Prefix Sum

input: $x[1]\ldots x[n]$
output: $s[1]\ldots s[n]$ with $s[i]=\sum_{j=1}^{i} x[i]$ (w.r.t. operator $*$)

```
Algorithm 6 PrefixSum($n, x[1]\ldots x[n]$)
// compute prefixsums; $n = 2^k$
if $n = 1$ then $s[1] \leftarrow x[1]$; return
for $1 \le i \le n/2$ pardo
    $a[i] \leftarrow x[2i-1] * x[2i]$
$z[1],\ldots,z[n/2] \leftarrow$ PrefixSum($n/2, a[1]\ldots a[n/2]$)
for $1 \le i \le n$ pardo
    $i$ even   : $s[i] \leftarrow z[i/2]$
    $i = 1$    : $s[1] = x[1]$
    $i$ odd    : $s[i] \leftarrow z[(i-1)/2] * x[i]$
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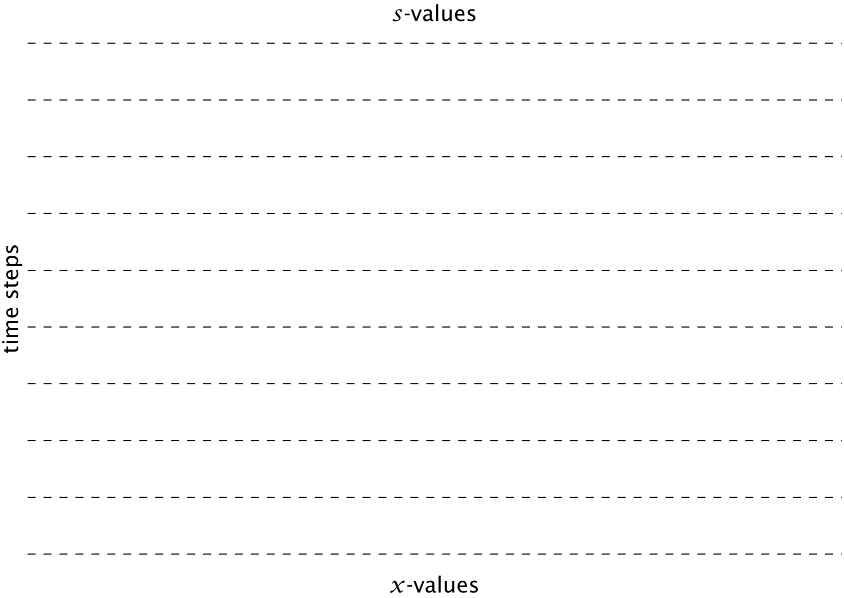
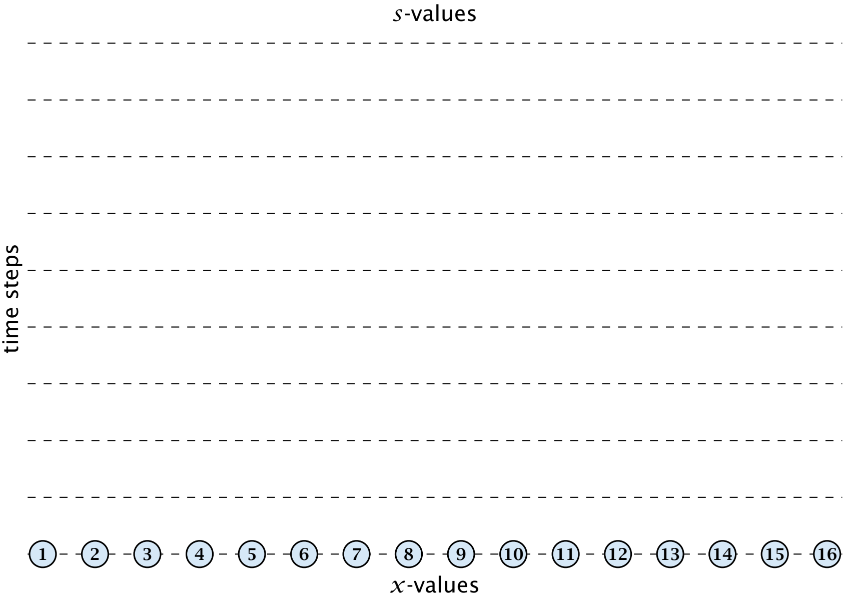
Prefix Sum



Prefix Sum



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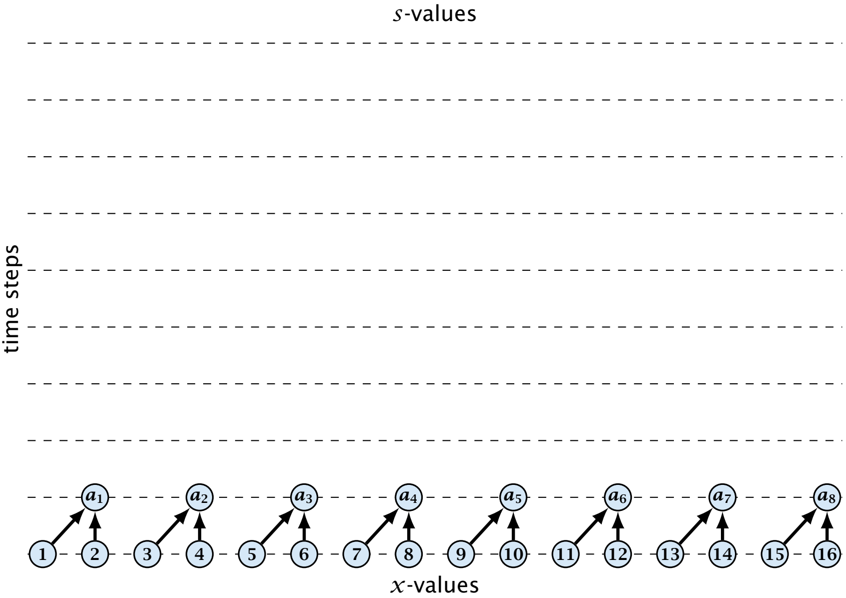

s-values
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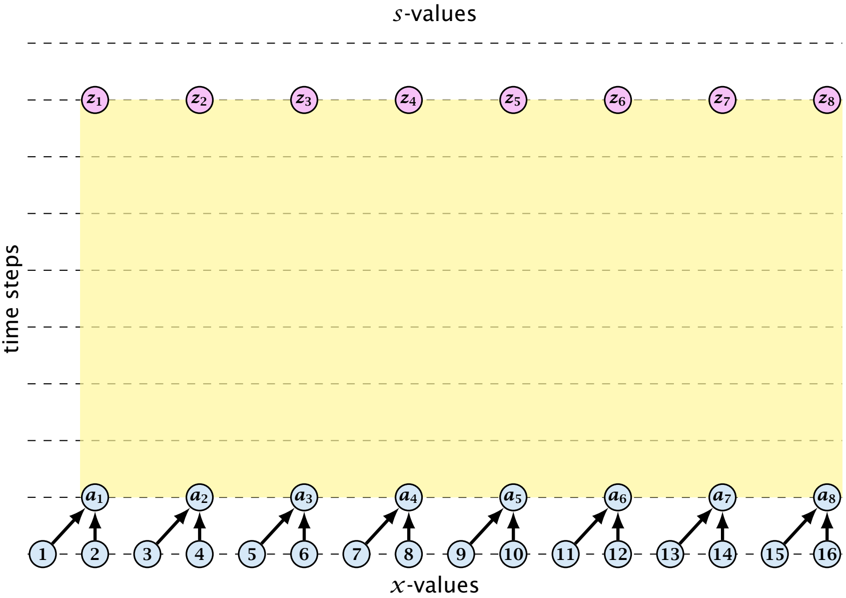
x-values

Prefix Sum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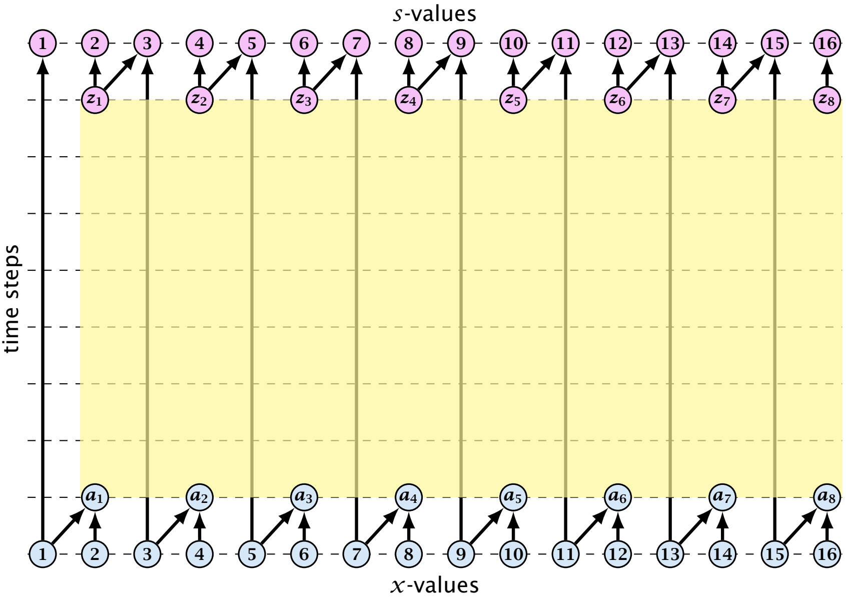
Prefix Sum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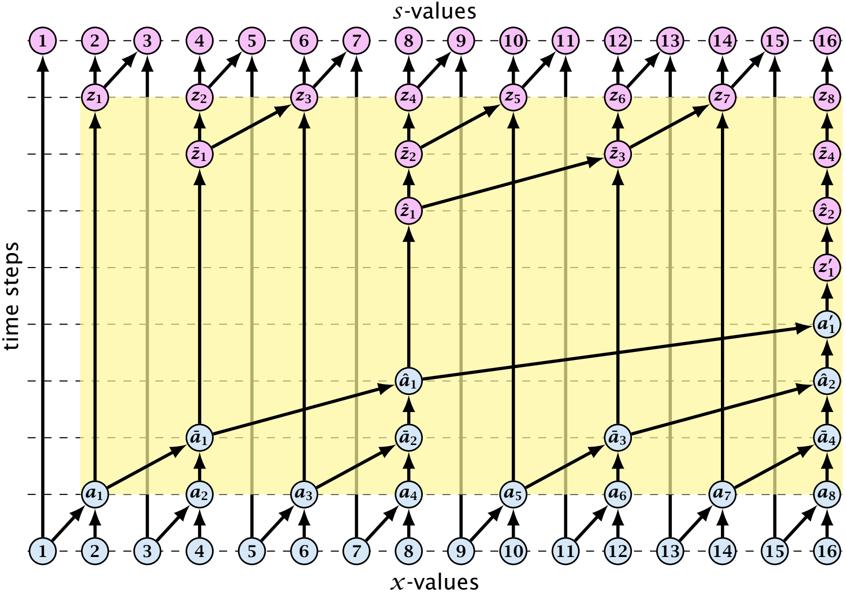
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Prefix Sum

The algorithm uses work $\mathcal{O}(n)$ and time $\mathcal{O}(\log n)$ for solving Prefix Sum on an EREW-PRAM with n processors.

It is clearly work-optimal.

Theorem 1
On a CREW PRAM a Prefix Sum requires running time $\Omega(\log n)$ regardless of the number of processors.

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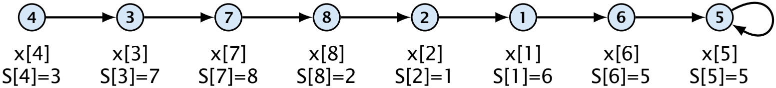
Theorem 1
On a CREW PRAM a Prefix Sum requires running time $\Omega(\log n)$ *regardless of the number of processors.*

Parallel Prefix

Input: a linked list given by successor pointers; a value $x[i]$ for every list element; an operator $*$;

Output: for every list position ℓ the sum (w.r.t. $*$) of elements after ℓ in the list (including ℓ)



Parallel Prefix

Algorithm 7 ParallelPrefix

```
for $1 \le i \le n$ pardo
    $P[i] \leftarrow S[i]$
    while $S[i] \neq S[S[i]]$ do
        $x[i] \leftarrow x[i] * x[S[i]]$
        $S[i] \leftarrow S[S[i]]$
    if $P[i] \neq i$ then $x[i] \leftarrow x[i] * x[S(i)]$
```

The algorithm runs in time $\mathcal{O}(\log n)$.

It has work requirement $\mathcal{O}(n \log n)$. non-optimal

This technique is also known as pointer jumping

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4.3 Divide & Conquer — Merging

Given two sorted sequences $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, compute the sorted squence $C = (c_1, \ldots, c_n)$.

Definition 2

Let $X = (x_1, \ldots, x_t)$ be a sequence. The rank $\text{rank}(y : X)$ of y in X is

$$\text{rank}(y : X) = |\{x \in X \mid x \leq y\}|$$

For a sequence $Y = (y_1, \ldots, y_s)$ we define $\text{rank}(Y : X) := (r_1, \ldots, r_s)$ with $r_i = \text{rank}(y_i : X)$.

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Observation:
We can assume wlog. that elements in A and B are different.

Then for $c_i \in C$ we have $i = \operatorname{rank}(c_i : A \cup B)$.

This means we just need to determine rank(x : $A \cup B$) for all elements!

Observe, that $\operatorname{rank}(x : A \cup B) = \operatorname{rank}(x : A) + \operatorname{rank}(x : B)$.

Clearly, for $x \in A$ we already know $\operatorname{rank}(x : A)$, and for $x \in B$ we know $\operatorname{rank}(x : B)$.

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can be done in $\mathcal{O}(\log n)$ time with $2n$ processors by binary search

Lemma 3

On a CREW PRAM, Merging can be done in $\mathcal{O}(\log n)$ *time and* $\mathcal{O}(n \log n)$ *work.*

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$A = (a_1, \dots, a_n)$; $B = (b_1, \dots, b_n)$;
$\log n$ integral; $k := n/\log n$ integral;

Algorithm 8 GenerateSubproblems

```
j_0 ← 0
j_k ← n
for 1 ≤ i ≤ k − 1 pardo
    j_i ← rank(b_{i log n} : A)
for 0 ≤ i ≤ k − 1 pardo
    B_i ← (b_{i log n+1}, ..., b_{(i+1) log n})
    A_i ← (a_{j_i+1}, ..., a_{j_{i+1}})
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If C_i is the merging of A_i and B_i then the sequence $C_0 \dots C_{k-1}$ is a sorted sequence.

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for $1 \le i \le k-1$ pardo
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for $0 \le i \le k-1$ pardo
    $B_i \leftarrow (b_{i \log n + 1}, \ldots, b_{(i+1)\log n})$
    $A_i \leftarrow (a_{j_i + 1}, \ldots, a_{j_{i+1}})$
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4.3 Divide & Conquer — Merging

We can generate the subproblems in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n)$.

Note that in a sub-problem B_i has length $\log n$.

If we run the algorithm again for every subproblem, (where A_i takes the role of B) we can in time $\mathcal{O}(\log\log n)$ and work $\mathcal{O}(n)$ generate subproblems where A_j and B_j have both length at most $\log n$.

Such a subproblem can be solved by a single processor in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(|A_i| + |B_i|)$.

Parallelizing the last step gives total work $\mathcal{O}(n)$ and time $\mathcal{O}(\log n)$.

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Note that in a sub-problem B_i has length $\log n$.

If we run the algorithm again for every subproblem, (where A_i takes the role of B) we can in time $\mathcal{O}(\log\log n)$ and work $\mathcal{O}(n)$ generate subproblems where A_j and B_j have both length at most $\log n$.

Such a subproblem can be solved by a single processor in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(|A_i| + |B_i|)$.

Parallelizing the last step gives total work $\mathcal{O}(n)$ and time $\mathcal{O}(\log n)$.

the resulting algorithm is work optimal

4.4 Maximum Computation

Lemma 4

On a CRCW PRAM the maximum of n numbers can be computed in time $\mathcal{O}(1)$ with n^2 processors.

proof on board...

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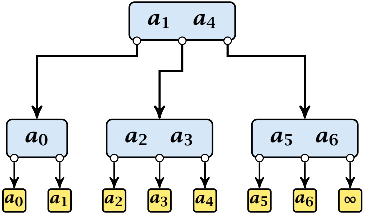
4.5 Inserting into a $(2,3)$-tree

Given a $(2,3)$-tree with n elements, and a sequence $x_0 < x_1 < x_2 < \cdots < x_k$ of elements. We want to insert elements $x_1, \ldots, x_k$ into the tree ($k \ll n$).

time: $\mathcal{O}(\log n)$; work: $\mathcal{O}(k \log n)$



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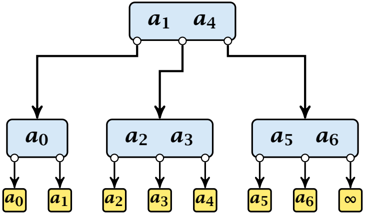

4.5 Inserting into a $(2,3)$-tree

1. determine for every x_i the leaf element before which it has to be inserted

   time: $\mathcal{O}(\log n)$; work: $\mathcal{O}(k \log n)$; CREW PRAM

   all x_i's that have to be inserted before the same element form a chain

2. determine the largest/smallest/middle element of every chain

   time: $\mathcal{O}(\log k)$; work: $\mathcal{O}(k)$;

3. insert the middle element of every chain

   compute new chains

   time: $\mathcal{O}(\log n)$; work: $\mathcal{O}(k_i \log n + k)$; k_i= #inserted elements

   (computing new chains is constant time)

4. repeat Step 3 for logarithmically many rounds

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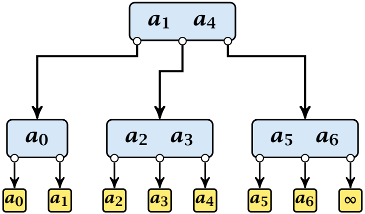

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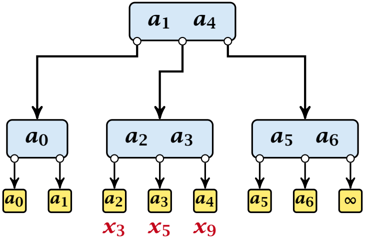

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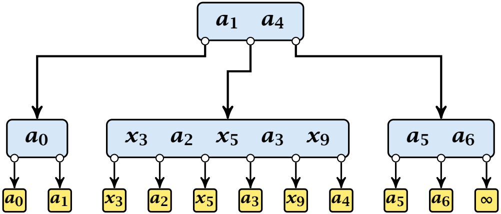

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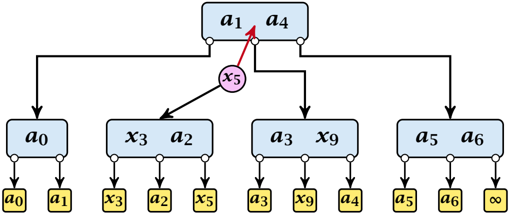

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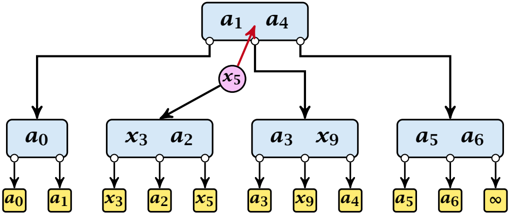

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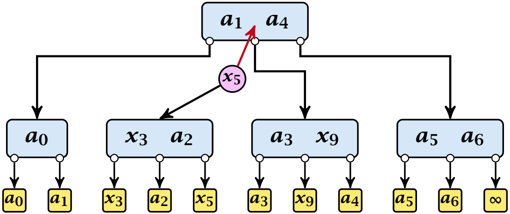

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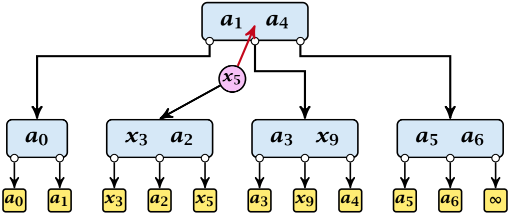

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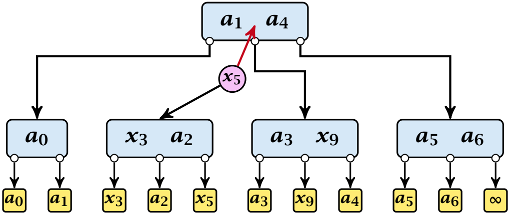

- each internal node is split into at most two parts
- each split operation promotes at most one element
- hence, on every level we want to insert at most one element per successor pointer
- we can use the same routine for every level

4.5 Inserting into a $(2,3)$-tree

- Step 3, works in phases; one phase for every level of the tree
- Step 4, works in rounds; in each round a different set of elements is inserted

Observation

We can start with phase i of round r as long as phase i of round $r-1$ and (of course), phase $i-1$ of round r has finished.

This is called Pipelining. Using this technique we can perform all rounds in Step 4 in just $\mathcal{O}(\log k + \log n)$ many parallel steps.

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4.6 Symmetry Breaking

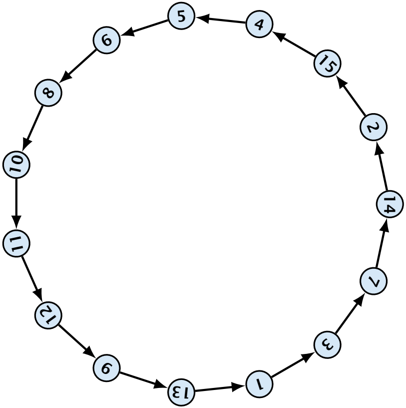
The following algorithm colors an n-node cycle with $\lceil \log n \rceil$ colors.

Algorithm 9 BasicColoring

```
for $1 \le i \le n$ pardo
    $\mathrm{col}(i) \leftarrow i$
    $k_i \leftarrow$ smallest bitpos where $\mathrm{col}(i)$ and $\mathrm{col}(S(i))$ differ
    $\mathrm{col}'(i) \leftarrow 2k_i + \mathrm{col}(i)_{k_i}$
```

(bit positions are numbered starting with 0)

4.6 Symmetry Breaking



v	col	k	col′
1	0001	1	2
3	0011	2	4
7	0111	0	1
14	1110	2	5
2	0010	0	0
15	1111	0	1
4	0100	0	0
5	0101	0	1
6	0110	1	3
8	1000	1	2
10	1010	0	0
11	1011	0	1
12	1100	0	0
9	1001	2	4
13	1101	2	5

4.6 Symmetry Breaking

Applying the algorithm to a coloring with bit-length t generates a coloring with largest color at most

$$2(t-1)+1$$

and bit-length at most

$$\lceil \log_2(2(t-1)+1) \rceil \le \lceil \log_2(2t) \rceil = \lceil \log_2(t) \rceil + 1$$

Applying the algorithm repeatedly generates a constant number of colors after $\mathcal{O}(\log^* n)$ operations.

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As long as the bit-length $t \geq 4$ the bit-length decreases.

Applying the algorithm with bit-length 3 gives a coloring with colors in the range $0, \ldots, 5 = 2t - 1$.

We can improve to a 3-coloring by successively re-coloring nodes from a color-class:

Algorithm 10 ReColor

```
for $\ell \leftarrow 5$ to 3
    for $1 \leq i \leq n$ pardo
        if $\mathrm{col}(i) = \ell$ then
            $\mathrm{col}(i) \leftarrow \min\{\{0,1,2\} \setminus \{\mathrm{col}(P[i]), \mathrm{col}(S[i])\}\}$
```

This requires time $\mathcal{O}(1)$ and work $\mathcal{O}(n)$.

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4.6 Symmetry Breaking

Lemma 7

We can color vertices in a ring with three colors in $\mathcal{O}(\log^* n)$ *time and with* $\mathcal{O}(n \log^* n)$ *work.*

not work optimal

4.6 Symmetry Breaking

Lemma 8

Given n integers in the range $0, \ldots, \mathcal{O}(\log n)$, there is an algorithm that sorts these numbers in $\mathcal{O}(\log n)$ time using a linear number of operations.

Proof: Exercise!

4.6 Symmetry Breaking

Algorithm 11 OptColor

```
for $1 \leq i \leq n$ pardo
    $\mathrm{col}(i) \leftarrow i$
apply BasicColoring once
sort vertices by colors
for $\ell = 2\lceil \log n \rceil$ to $3$ do
    for all vertices $i$ of color $\ell$ pardo
        $\mathrm{col}(i) \leftarrow \min\{\{0,1,2\} \setminus \{\mathrm{col}(P[i]), \mathrm{col}(S[i])\}\}$
```

Lemma 9

A ring can be colored with 3 *colors in time* $\mathcal{O}(\log n)$ *and with work* $\mathcal{O}(n)$*.*

work optimal but not too fast