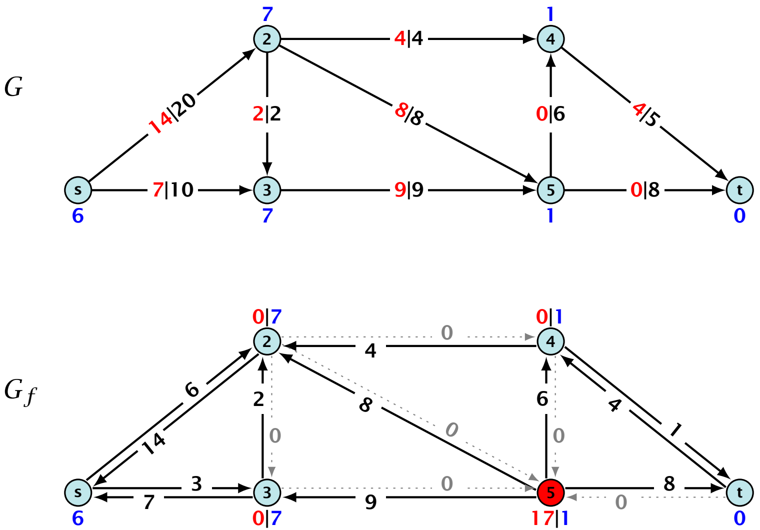
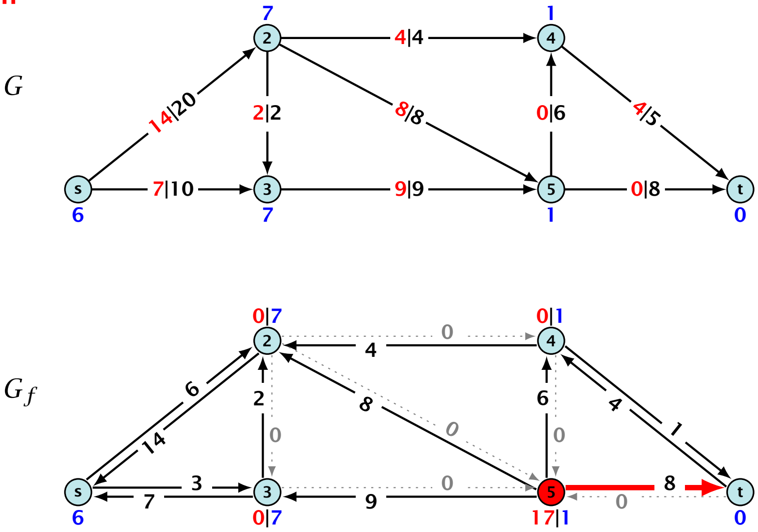


Preflow Push Algorithm

push



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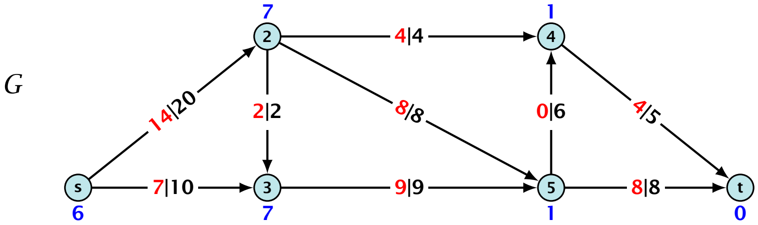
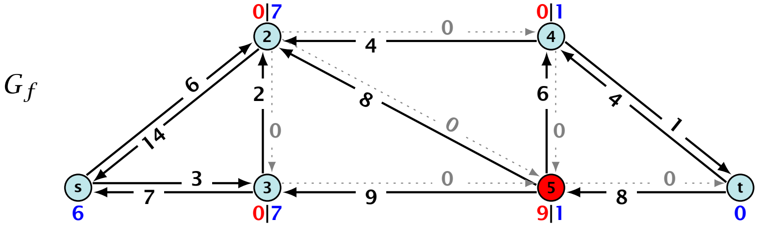
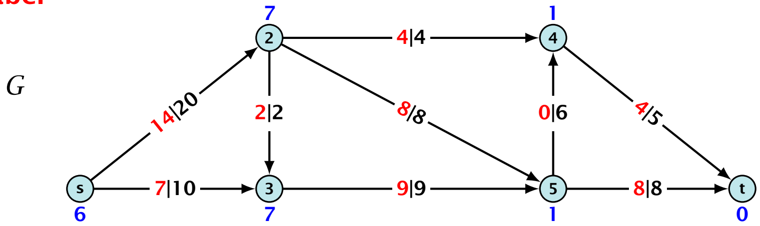
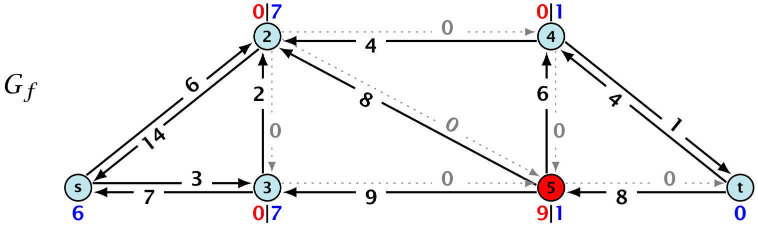




Preflow Push Algorithm

relabel





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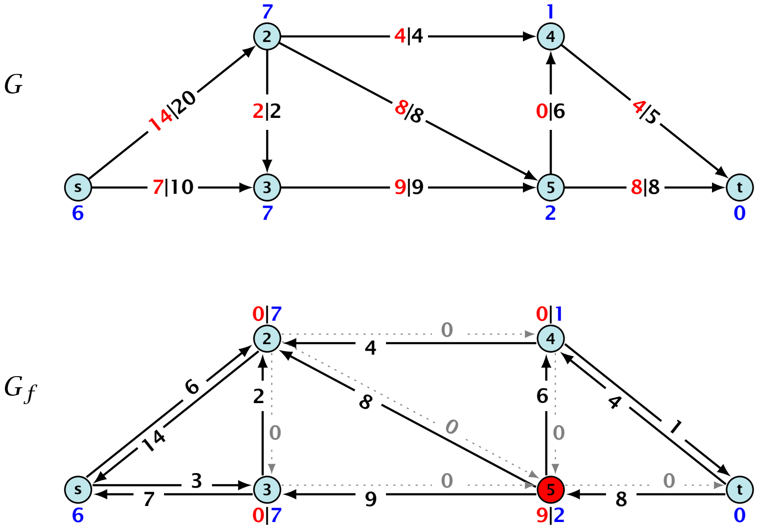
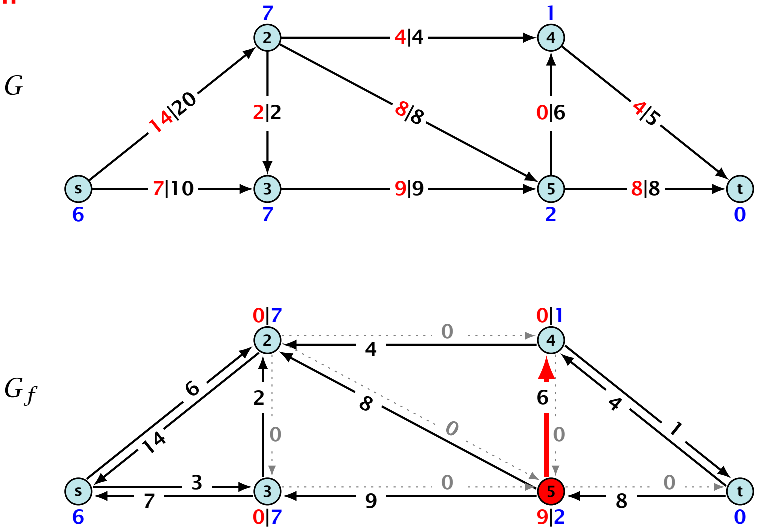


Preflow Push Algorithm

push



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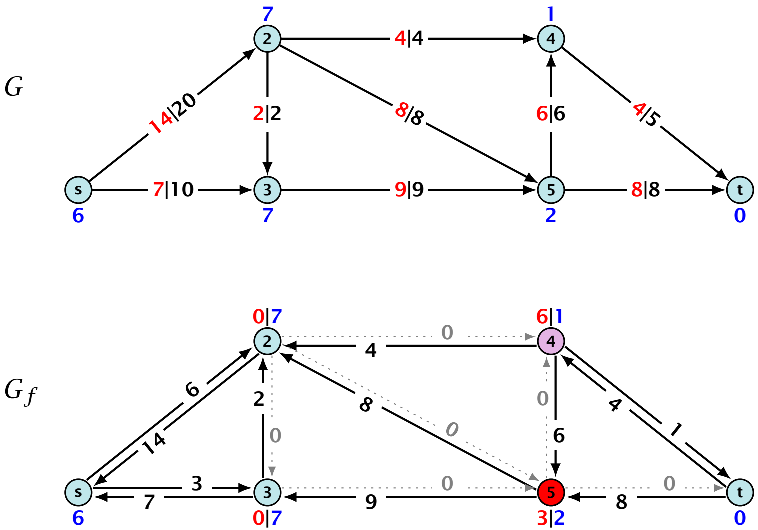
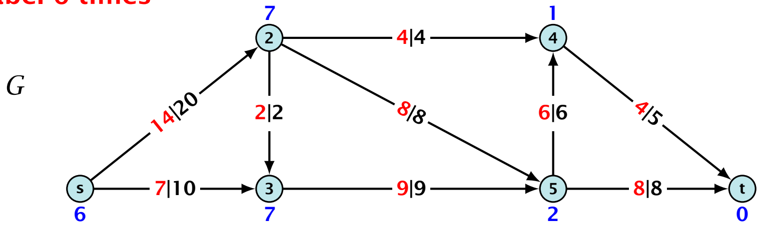
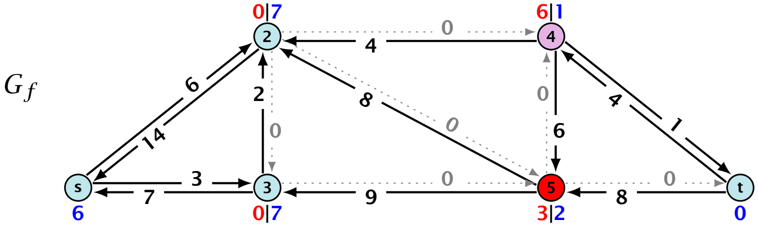


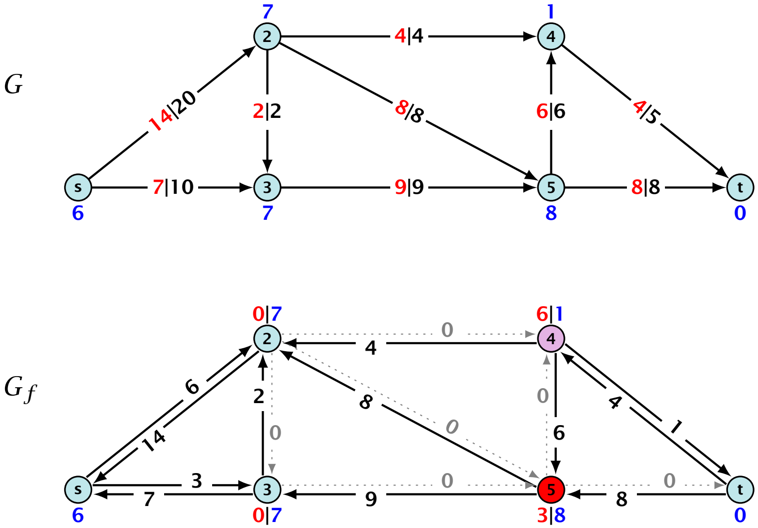
Preflow Push Algorithm

relabel 6 times



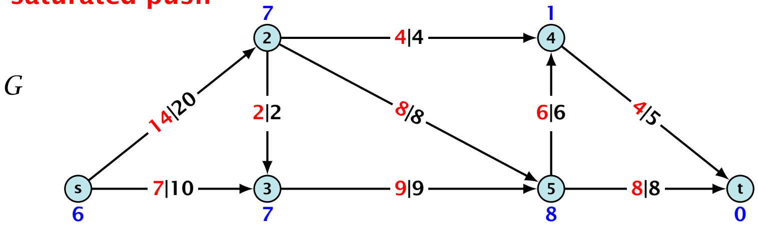


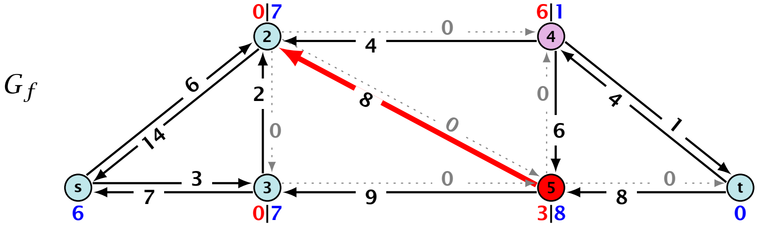
Preflow Push Algorithm



Preflow Push Algorithm

non-saturated push





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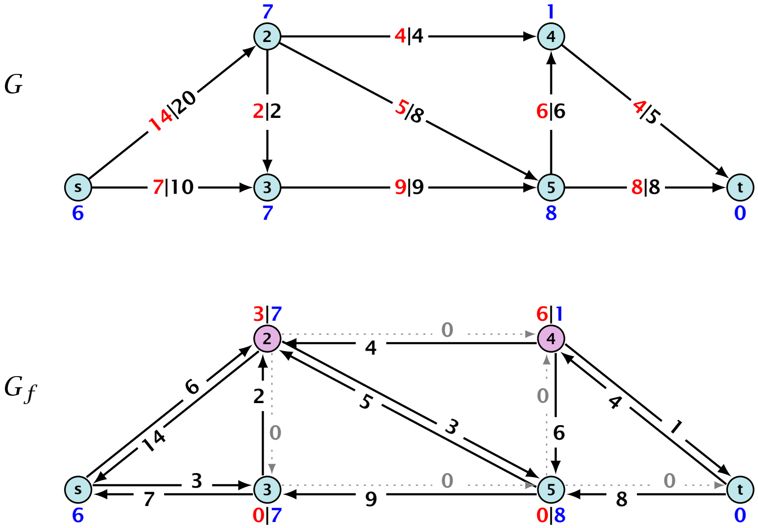

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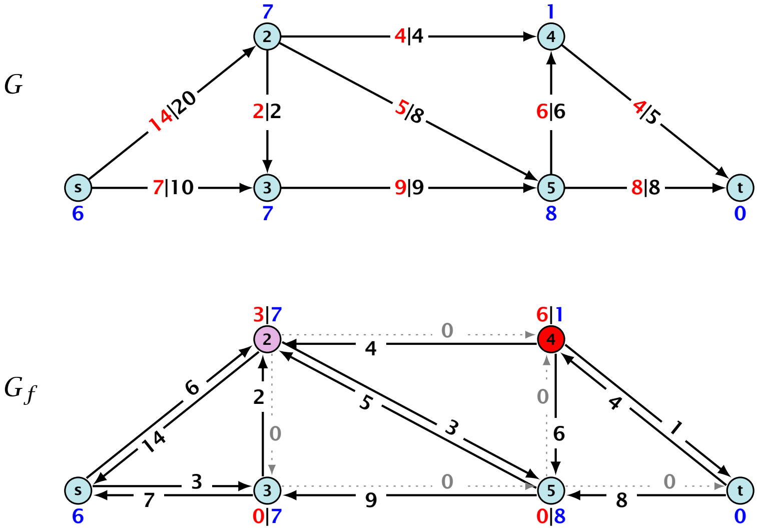
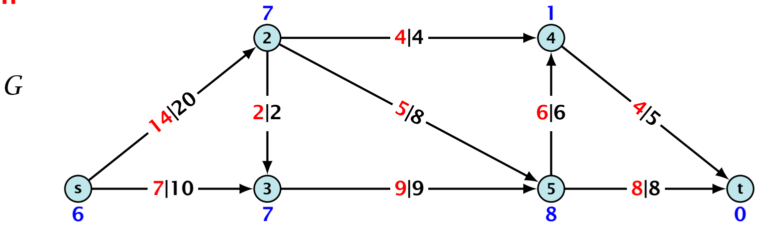
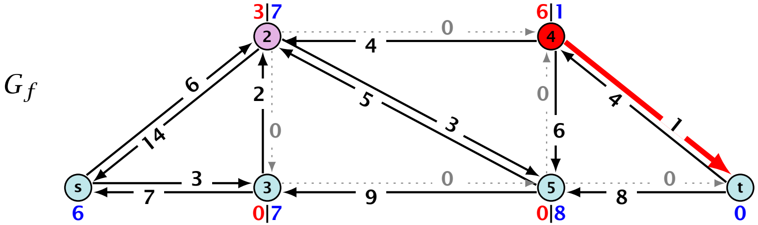


Preflow Push Algorithm

push





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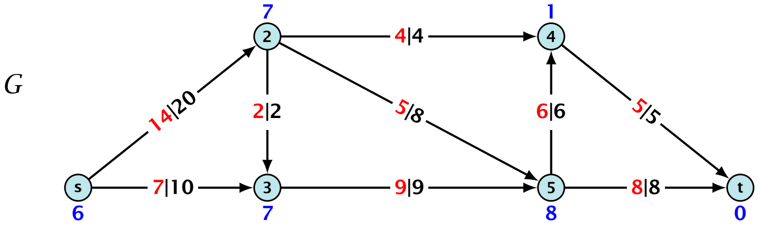
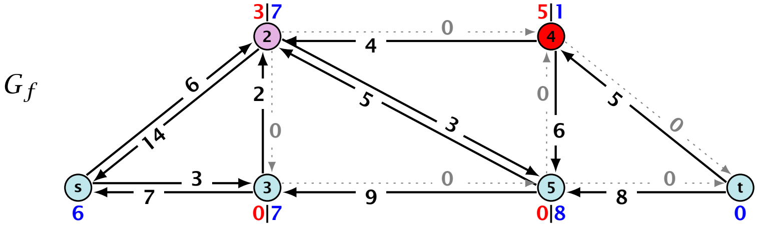
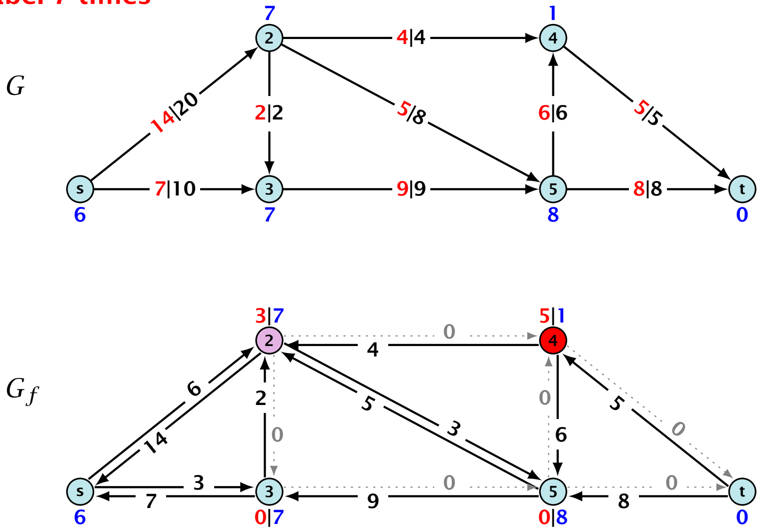


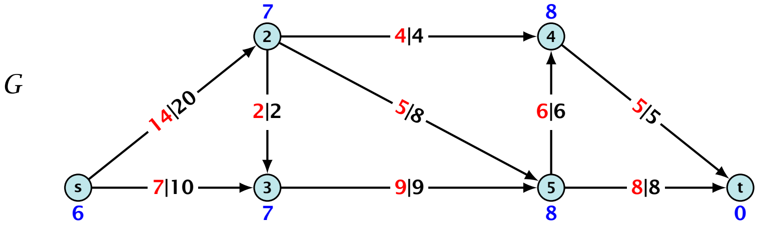


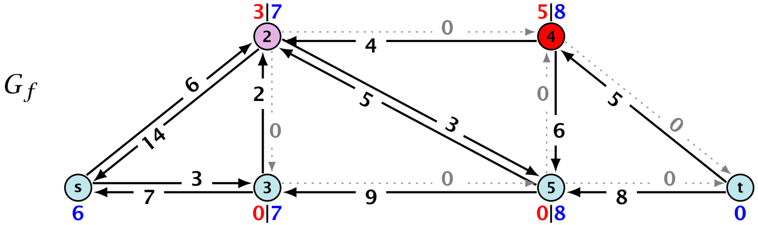
Preflow Push Algorithm

relabel 7 times



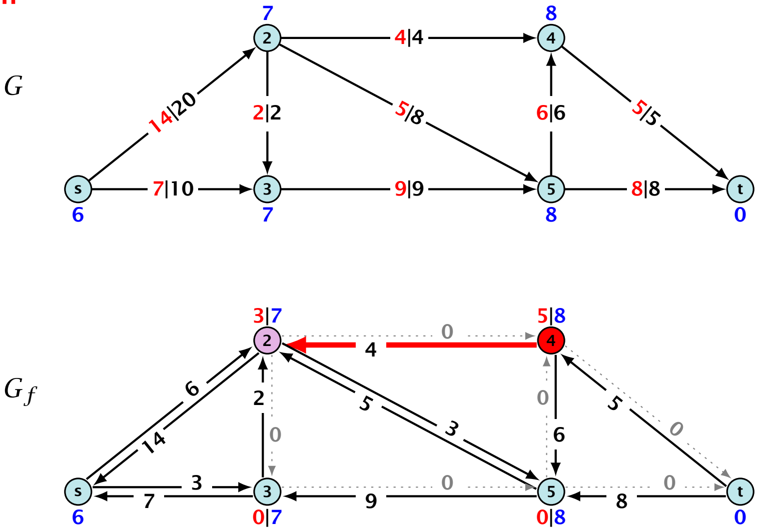
Preflow Push Algorithm



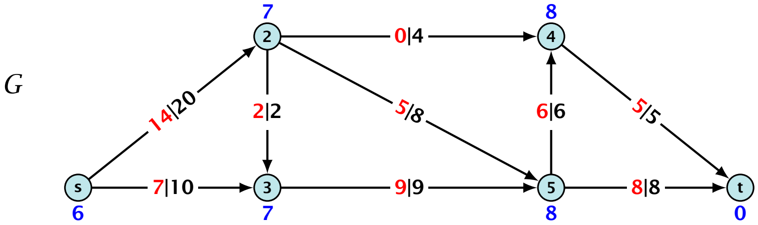


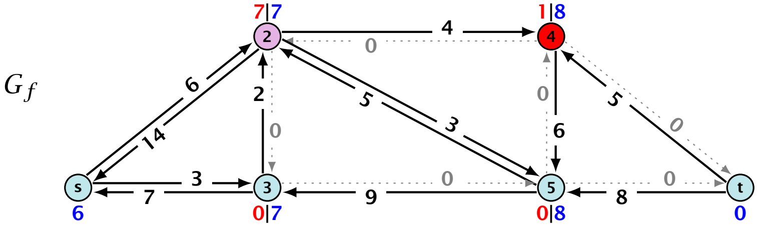
Preflow Push Algorithm

push



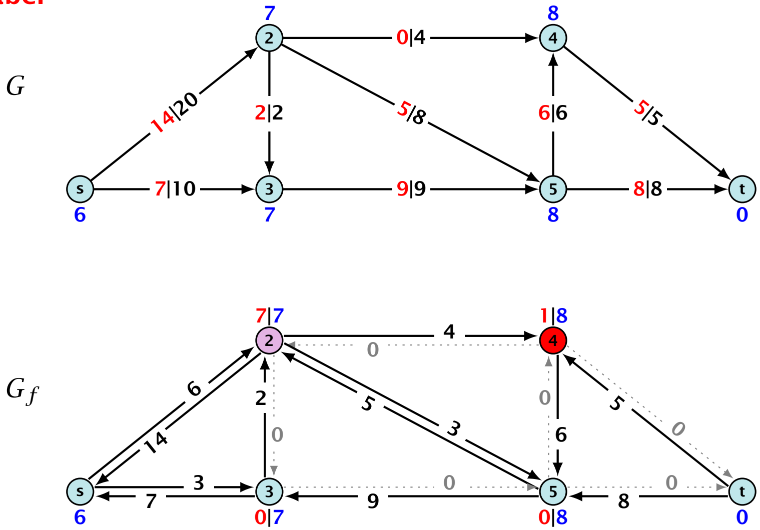
Preflow Push Algorithm



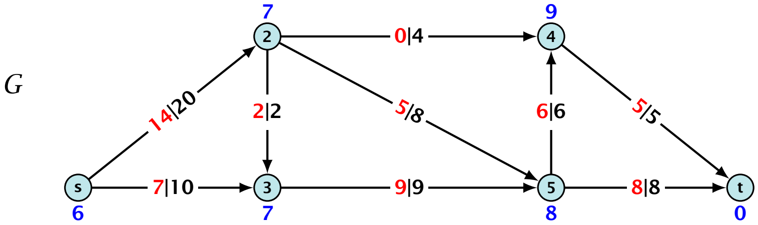


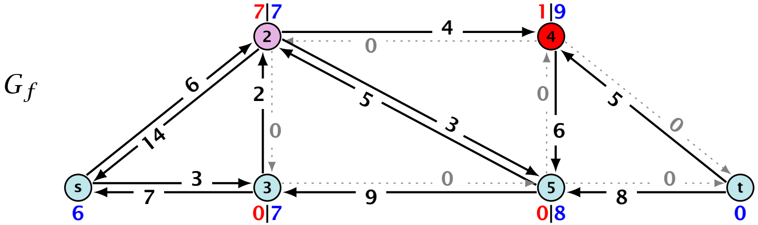
Preflow Push Algorithm

relabel



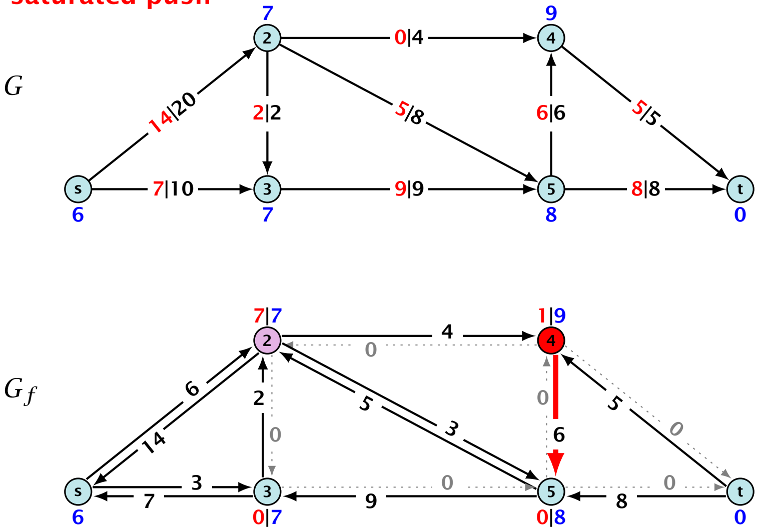
Preflow Push Algorithm



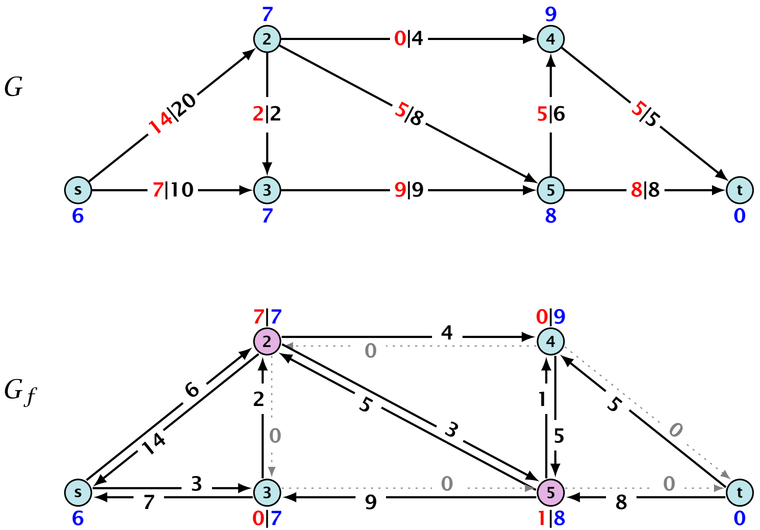


Preflow Push Algorithm

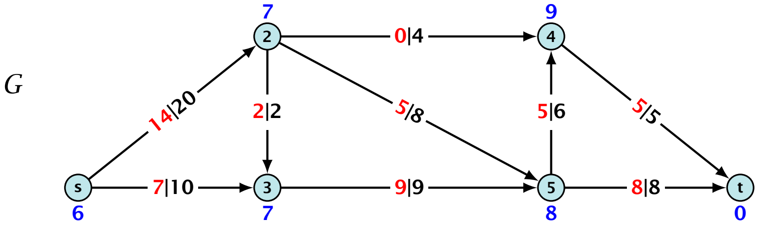
non-saturated push

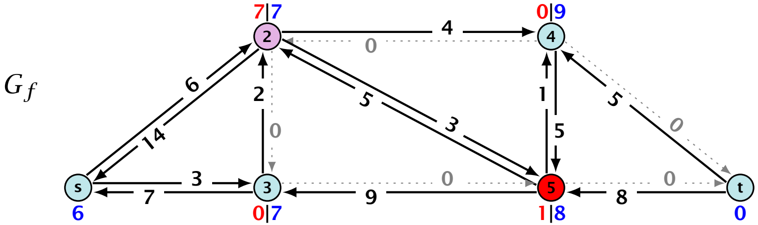


Preflow Push Algorithm



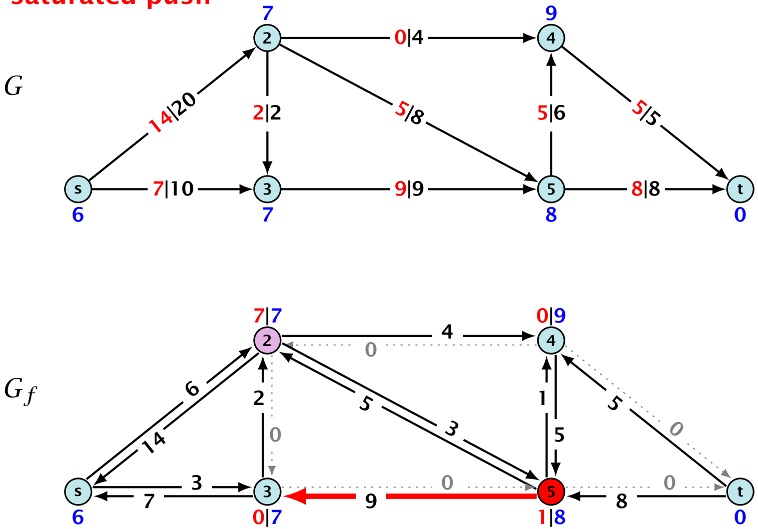
Preflow Push Algorithm





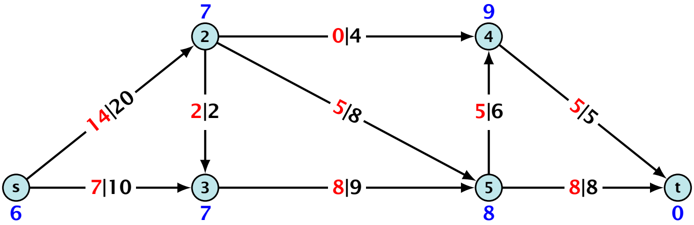
Preflow Push Algorithm

non-saturated push

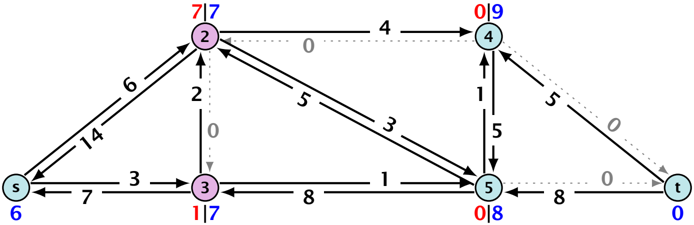


Preflow Push Algorithm

G



G_f



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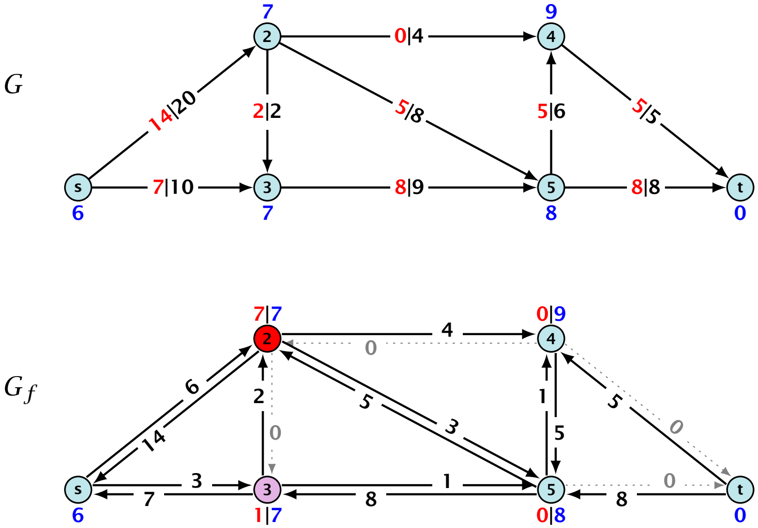
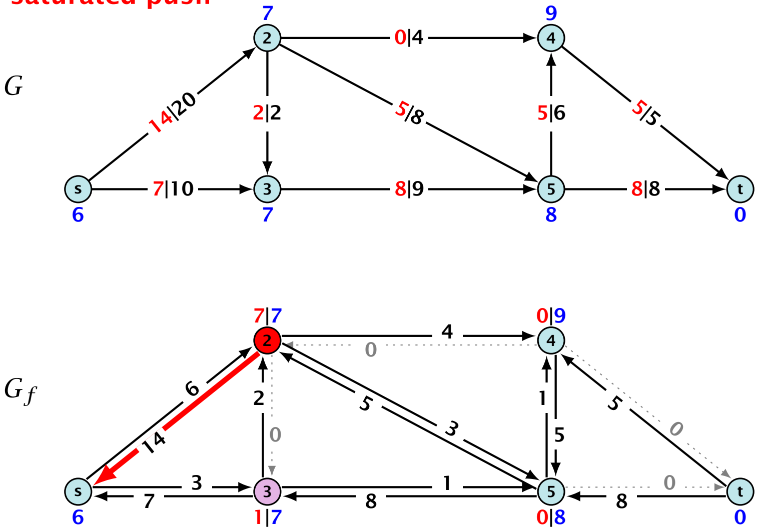


Preflow Push Algorithm

non-saturated push



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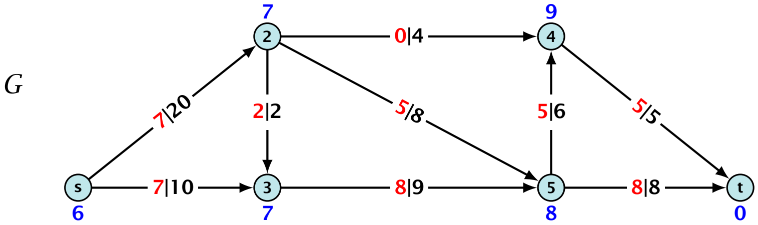

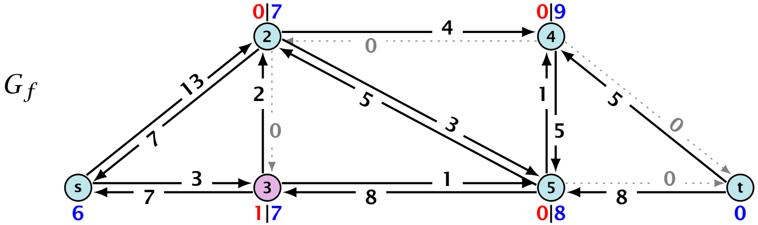

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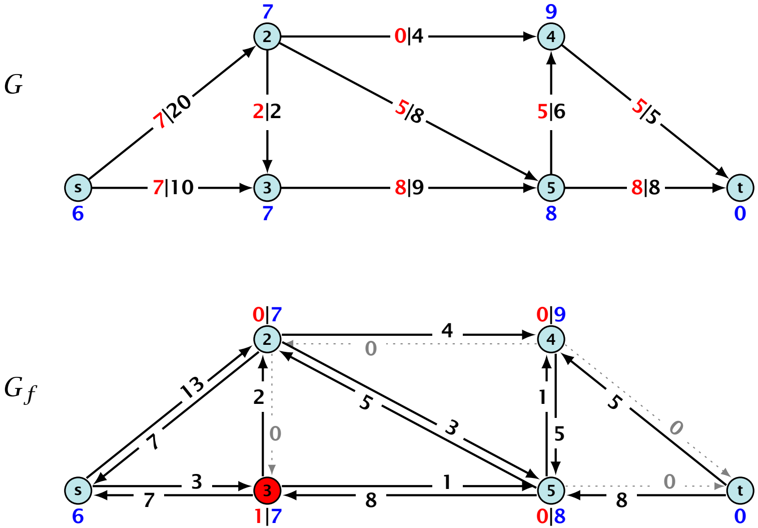

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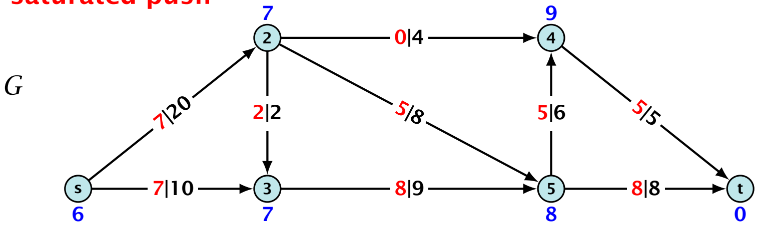

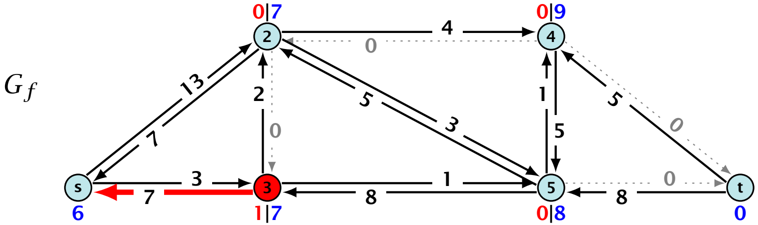

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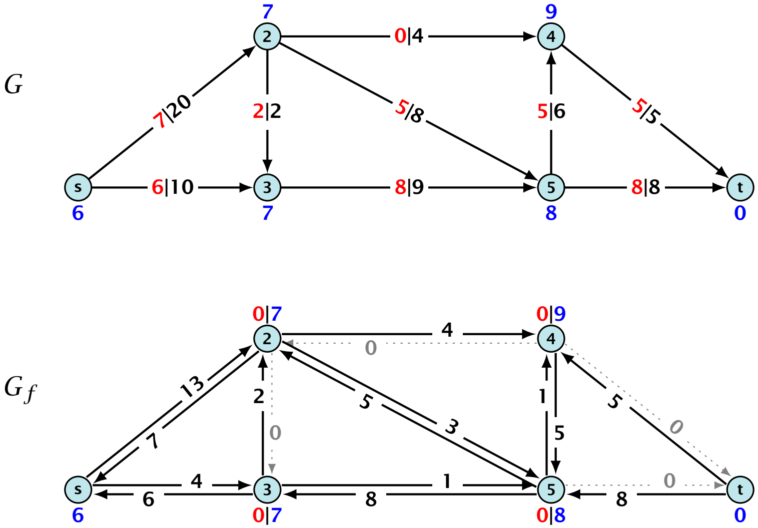
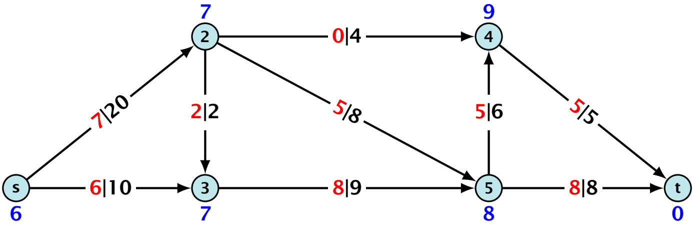
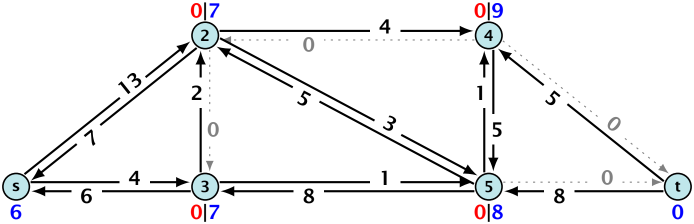


Preflow Push Algorithm

G



G_f



Analysis

Lemma 68

An active node has a path to s *in the residual graph.*

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- In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.
- In the residual graph there are no edges into A, and, hence, no edges leaving A/entering B can carry any flow.
- Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x,y) = \begin{cases} 0 & (x,y) \notin E \\ f((x,y)) & (x,y) \in E \end{cases}$$

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Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

Analysis

Lemma 69

The label of a node cannot become larger than $2n - 1$.

Proof.

- When increasing the label at a node u there exists a path from u to s of length at most $n - 1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

Analysis

Lemma 70

There are only $\mathcal{O}(n^3)$ calls to discharge when using the relabel-to-front heuristic.

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- Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

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- Hence,

$$\begin{aligned}\#\text{non-saturating\_pushes} &\le \#\text{relabels} + 2n \cdot \#\text{saturating\_pushes} \\ &\le \mathcal{O}(n^2m) \ .\end{aligned}$$

Analysis

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2m)$.

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u, v) can be performed in constant time

- check whether edge (v, u) needs to be added to G_f
- check whether (u, v) needs to be deleted (saturating push)
- check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissable
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13.2 Relabel to front

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 48 discharge(u)

```
while u is active do
    v ← u.current-neighbour
    if v = null then
        relabel(u)
        u.current-neighbour ← u.neighbour-list-head
    else
        if (u, v) admissable then push(u, v)
        else u.current-neighbour ← v.next-in-list
```

13.2 Relabel to front

Lemma 73

If $v = \mathrm{null}$ in line 3, then there is no outgoing admissable edge from u.

The lemma holds because push- and relabel-operations on nodes different from u cannot make edges outgoing from u admissable.

This shows that discharge(u) is correct, and that we can perform a relabel in line 4.

13.2 Relabel to front

Algorithm 49 relabel-to-front(G, s, t)

```
initialize preflow
initialize node list L containing V \ {s,t} in any order
foreach u ∈ V \ {s,t} do
    u.current-neighbour ← u.neighbour-list-head
u ← L.head
while u ≠ null do
    old-height ← ℓ(u)
    discharge(u)
    if ℓ(u) > old-height then
        move u to the front of L
    u ← u.next
```

13.2 Relabel to front

Lemma 74 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissable edges; this means for an admissable edge (x, y) the node x appears before y in sequence L.*
2. *No node before u in the list L is active.*

Proof:

- Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissable, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active
- Maintenance:
  1. - Pushes do no create any new admissable edges. Therefore, not relabeling u leaves L topologically sorted.
     - After relabeling, u cannot have admissable incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissable edges leaving u that were generated by the relabeling.

13.2 Relabel to front

Proof:

- Maintenance:

  2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

     For the case that we do a relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissable arc. However, all admissable arc point to successors of u.

Note that the invariant for $u = \text{null}$ means that we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to front

Lemma 75
There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to front

Lemma 76
The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting $u.current\text{-}neighbour$). In total we have $\mathcal{O}(n^2)$ relabel-operations.

13.2 Relabel to front

Note that by definition a saturing push operation $(\min\{c_f(e), f(u)\} = c_f(e))$ can at the same time be a non-saturating push operation $(\min\{c_f(e), f(u)\} = f(u))$.

Lemma 77

*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.\mathit{current\text{-}neighbour}$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $\mathit{degree}(u) + 1$ many entries ($+1$ for null-entry).

13.2 Relabel to front

Lemma 78

The cost for all non-saturating push-operations is only $\mathcal{O}(n^3)$.

A non-saturating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 79

The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

13.3 Highest label

Algorithm 50 highest-label(G, s, t)

```
initialize preflow
foreach u ∈ V \ {s, t} do
    u.current-neighbour ← u.neighbour-list-head
while ∃ active node u do
    select active node u with highest label
    discharge(u)
```

13.3 Highest label

Lemma 80
When using highest label the number of non-saturating pushes is only $\mathcal{O}(n^3)$.

After a non-saturating push from u a relabel is required to make a currently non-active node x, with $\ell(x) \geq \ell(u)$ active again (note that this includes u).

Hence, after n non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

13.3 Highest label

Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of relabel-to-front.

Question:

How do we find the next node for a discharge operation?

13.3 Highest label

Maintain lists L_i, $i \in \{0, \ldots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $k-1, \ldots, 0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to s or t the list $k-1$ must be non-empty (i.e., the search takes constant time).

13.3 Highest label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#\textit{non-saturating-pushes-to-s-or-t})$$

Lemma 81

The number of non-saturating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 82

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

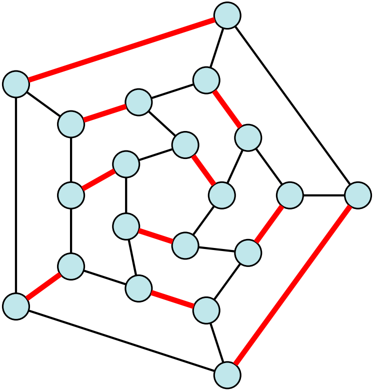
13.3 Highest label

Proof of the Lemma.

- We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- After a node v (which must have $\ell(v) = n + 1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\leq n + 1$ to $n + 2$ before v can become active again.
- This happens for every push that v makes to the source. Since, every node can pass the threshold $n + 2$ at most once, v can make at most n pushes to the source.
- As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

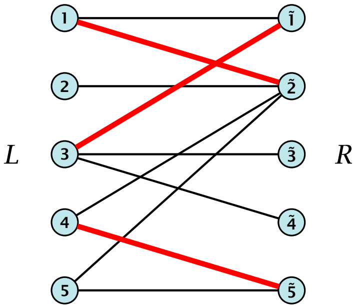
Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



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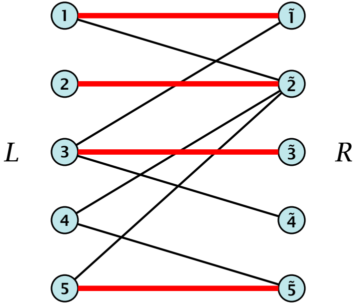
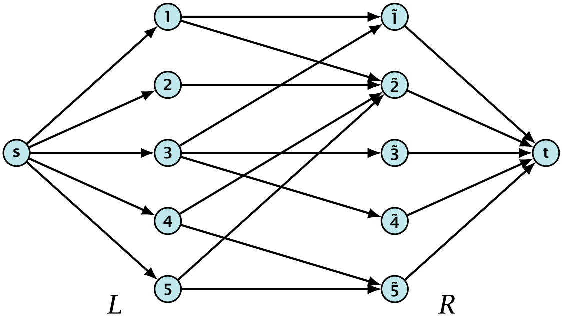


Maxflow Formulation

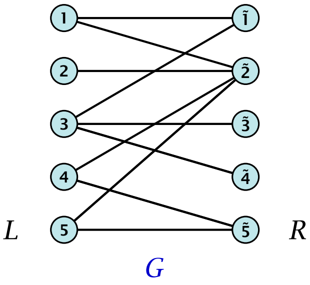
- Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s,t\}, E')$.
- Direct all edges from L to R.
- Add source s and connect it to all nodes on the left.
- Add t and connect all nodes on the right to t.
- All edges have unit capacity.

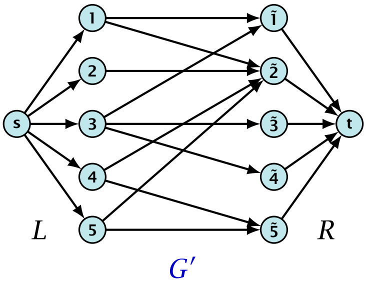


Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.





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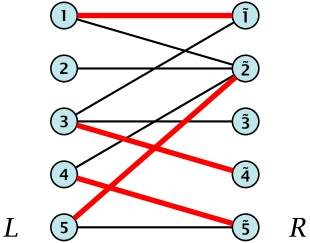
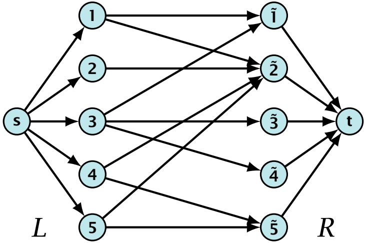


G



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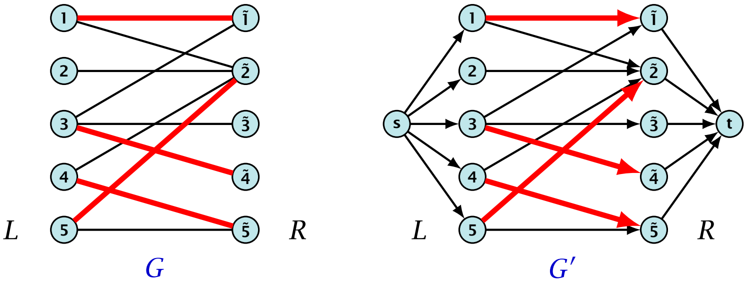

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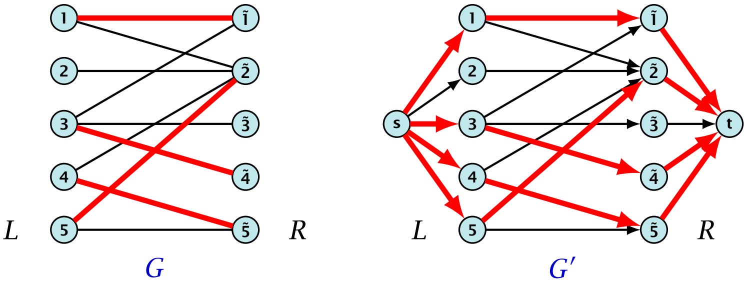
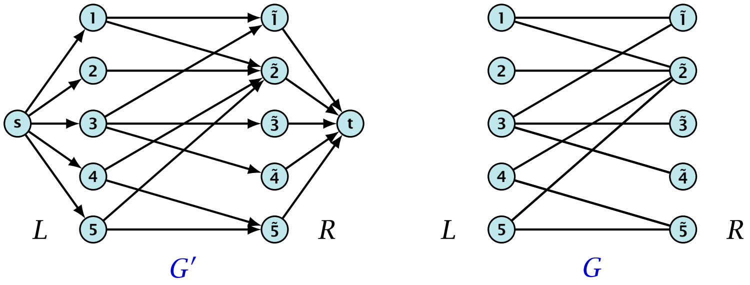


Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

- Let f be a maxflow in G' of value k
- Integrality theorem $\Rightarrow$ k integral; we can assume f is 0/1.
- Consider M= set of edges from L to R with $f(e) = 1$.
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.



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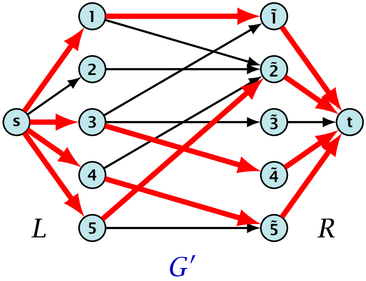

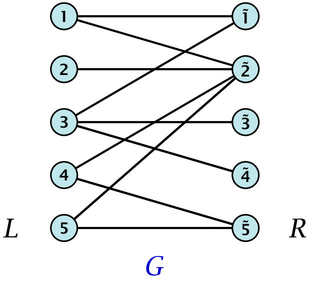

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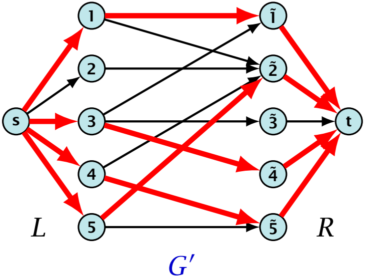

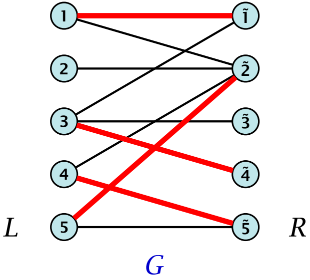

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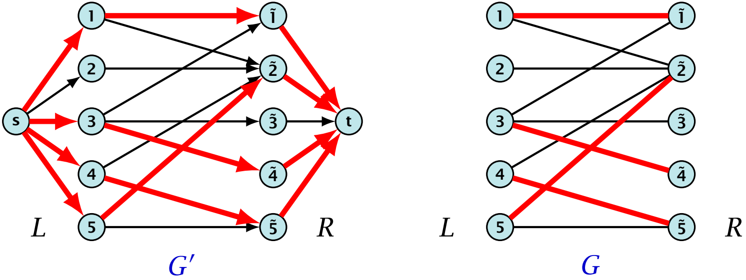


14.1 Matching

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

Baseball Elimination

team i	wins w_i	losses ℓ_i	remaining games Atl	 Phi	 NY	 Mon
Atlanta	83	71	-	1	6	1
Philadelphia	80	79	1	-	0	2
New York	78	78	6	0	-	0
Montreal	77	82	1	2	0	-

Which team can end the season with most wins?

- Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- But also Philadelphia is eliminated. Why?

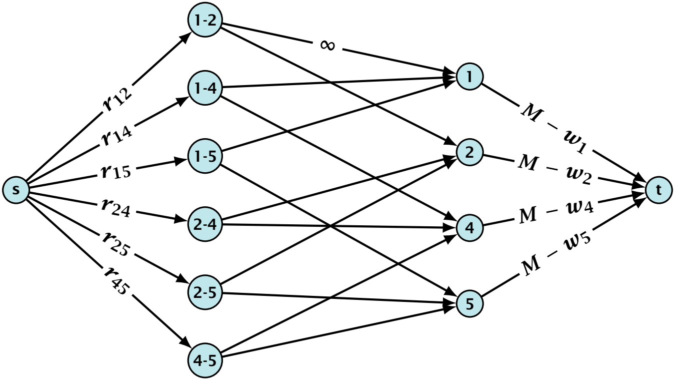
Baseball Elimination

Formal definition of the problem:

- Given a set S of teams, and one specific team $z \in S$.
- Team x has already won w_x games.
- Team x still has to play team y, r_{xy} times.
- Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow networks for $z = 3$. M is number of wins Team 3 can still obtain.



Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \qquad r(T) := \sum_{i,j \in T, i<j} r_{ij}$$

wins of teams in T

remaining games among teams in T

If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 83

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij\in S\setminus\{z\},i<j} r_{ij}$.

Proof (⇐)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.
- If for a node x-y not both team nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.
- We don't find a flow that saturates all source edges:

$$
\begin{aligned}
r(S\setminus\{z\}) &> \mathrm{cap}(S, V\setminus S)\\
&\geq \sum_{i<j:\, i\notin T \vee j\notin T} r_{ij} + \sum_{i\in T}(M - w_i)\\
&\geq r(S\setminus\{z\}) - r(T) + |T|M - w(T)
\end{aligned}
$$

- This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Project Selection

Project selection problem:

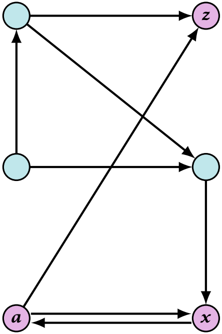
- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- Some projects have requirements (taking course EA2 requires course EA1).
- Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

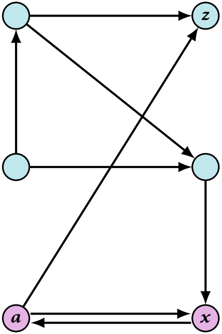
Goal: Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

- $\{x, a, z\}$ is a feasible subset.
- $\{x, a\}$ is infeasible.

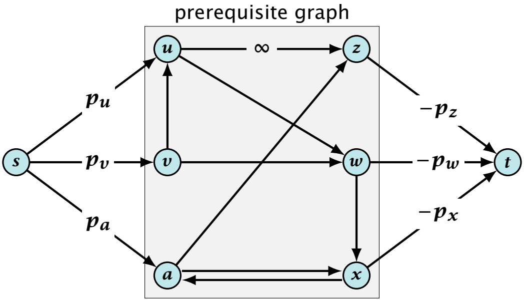




Project Selection

Mincut formulation:

- Edges in the prerequisite graph get infinite capacity.
- Add edge (s, v) with capacity p_v for nodes v with positive profit.
- Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.

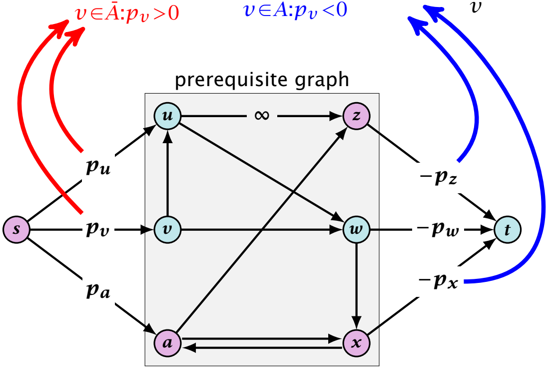


Theorem 84

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

- A is feasible because of capacity infinity edges.
- $\text{cap}(A, V \setminus A) = \sum_{v \in \bar{A}: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v) = \sum_v p_v - \sum_{v \in A} p_v$



Mincost Flow

Consider the following problem:

$$\begin{aligned} \text{min} \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \quad 0 \leq f(e) \leq u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{aligned}$$

- $G = (V, E)$ is an oriented graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

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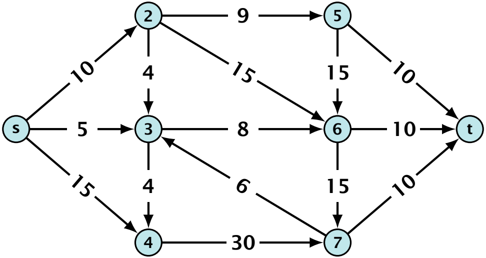
Solve Maxflow Using Mincost Flow



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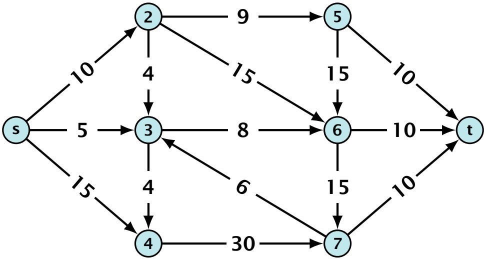
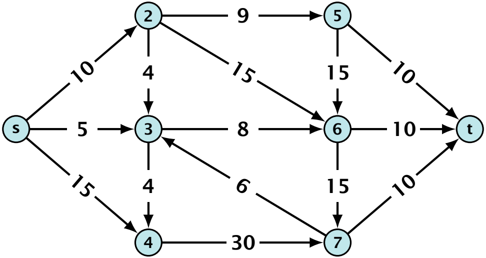


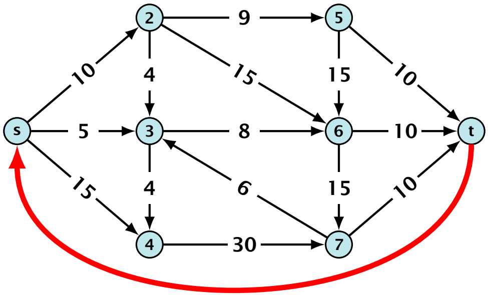
- Given a flow network for a standard maxflow problem.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.

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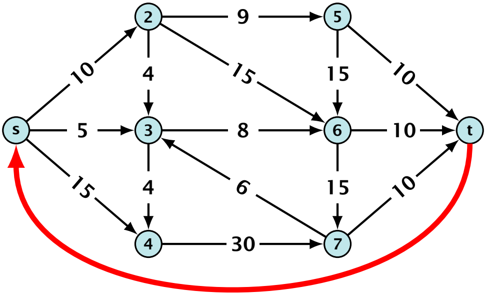

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- Add an edge from t to s with infinite capacity and cost -1.
- Then, $\mathrm{val}(f^*) = -\mathrm{cost}(f_{\min})$, where f^* is a maxflow, and $f_{\min}$ is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

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Generalization

Our model:

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where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

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$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & a(v) \le f(v) \le b(v) \end{array}$$

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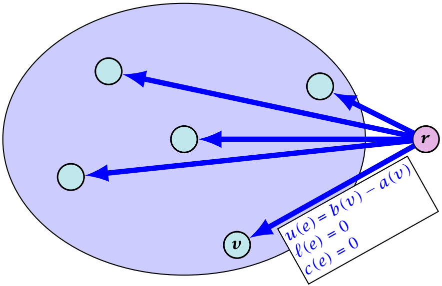

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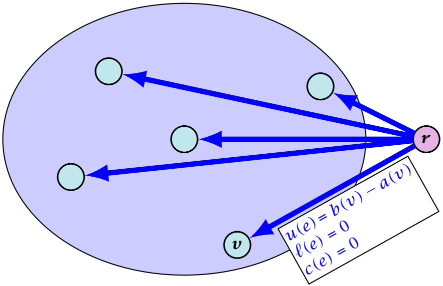

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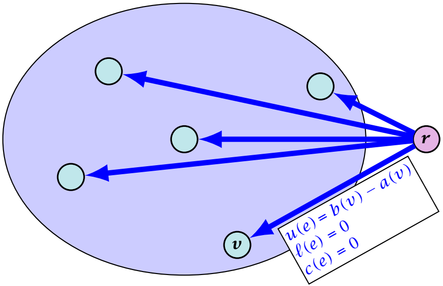

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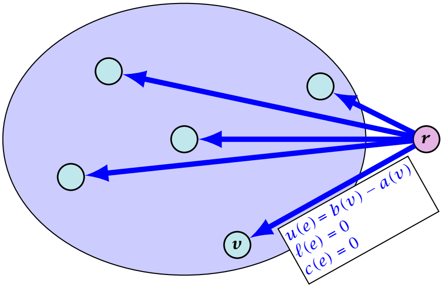

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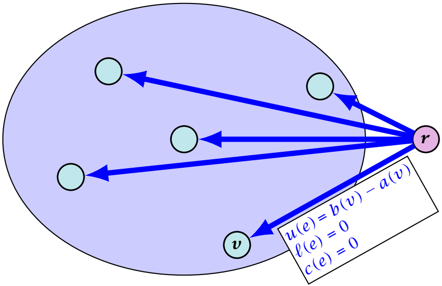

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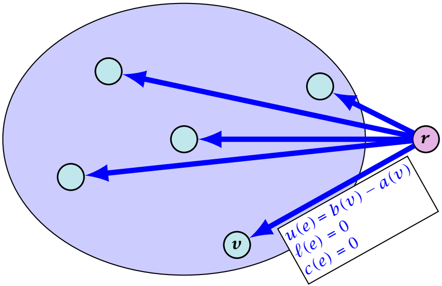

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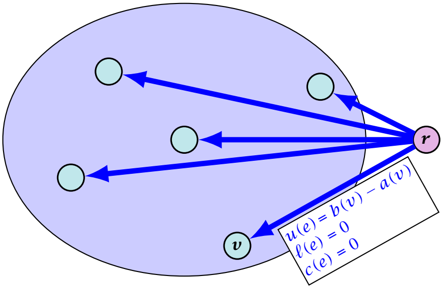

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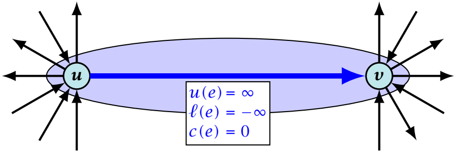
We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:



If $c(e) = 0$ we can simply contract the edge/identify nodes u and v

Reduction III

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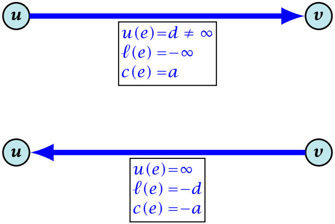

Replace the edge by an edge in opposite direction.

Reduction IV

$$\begin{array}{lll} \min & \sum_e c(e) f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

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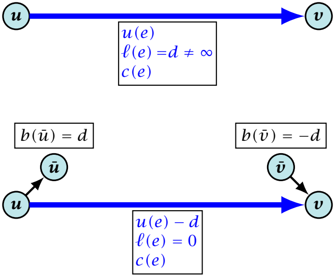

The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
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Residual Graph

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

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15 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$ (G may be a directed graph instead of just an oriented graph).

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Lemma 85

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

$\Rightarrow$ Suppose that g is a feasible circulation of negative cost in the residual graph.

$\Leftarrow$ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

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A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

- ⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

  Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

- ⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

  Clearly, $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph.

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A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
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Algorithm 51 CycleCanceling($G = (V, E), c, u, b$)

1: establish a feasible flow f in G
2: **while** G_f contains negative cycle **do**
3: use Bellman-Ford to find a negative circuit Z
4: $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
5: augment δ units along Z and update G_f

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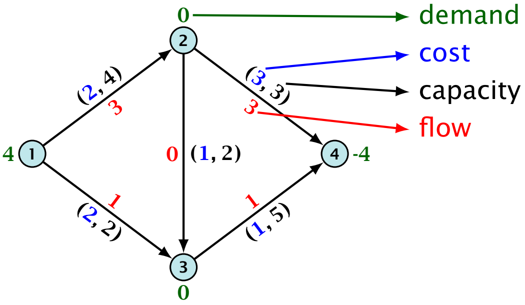

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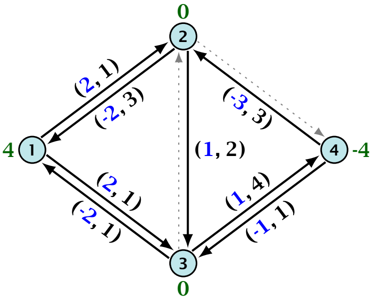

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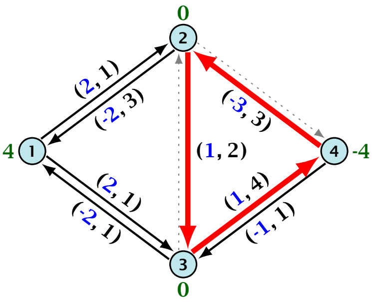

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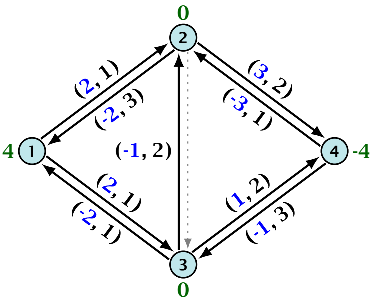

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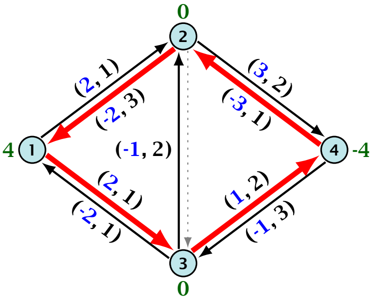

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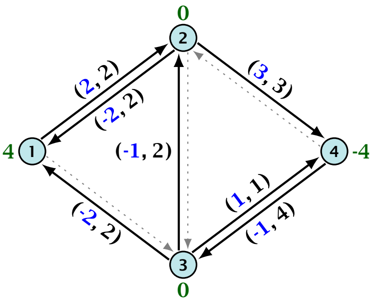

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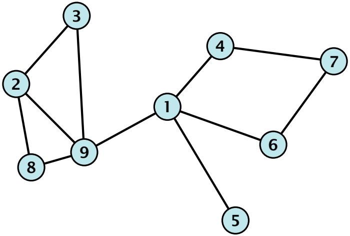
Lemma 87

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, *for integer capacities and costs, when for all edges* e, $|c(e)| \le C$ *and* $|u(e)| \le U$.

16 Global Mincut

Given an undirected, capacitated graph $G = (V, E, c)$ find a partition of V into two non-empty sets $S, V \setminus S$ s.t. the capacity of edges between both sets is minimized.



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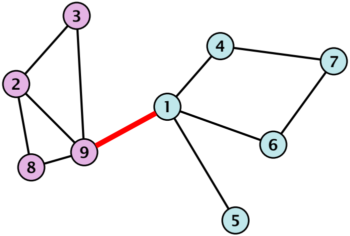

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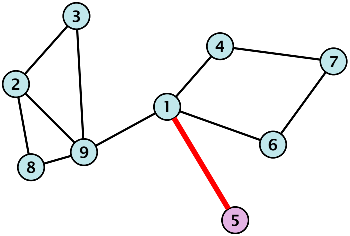

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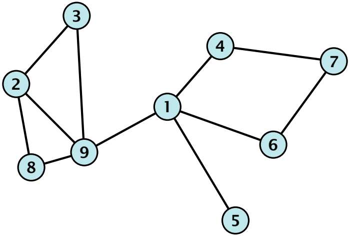
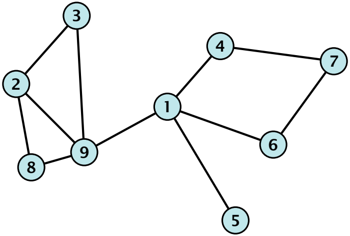
Given an undirected, capacitated graph $G = (V, E, c)$ find a partition of V into two non-empty sets $S, V \setminus S$ s.t. the capacity of edges between both sets is minimized.



16 Global Mincut

We can solve this problem using standard maxflow/mincut.



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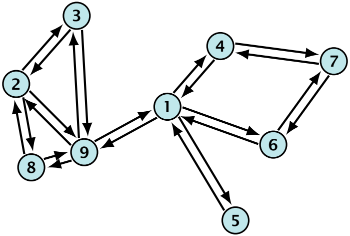
We can solve this problem using standard maxflow/mincut.

- Construct a directed graph $G' = (V, E')$ that has edges (u, v) and (v, u) for every edge $\{u, v\} \in E$.



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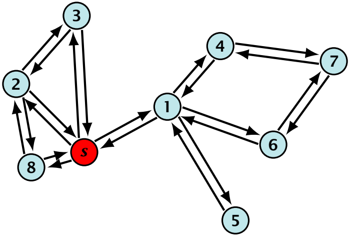

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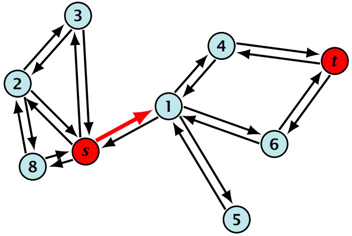

Edge Contractions

- Given a graph $G = (V, E)$ and an edge $e = \{u, v\}$.
- The graph G/e is obtained by "identifying" u and v to form a new node.
- Resulting parallel edges are replaced by a single edge, whose capacity equals the sum of capacities of the parallel edges.

Example 88

- Edge-contractions do no decrease the size of the mincut.

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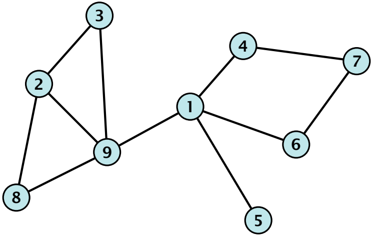

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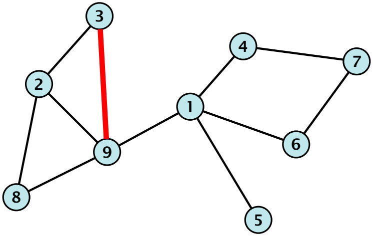

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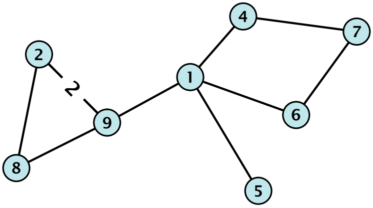

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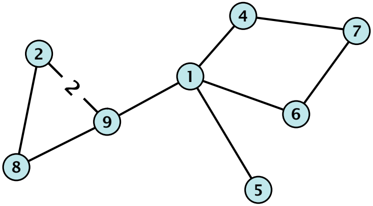

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Edge Contractions

We can perform an edge-contraction in time $\mathcal{O}(n)$.

Randomized Mincut Algorithm

Algorithm 52 KargerMincut($G = (V, E, c)$)

```
for i = 1 → n − 2 do
    choose e ∈ E randomly with probability c(e)/C(E)
    G ← G/e
return only cut in G
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- Let G_t denote the graph after the $(n - t)$-th iteration, when t nodes are left.

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- What is the probability that this algorithm returns a mincut?

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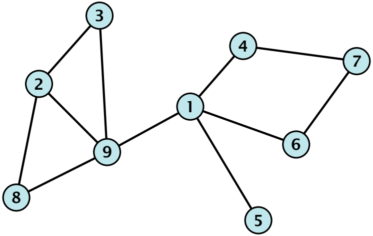

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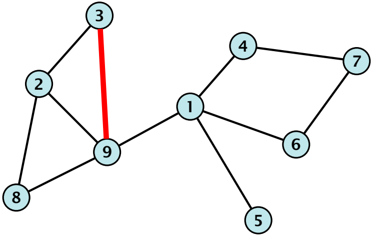

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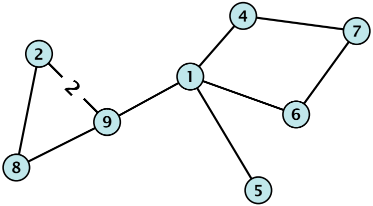

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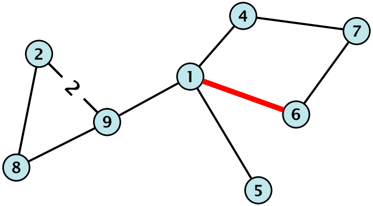

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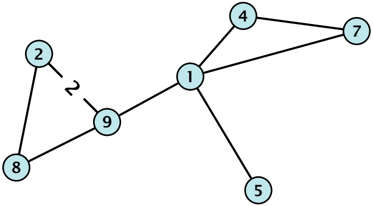

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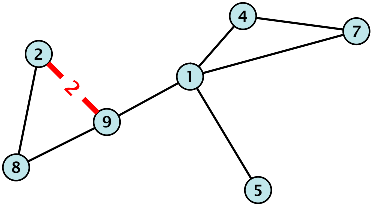

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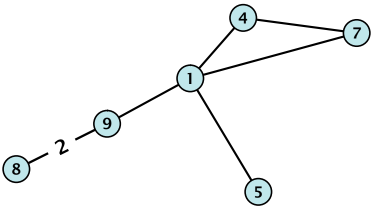

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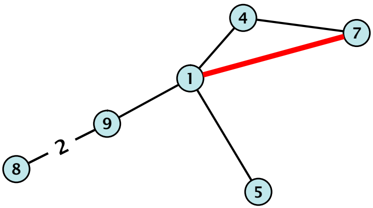

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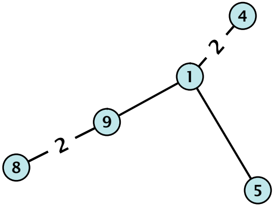

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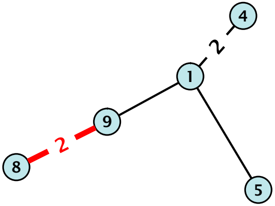

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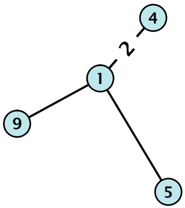

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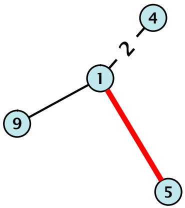

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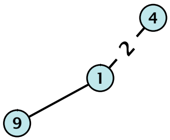

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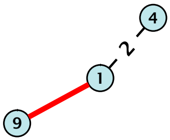

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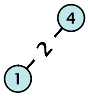

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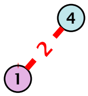

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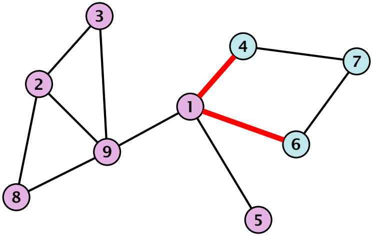

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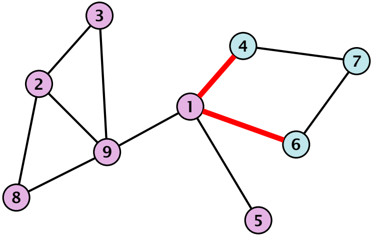

What is the probability that this algorithm returns a mincut?

Analysis

What is the probability that a given mincut A is still possible after round i?

- It is still possible to obtain cut A in the end if so far no edge in $(A, V \setminus A)$ has been contracted.

Analysis

What is the probability that we select an edge from A in iteration i?

- Let $\min = \text{cap}(A, V \setminus A)$ denote the capacity of a mincut.
- Let $\text{cap}(v)$ be capacity of edges incident to vertex $v \in V_{n-i+1}$.
- Clearly, $\text{cap}(v) \geq \min$.
- Summing $\text{cap}(v)$ over all edges gives

$$2c(E) = 2\sum_{e \in E} c(e) = \sum_{v \in V} \text{cap}(v) \geq (n - i + 1) \cdot \min$$

- Hence, the probability of choosing an edge from the cut is at most $\min / c(E) \leq 2/(n - i + 1)$.

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The probability that we do not choose an edge from the cut in iteration i is

$$1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1} \ .$$

The probability that the cut is alive after iteration $n-t$ (after which t nodes are left) is

$$\prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1} = \frac{t(t-1)}{n(n-1)} \ .$$

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Repeating the algorithm $c\ln n\binom{n}{2}$ times gives that the probability that we are never successful is

$$\left(1-\frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}c\ln n} \leq \left(e^{-1/\binom{n}{2}}\right)^{\binom{n}{2}c\ln n} \leq n^{-c}\enspace,$$

where we used $1-x \leq e^{-x}$.

Theorem 89

The randomized mincut algorithm computes an optimal cut with high probability. The total running time is $\mathcal{O}(n^4\log n)$.

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Improved Algorithm

Algorithm 53 RecursiveMincut$(G = (V, E, c))$

for $i = 1 \to n - n/\sqrt{2}$ **do**
    choose $e \in E$ randomly with probability $c(e)/C(E)$
    $G \leftarrow G/e$
if $|V| = 2$ **return** cut-value;
$cuta \leftarrow$ RecursiveMincut(G);
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return $\min\{cuta, cutb\}$

Running time:

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Probability of Success

The probability of contracting an edge from the mincut during one iteration through the for-loop is only

$$\frac{t(t-1)}{n(n-1)} \approx \frac{t^2}{n^2} = \frac{1}{2} \ ,$$

as $t = \frac{n}{\sqrt{2}}$.

For the following analysis we ignore the slight error and assume that this probability is at most $\frac{1}{2}$.

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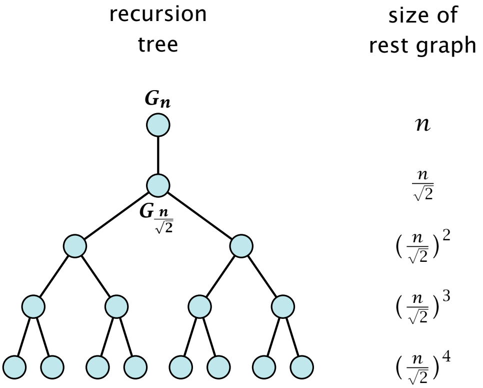

We can estimate the success probability by using the following game on the recursion tree. Delete every edge with probability $\frac{1}{2}$. If in the end you have a path from the root to at least one leaf node you are successful.

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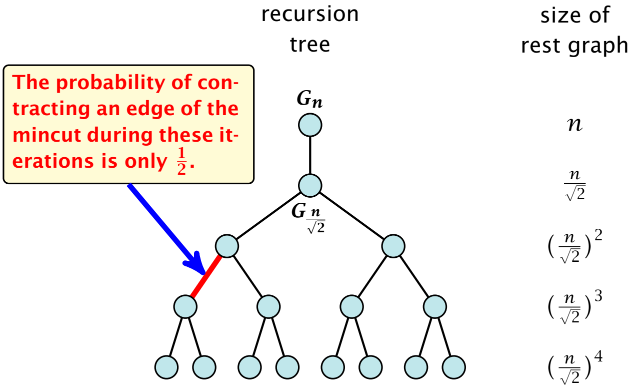

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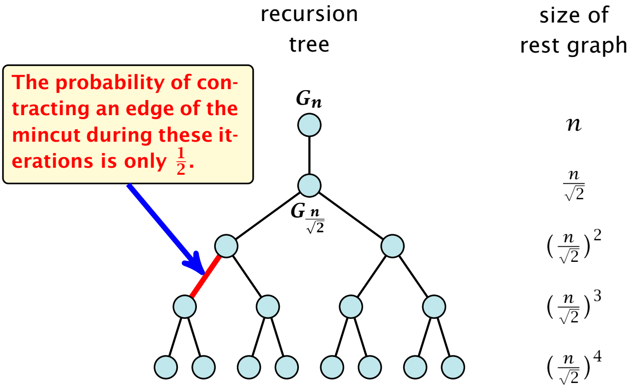

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Let for an edge e in the recursion tree, $h(e)$ denote the height (distance to leaf level) of the parent-node of e (end-point that is higher up in the tree). Let h denote the height of the root node.

Call an edge e alive if there exists a path from the parent-node of e to a descendant leaf, after we randomly deleted edges. Note that an edge can only be alive if it hasn't been deleted.

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16 Global Mincut

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One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ and has a success probability of $\Omega(\frac{1}{\log n})$.

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17 Gomory Hu Trees

Given an undirected, weighted graph $G = (V, E, c)$ a cut-tree $T = (V, F, w)$ is a tree with edge-set F and capacities w that fulfills the following properties.

1. **Equivalent Flow Tree:** For any pair of vertices $s, t \in V$, $f(s,t)$ in G is equal to $f_T(s,t)$.
2. **Cut Property:** A minimum s-t cut in T is also a minimum cut in G.

Here, $f(s,t)$ is the value of a maximum s-t flow in G, and $f_T(s,t)$ is the corresponding value in T.

Overview of the Algorithm

The algorithm maintains a partition of V, (sets $S_1, \ldots, S_t$), and a spanning tree T on the vertex set $\{S_1, \ldots, S_t\}$.

Initially, there exists only the set $S_1 = V$.

Then the algorithm performs $n - 1$ split-operations:

In the end this gives a tree on the vertex set V.

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- In each such split-operation it chooses a set S_i with $|S_i| \geq 2$ and splits this set into two non-empty parts X and Y.
- S_i is then removed from T and replaced by X and Y.
- X and Y are connected by an edge, and the edges that before the split were incident to S_i are attached to either X or Y.

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Details of the Split-operation

- Select S_i that contains at least two nodes a and b.
- Compute the connected components of the forest obtained from the current tree T after deleting S_i. Each of these components corresponds to a set of vertices from V.
- Consider the graph H obtained from G by contracting these connected components into single nodes.
- Compute a minimum a-b cut in H. Let A, and B denote the two sides of this cut.
- Split S_i in T into two sets/nodes $S_i^a \coloneqq S_i \cap A$ and $S_i^b \coloneqq S_i \cap B$ and add edge $\{S_i^a, S_i^b\}$ with capacity $f_H(a,b)$.
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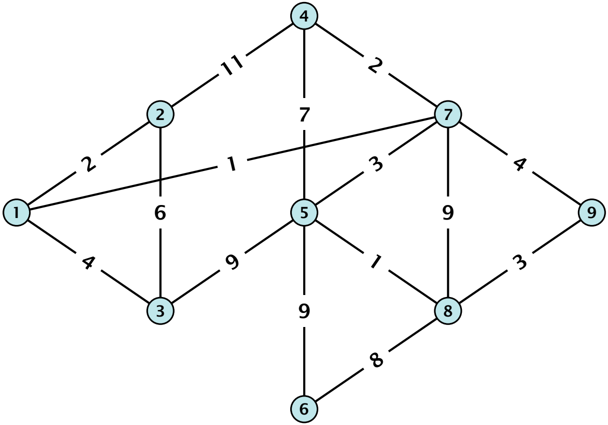

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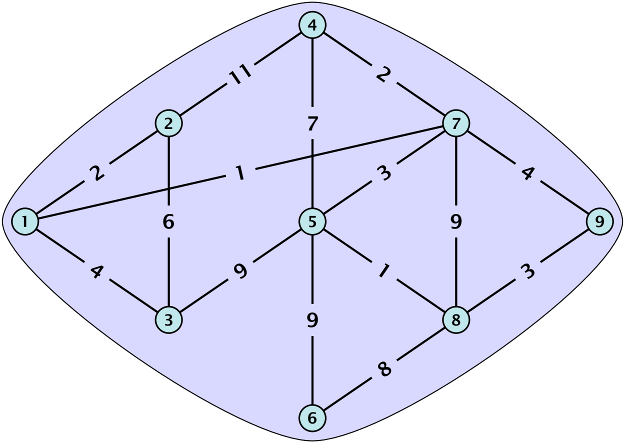

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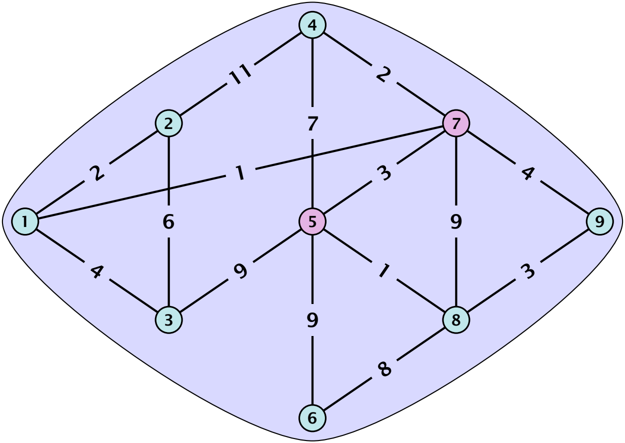

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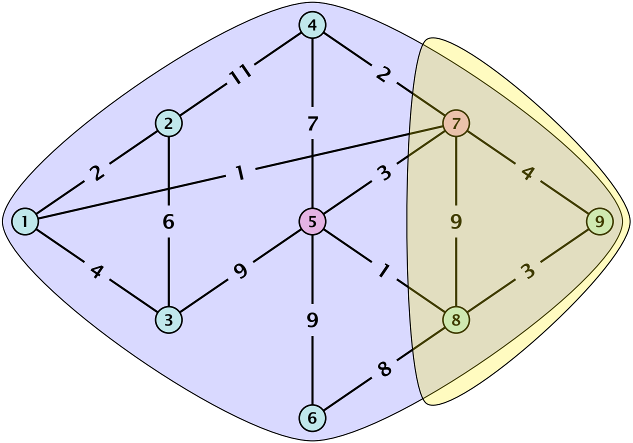

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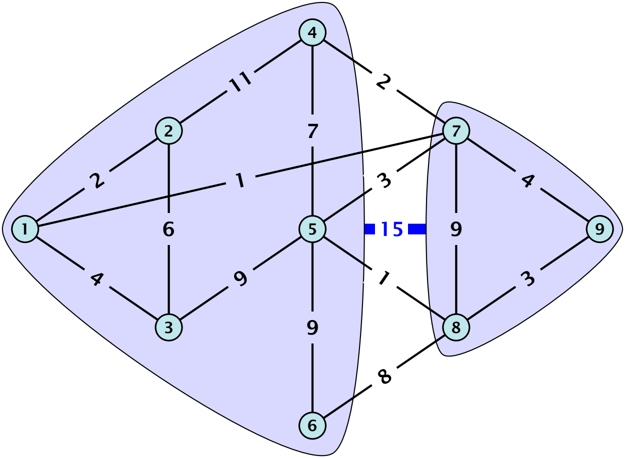

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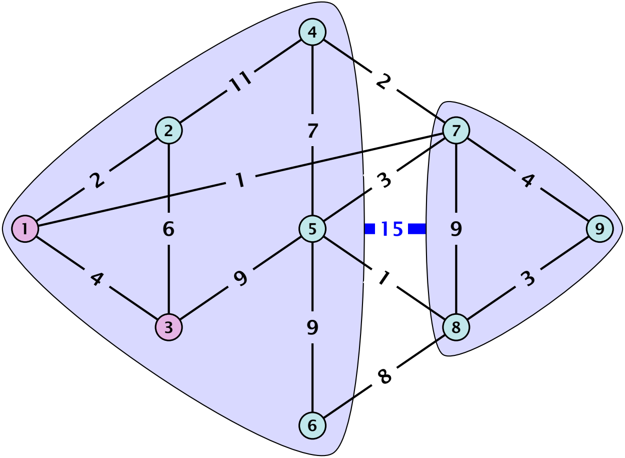

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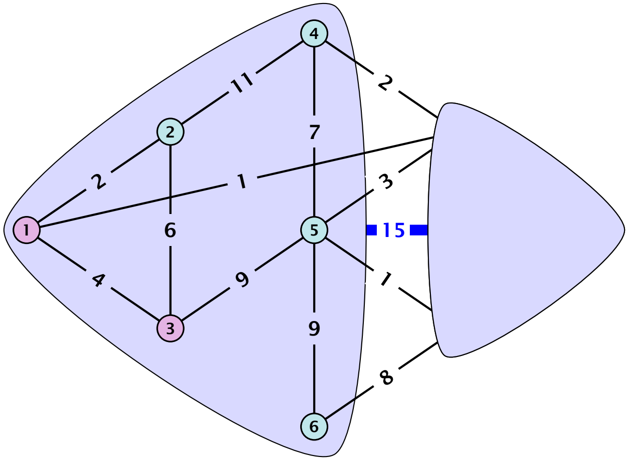

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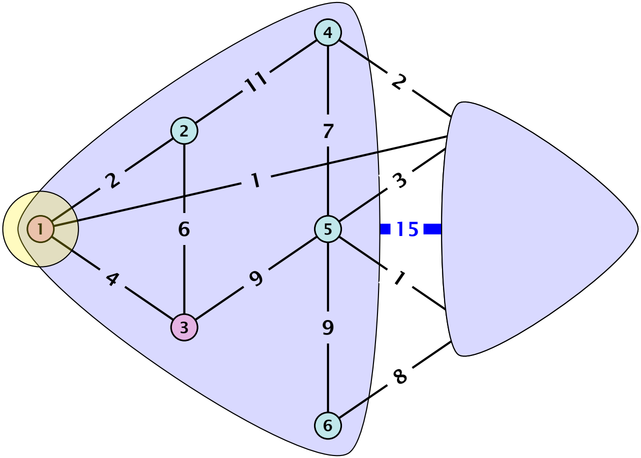

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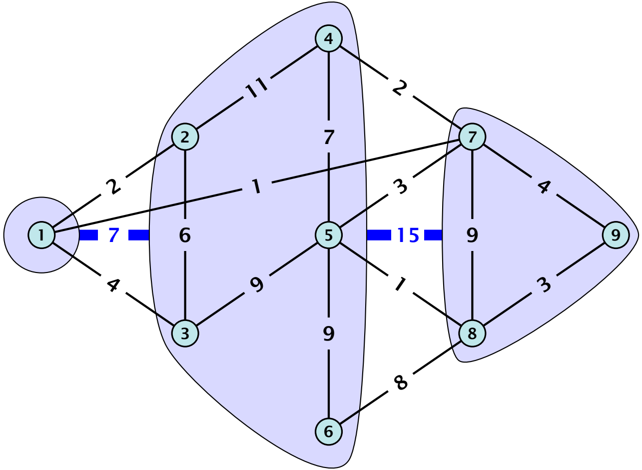

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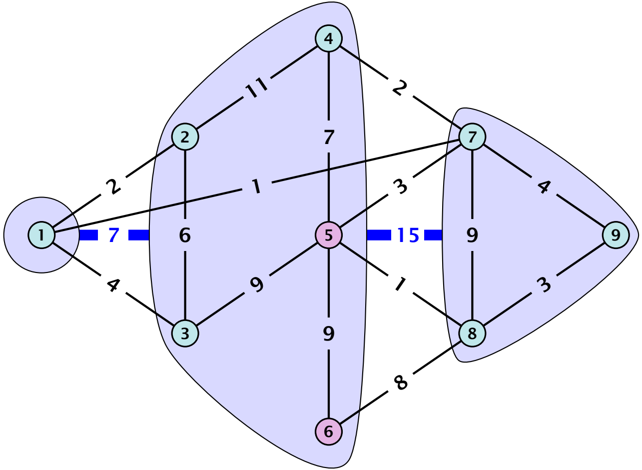

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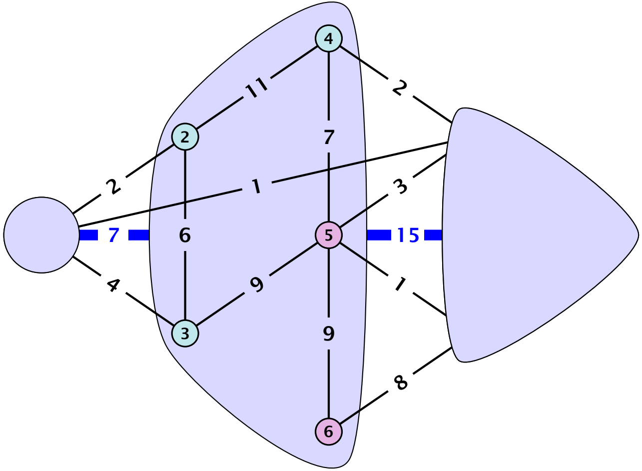

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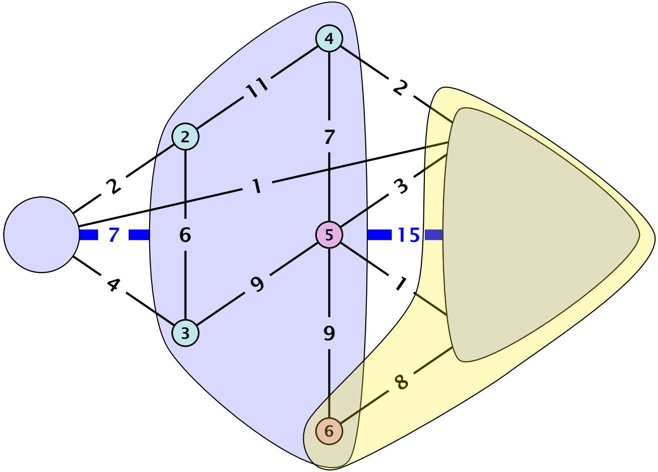

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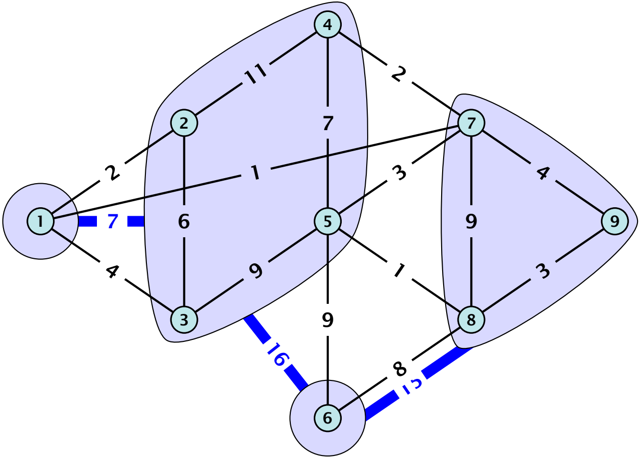

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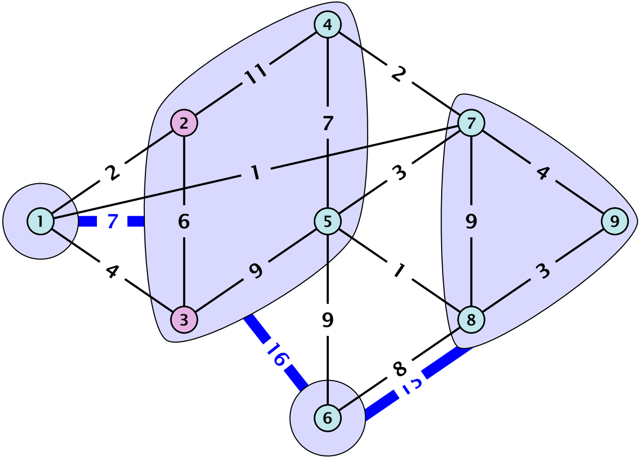

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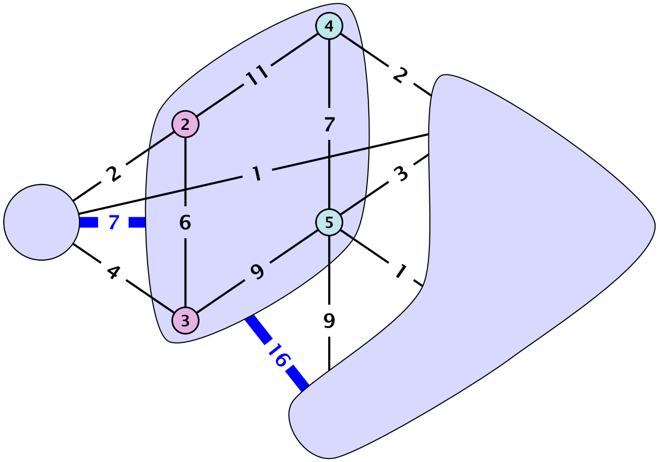

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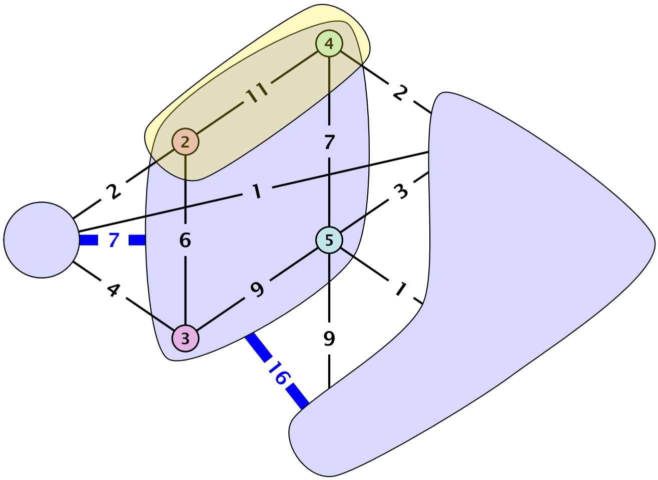

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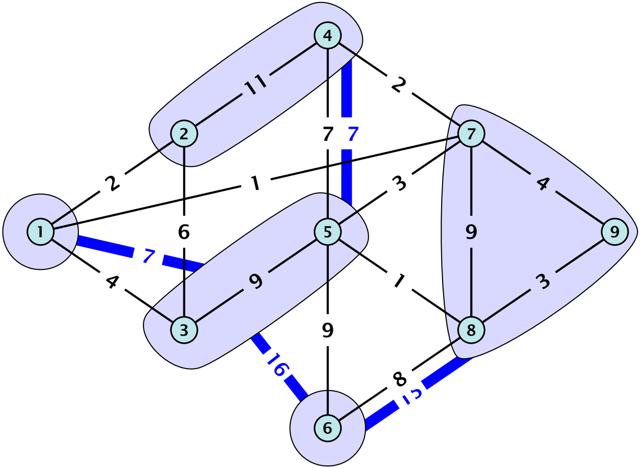

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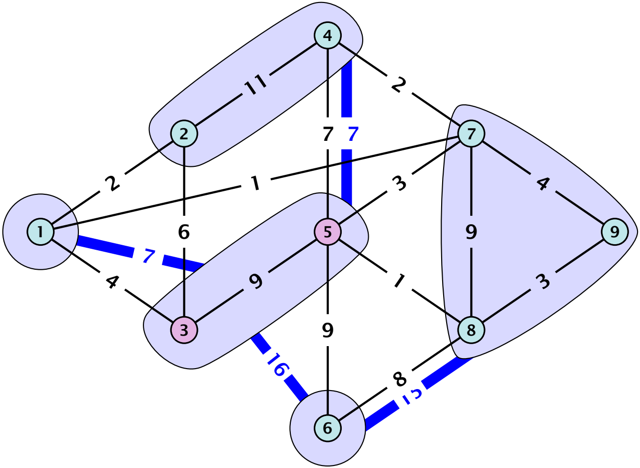

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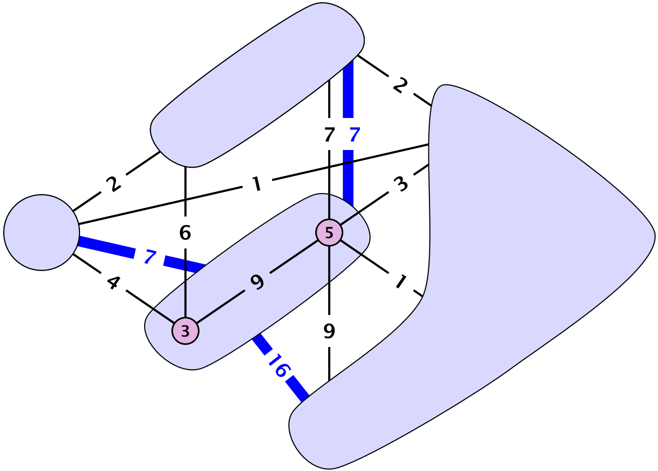

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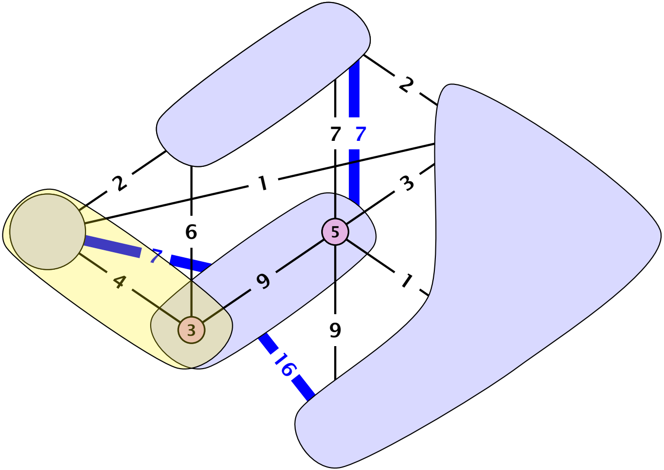

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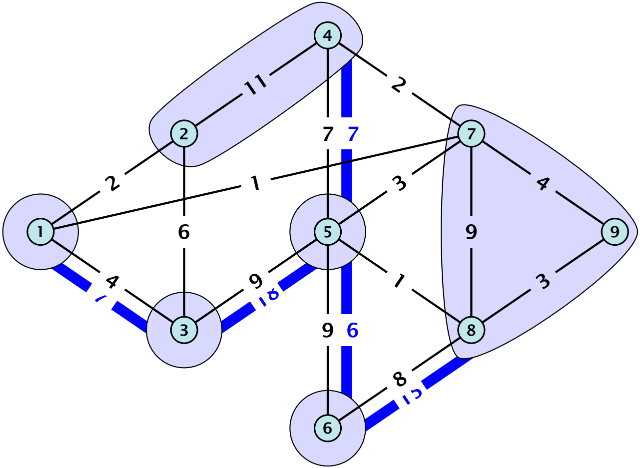

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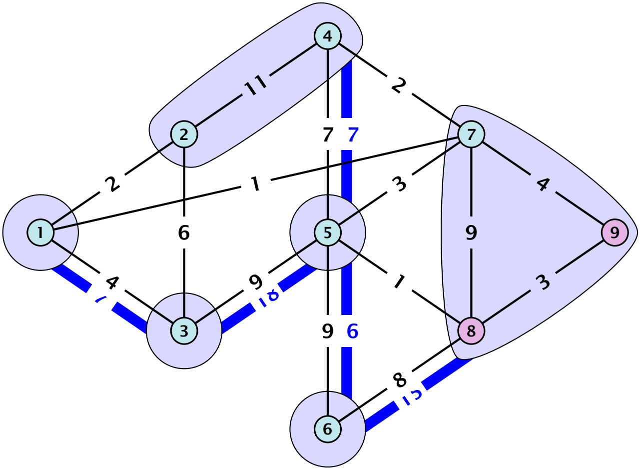

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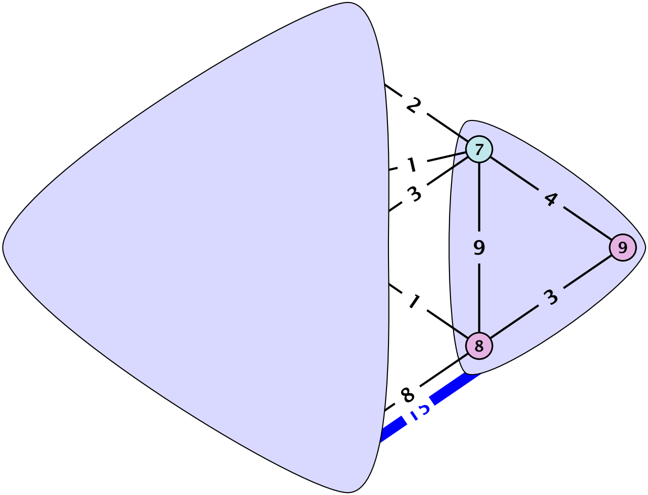

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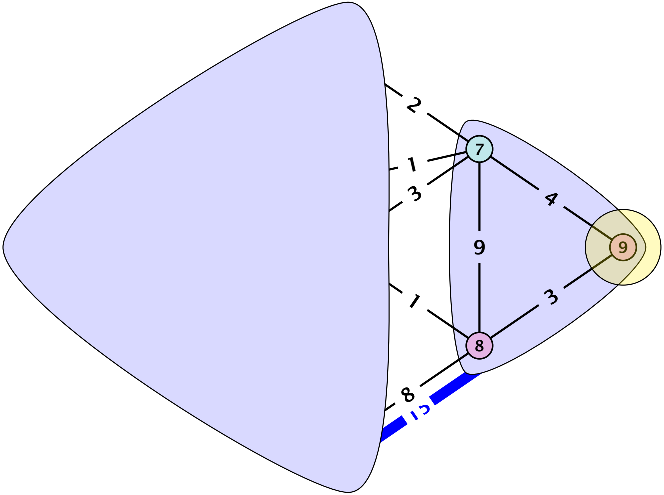

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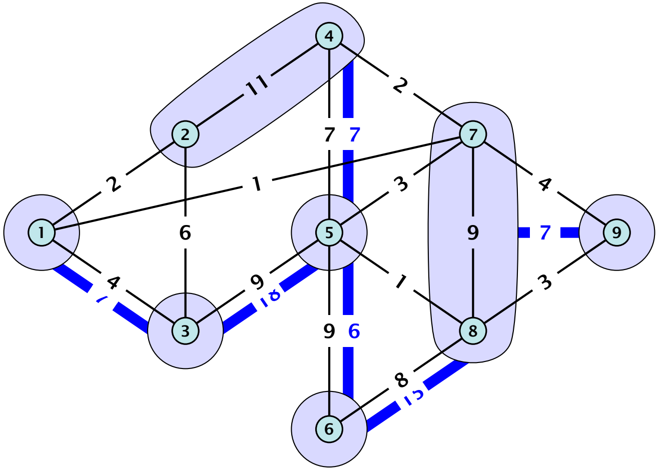

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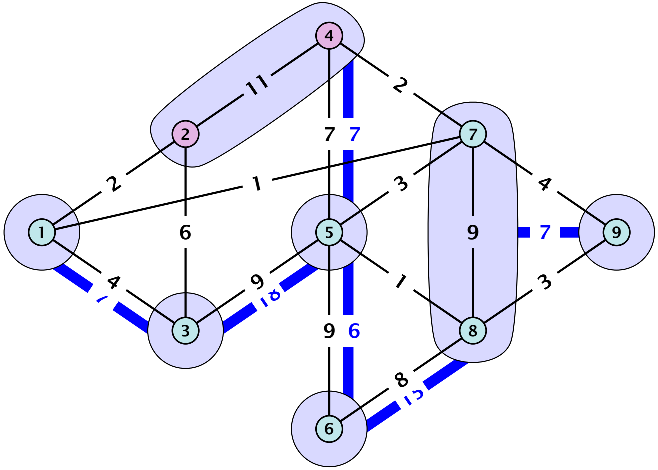

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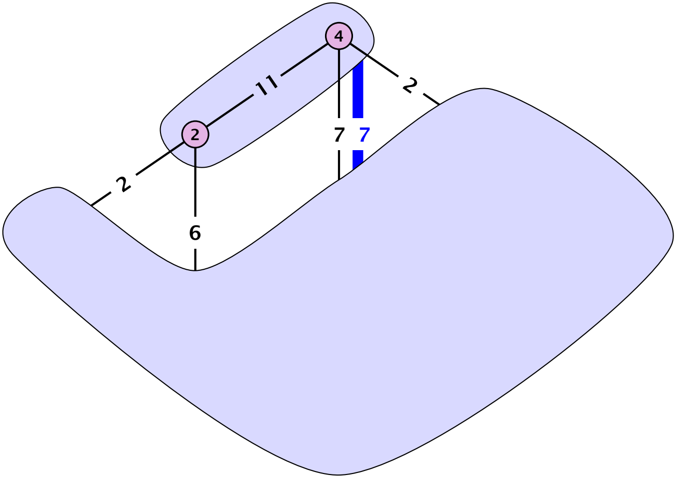

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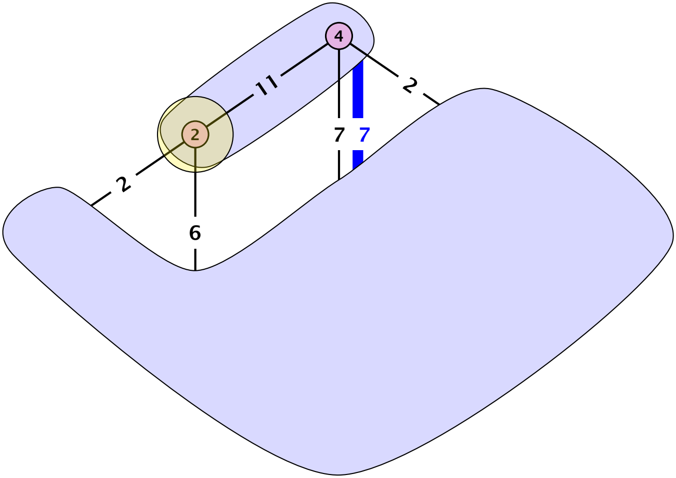

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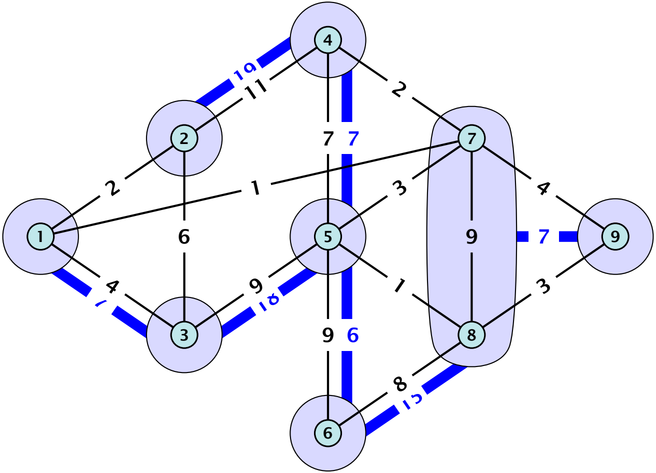

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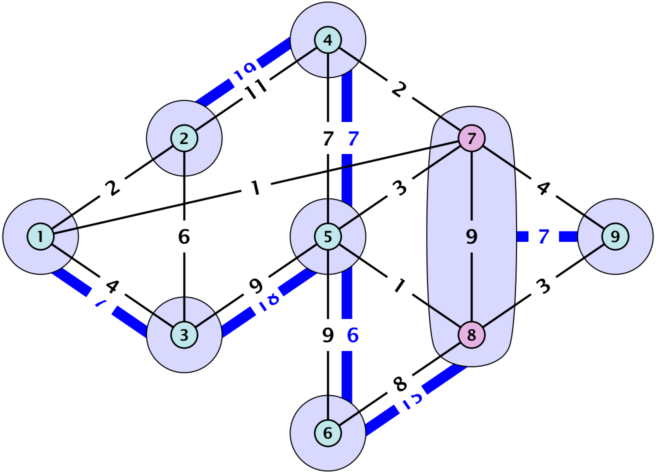

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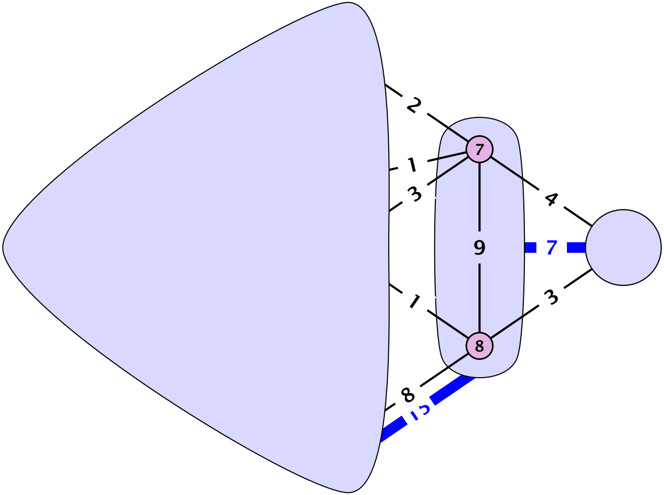

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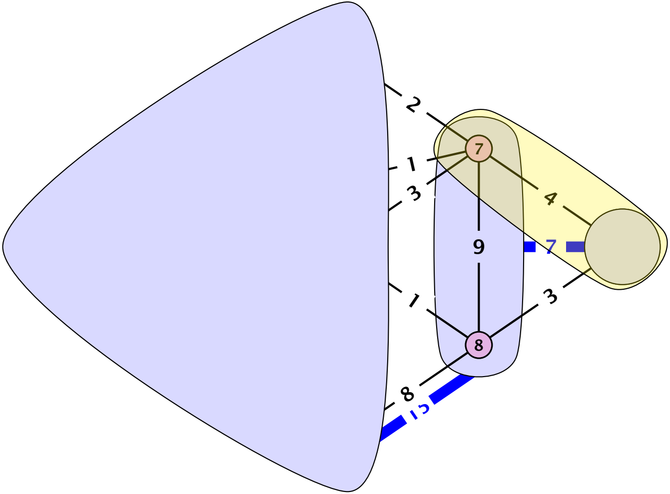

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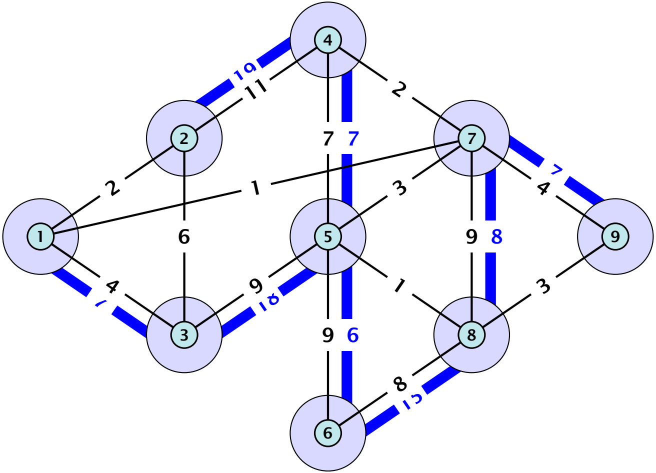

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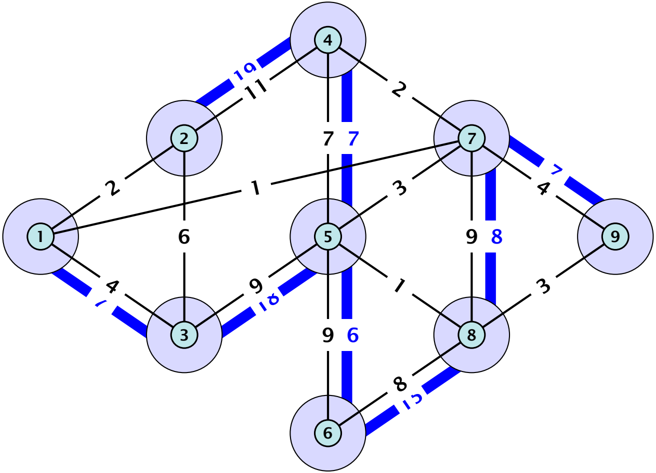

Analysis

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For nodes $s, t, x \in V$ *we have* $f(s,t) \geq \min\{f(s,x), f(x,t)\}$

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Lemma 94

Let S be some minimum r-s cut for some nodes $r, s \in V$ ($s \in S$), and let $v, w \in S$. Then there is a minimum v-w-cut T with $T \subset S$.

Proof: Let X be a minimum v-w cut with $X \cap S \neq \emptyset$ and $X \cap (V \setminus S) \neq \emptyset$. Note that $S \setminus X$ and $S \cap X$ are v-w cuts inside S.

We may assume w.l.o.g. $s \in X$.

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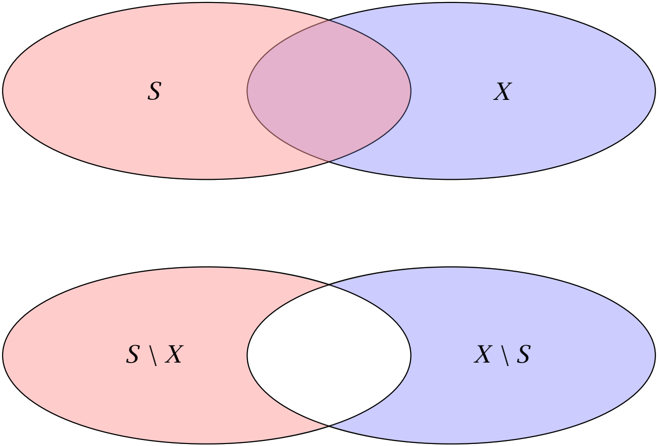

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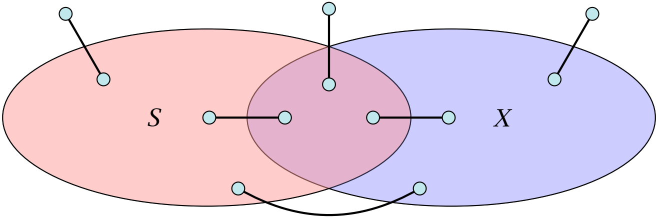

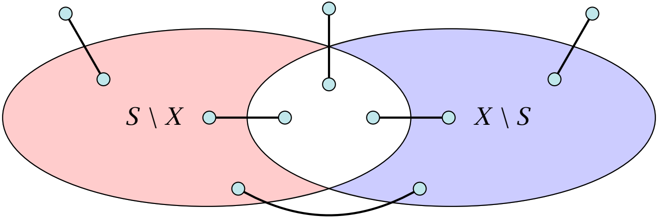

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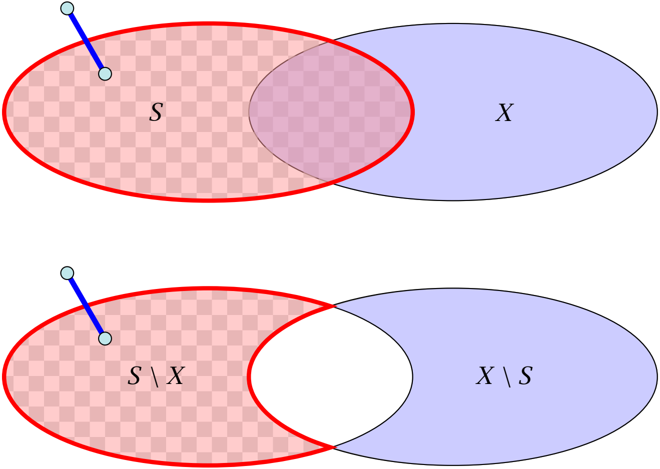

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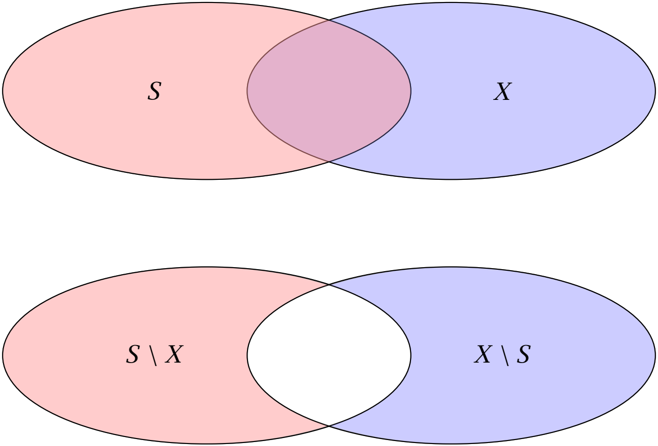

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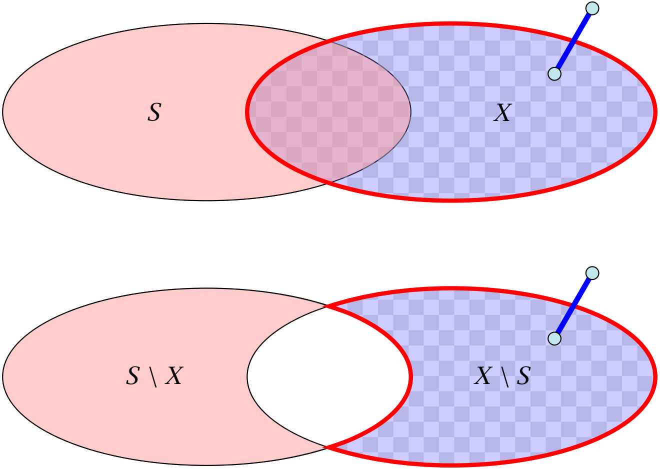

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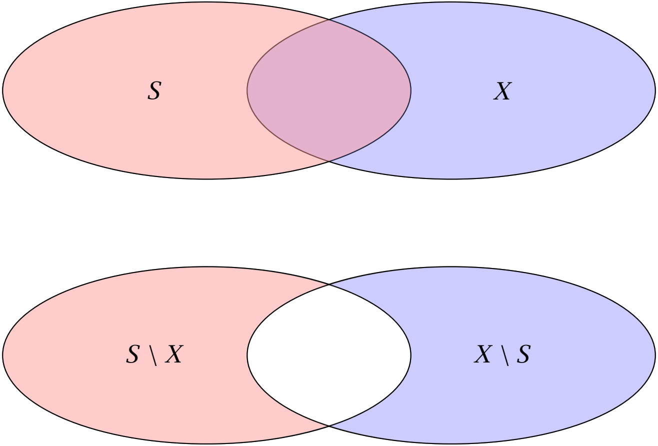

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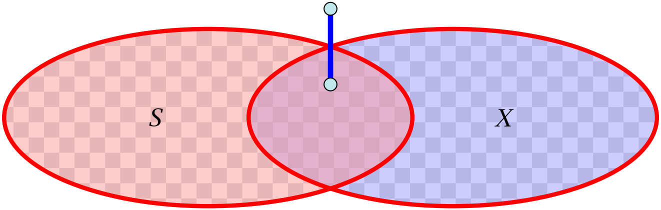

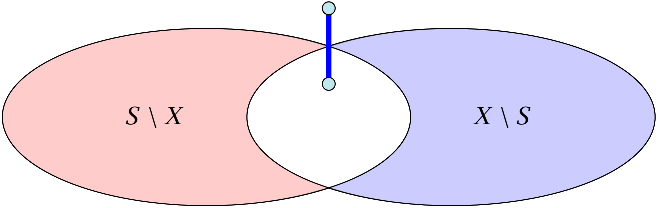

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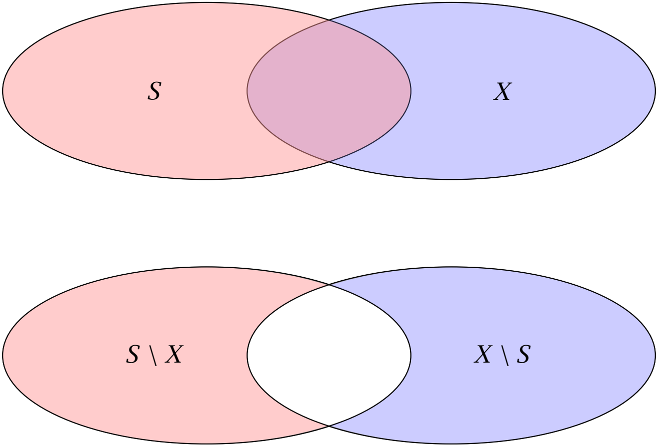

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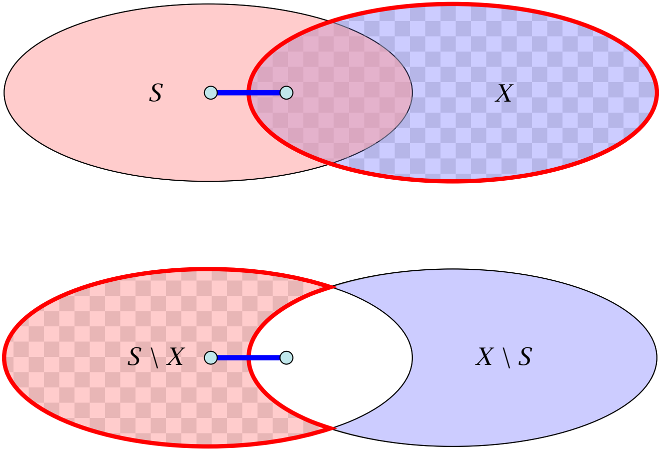

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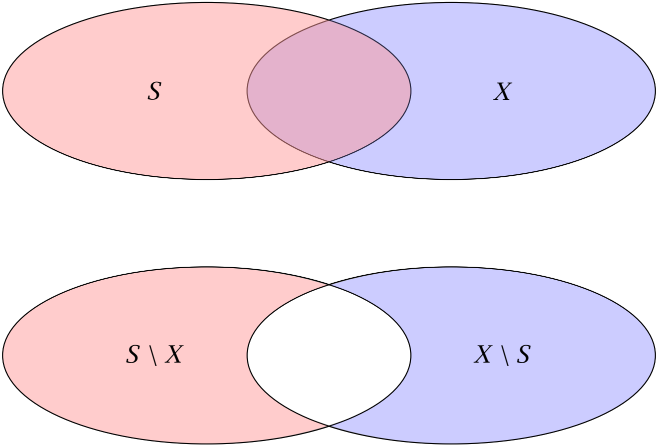

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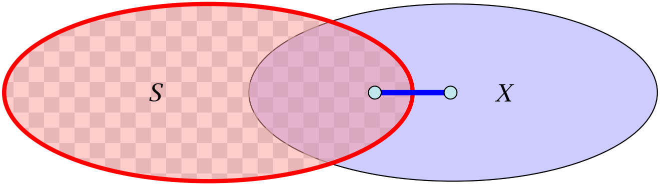

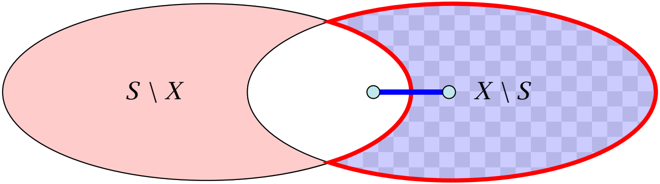

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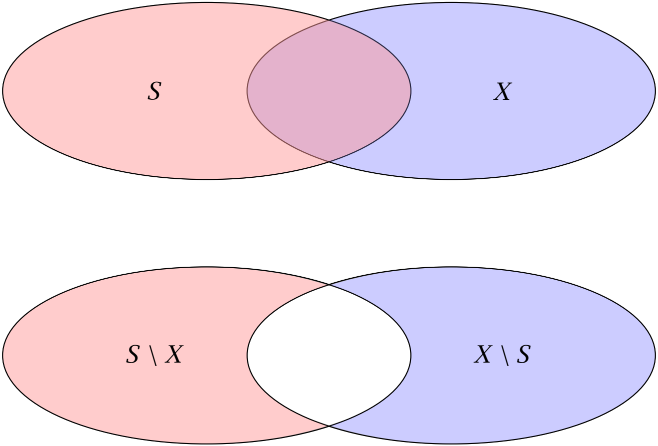

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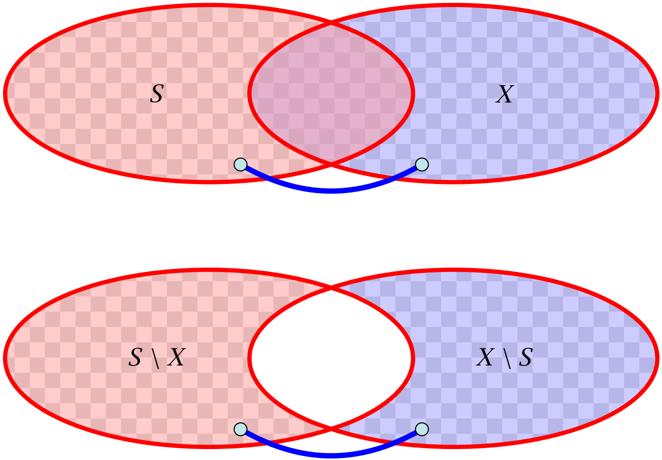

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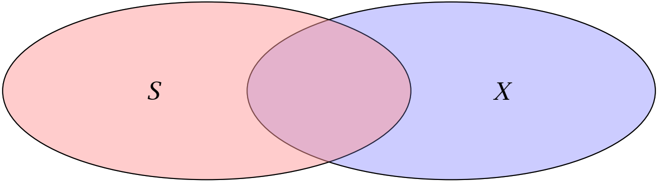

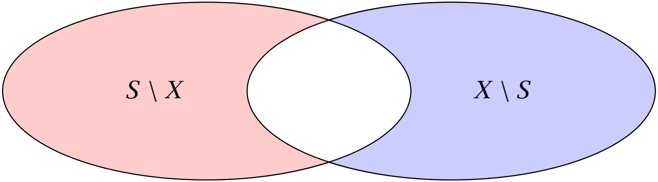

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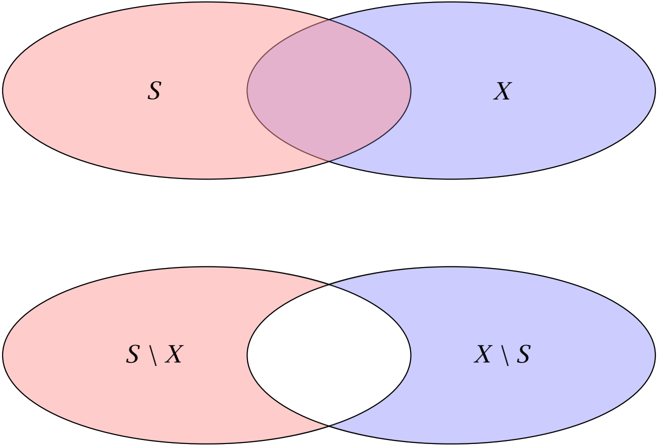

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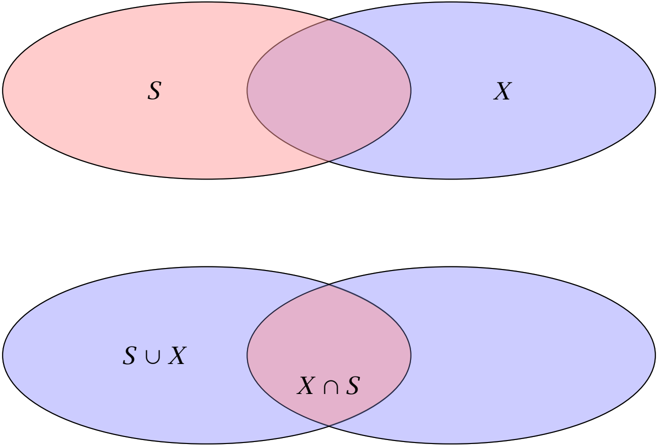

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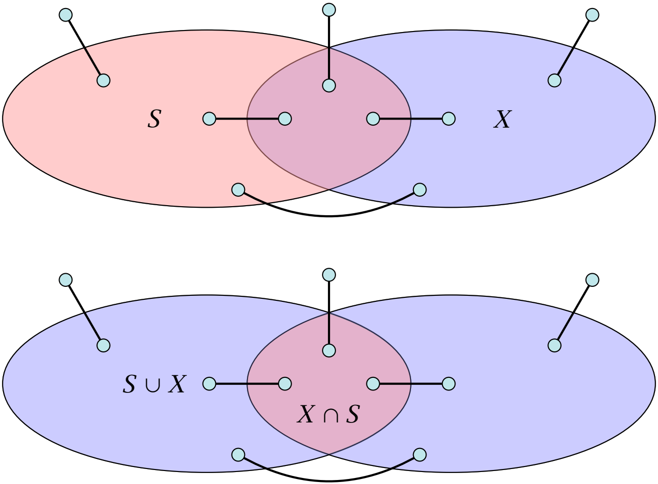

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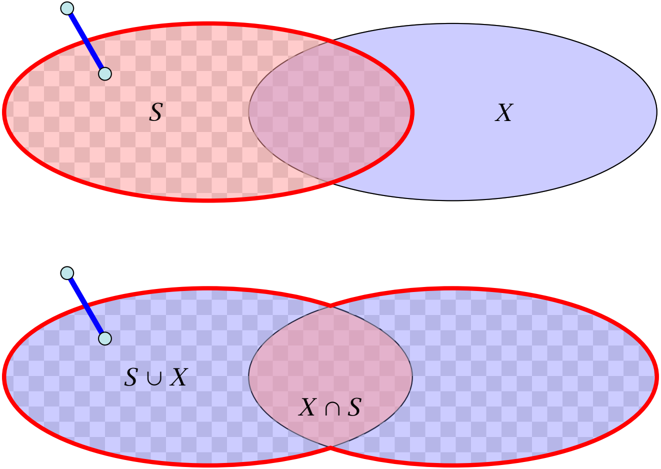

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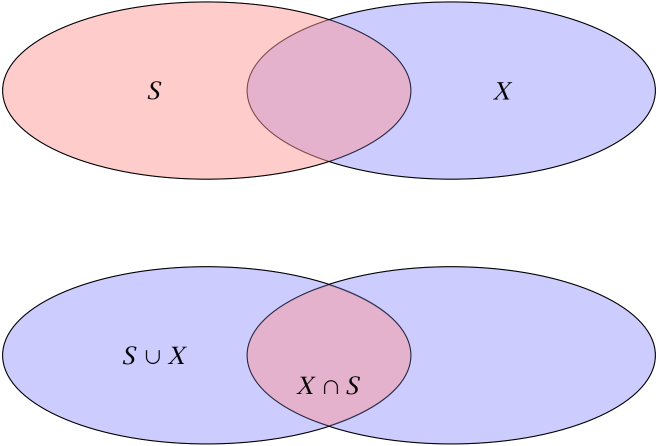

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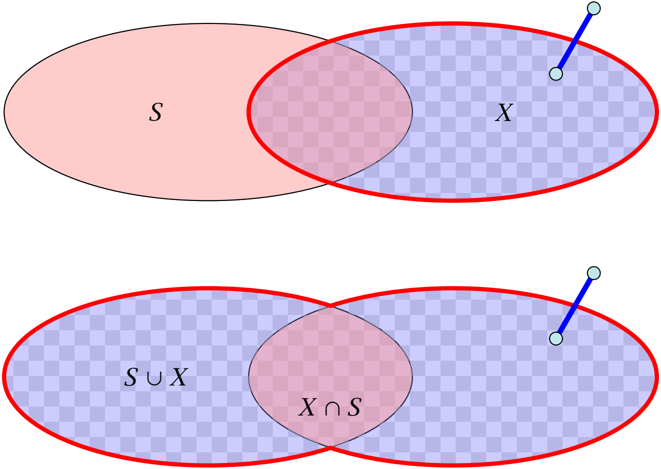

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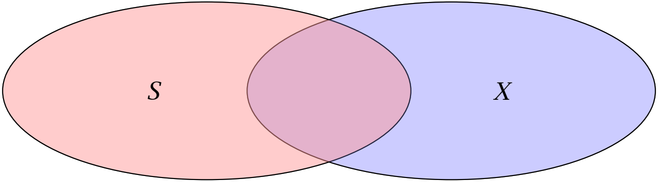

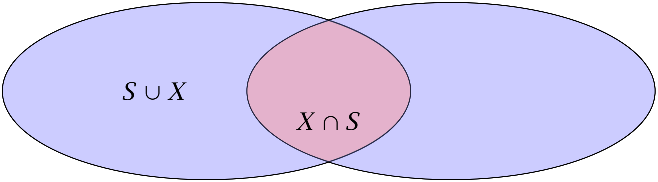

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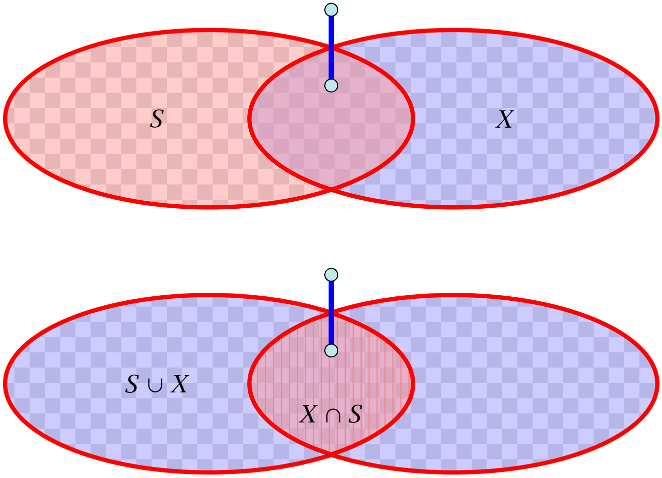

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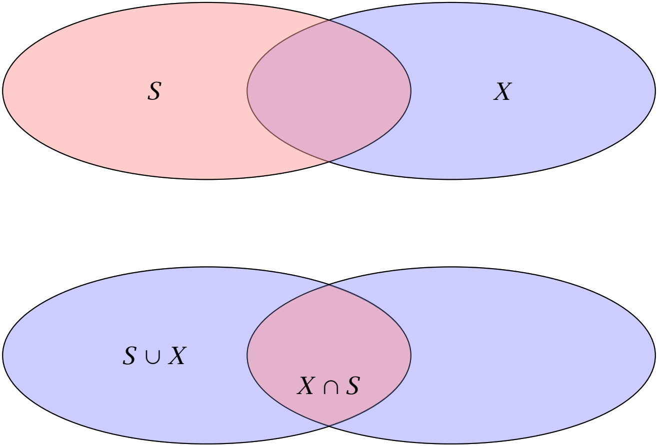

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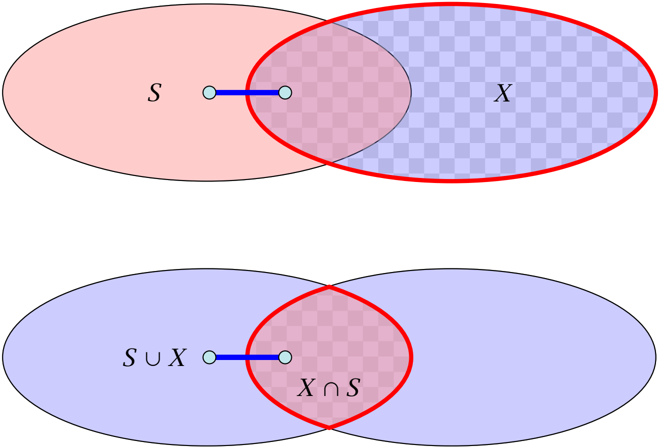

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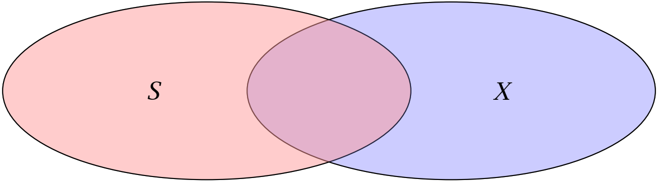

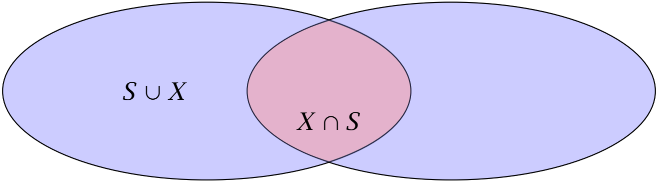

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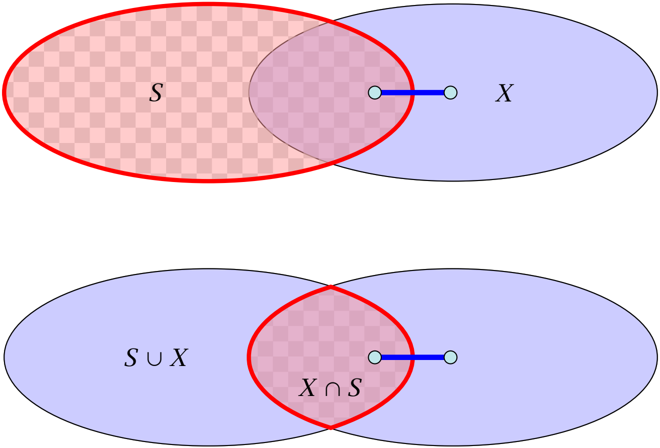

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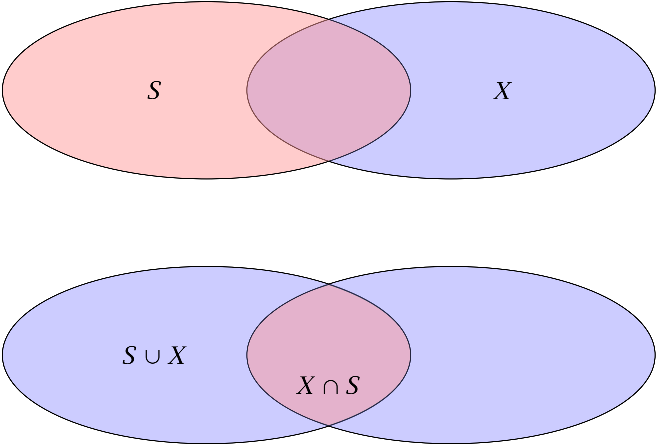

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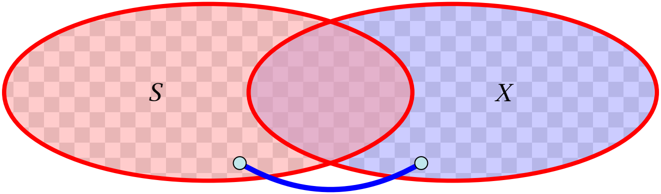

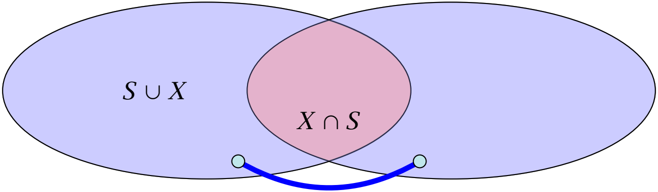

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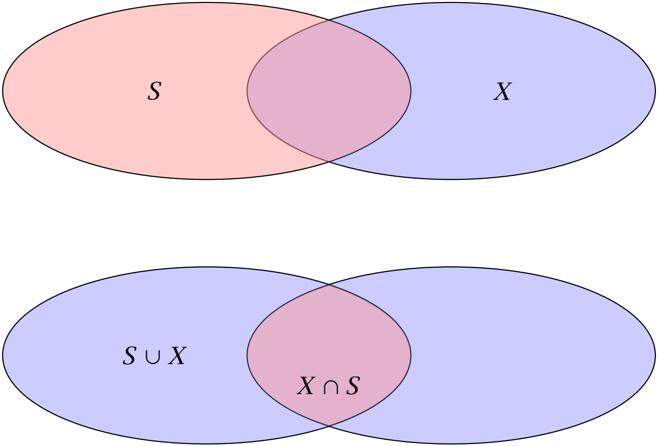

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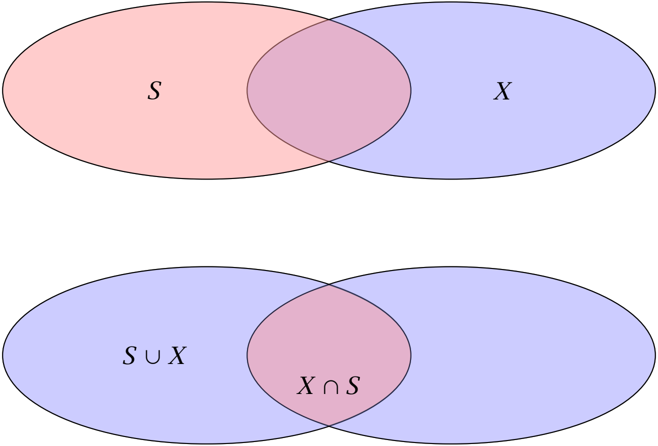

Analysis

Lemma 94 tells us that if we have a graph $G = (V, E)$ and we contract a subset $X \subset V$ that corresponds to some mincut, then the value of $f(s,t)$ does not change for two nodes $s, t \notin X$.

We will show (later) that the connected components that we contract during a split-operation each correspond to some mincut and, hence, $f_H(s,t) = f(s,t)$, where $f_H(s,t)$ is the value of a minimum s-t mincut in graph H.

Analysis

Invariant [existence of representatives]:
For any edge $\{S_i, S_j\}$ in T, there are vertices $a \in S_i$ and $b \in S_j$ such that $w(S_i, S_j) = f(a, b)$ and the cut defined by edge $\{S_i, S_j\}$ is a minimum a-b cut in G.

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We first show that the invariant implies that at the end of the algorithm T is indeed a cut-tree.

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- Let $\{x_j, x_{j+1}\}$ be the edge with minimum weight on the path.
- Since by the invariant this edge induces an s-t cut with capacity $f(x_j, x_{j+1})$ we get $f(s,t) \leq f(x_j, x_{j+1}) = f_T(s,t)$.

Analysis

- Hence, $f_T(s,t) = f(s,t)$ (flow equivalence).
- The edge $\{x_j, x_{j+1}\}$ is a mincut between s and t in T.
- By invariant, it forms a cut with capacity $f(x_j, x_{j+1})$ in G (which separates s and t).
- Since, we can send a flow of value $f(x_j, x_{j+1})$ btw. s and t, this is an s-t mincut (cut property).

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Proof of Invariant

The invariant obviously holds at the beginning of the algorithm.

Now, we show that it holds after a split-operation provided that it was true before the operation.

Let S_i denote our selected cluster with nodes a and b. Because of the invariant all edges leaving $\{S_i\}$ in T correspond to some mincuts.

Therefore, contracting the connected components does not change the mincut btw. a and b due to Lemma 94.

After the split we have to choose representatives for all edges. For the new edge $\{S_i^a, S_i^b\}$ with capacity $w(S_i^a, S_i^b) = f_H(a, b)$ we can simply choose a and b as representatives.

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For edges that are not incident to S_i we do not need to change representatives as the neighbouring sets do not change.

Consider an edge $\{X, S_i\}$, and suppose that before the split it used representatives $x \in X$, and $s \in S_i$. Assume that this edge is replaced by $\{X, S_i^a\}$ in the new tree (the case when it is replaced by $\{X, S_i^b\}$ is analogous).

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The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 94 we know that $f'(x,a) = f(x,a)$ as $x, a \notin B$.

We further have $f'(x,a) \geq \min\{f'(x,v_B), f'(v_B,a)\}$.

Since $s \in B$ we have $f'(v_B,x) \geq f(s,x)$.

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Also, $f'(a,v_B) \ge f(a,b) \ge f(x,s)$ since the a-b cut that splits S_i into S_i^a and S_i^b also separates s and x.

Proof of Invariant

Because the invariant was true before the split we know that the edge $\{X, S_i\}$ induces a cut in G of capacity $f(x,s)$. Since, x and a are on opposite sides of this cut, we know that $f(x,a) \le f(x,s)$.

The set B forms a mincut separating a from b. Contracting all nodes in this set gives a new graph G' where the set B is represented by node v_B. Because of Lemma 94 we know that $f'(x,a) = f(x,a)$ as $x, a \notin B$.

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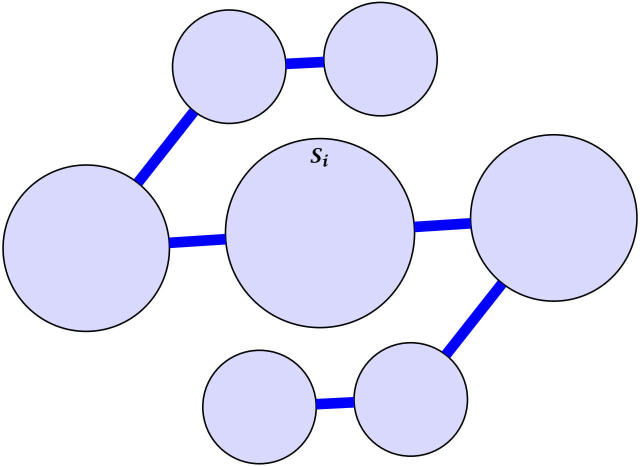
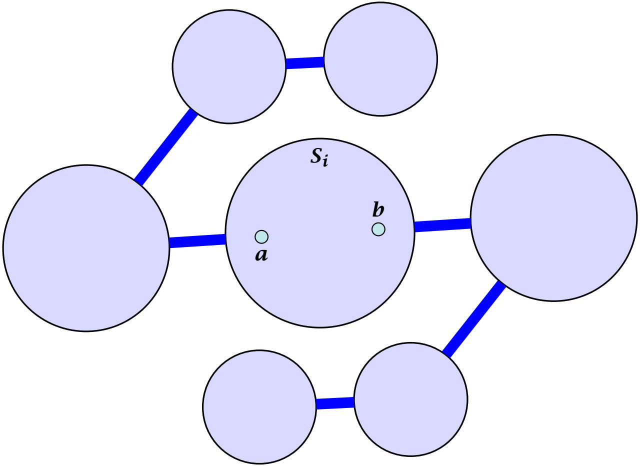
Analysis



Analysis



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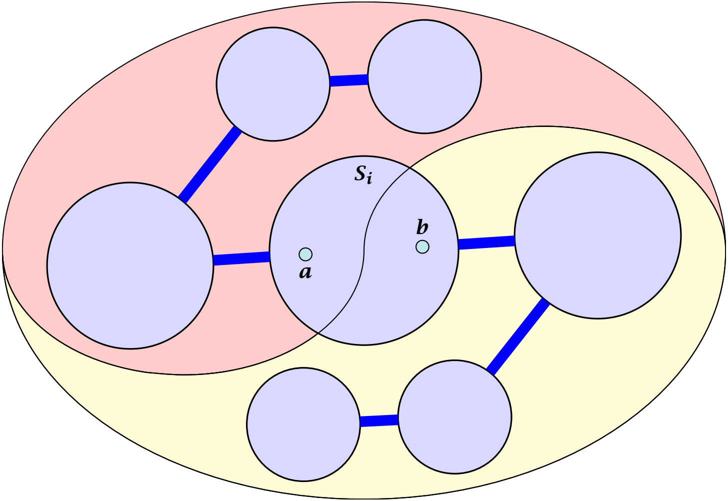

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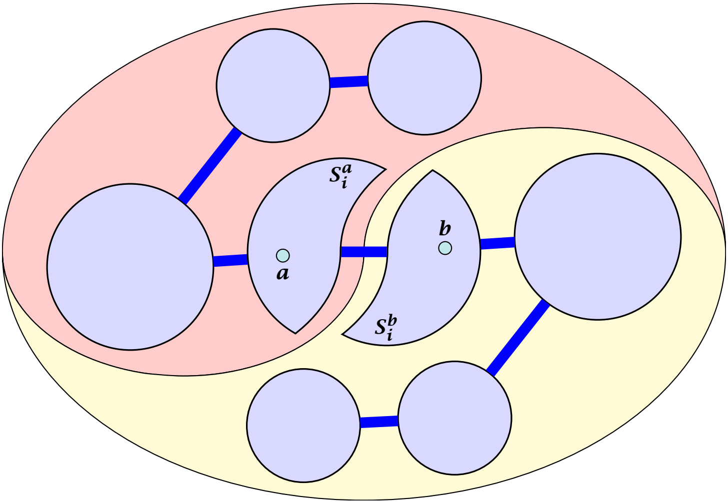

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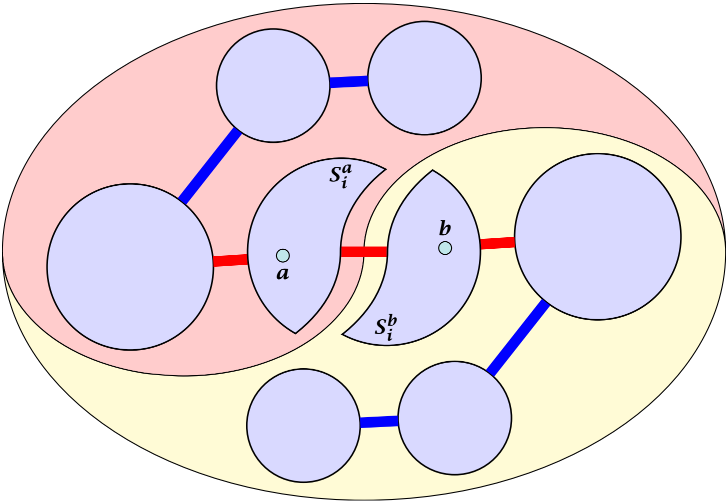

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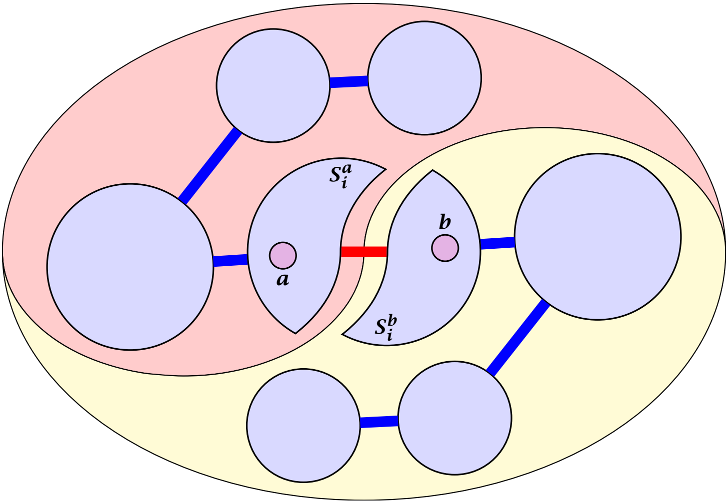

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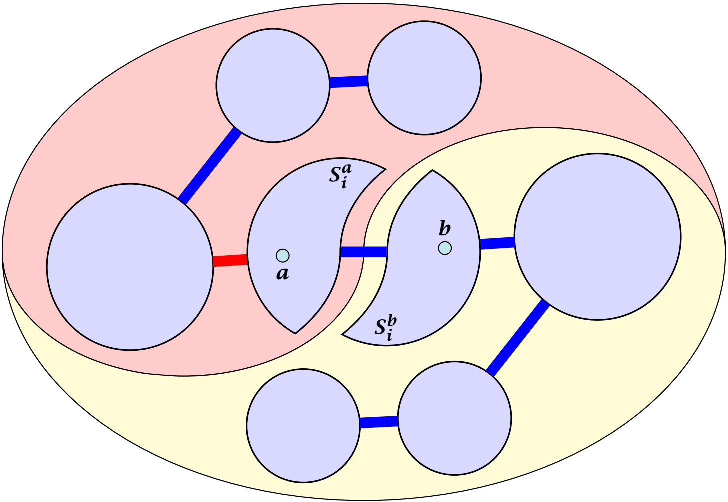

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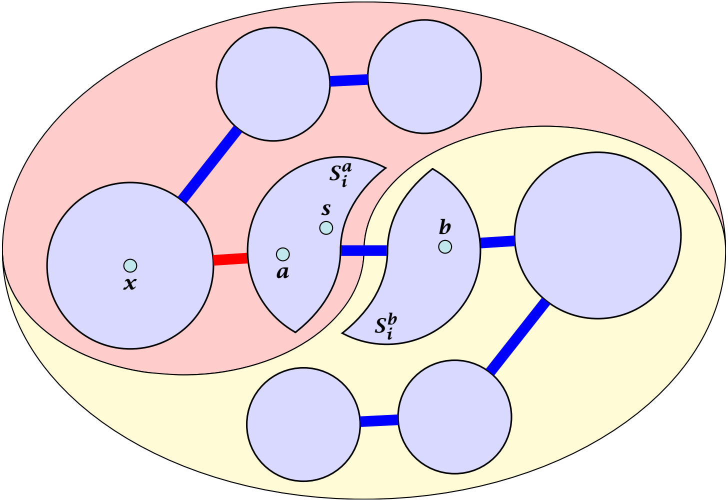

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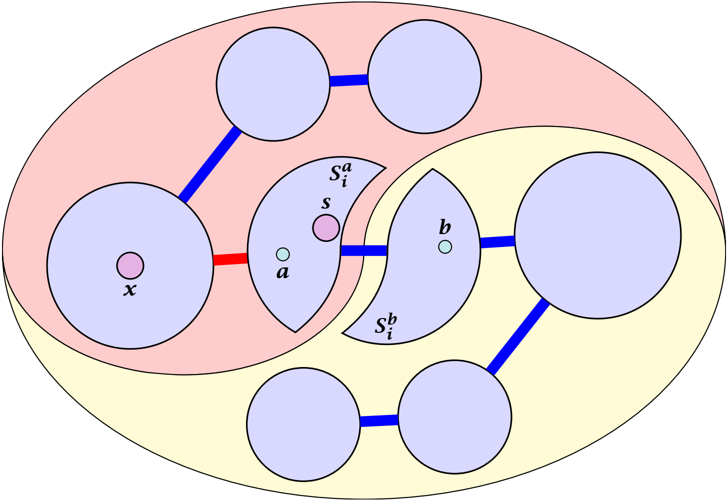

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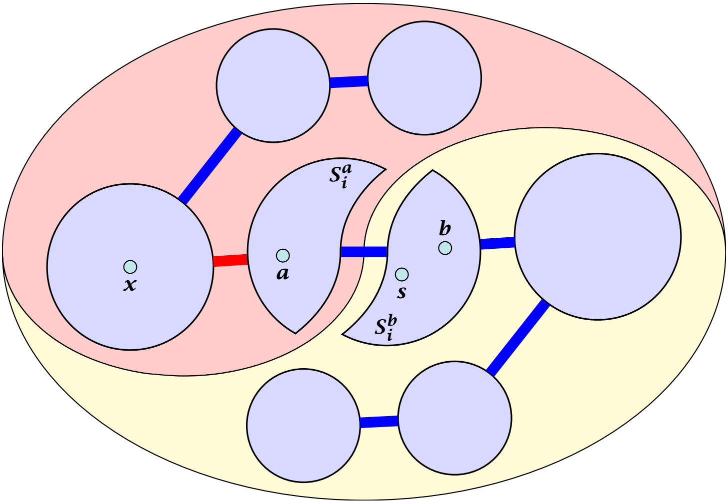

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