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Randomized Algorithms

Exercise Sheet 10

Due: January 11, 2016
at 10:15, in class

Exercise 10.1 (10 points)
We throw n balls uniformly at random into n bins. By using a Chernoff bound, show that the probability that a bin contains at least $\frac{\ln n}{\ln \ln n}$ balls is at most $\frac{1}{n}$ for large n.

Exercise 10.2 (10 points)
We plan to conduct an opinion poll to find the percentage of people in a community who want its president impeached. Assume that every person answers either yes or no. If the actual fraction of people who want the president impeached is p, we want to find an estimate X of p such that

$$P[|X - p| \leq \epsilon p] > 1 - \delta,$$

for a given ϵ and δ, with $0 < \epsilon, \delta < 1$.

We query N people chosen independently and uniformly at random from the community and output the fraction of them who want the president impeached. How large should N be for our result to be a suitable estimator of p? Use Chernoff bounds, and express N in terms of p, ϵ, and δ.

Calculate the value of N from your bound if $\epsilon = 0.1$ and $\delta = 0.05$ and if you know that p is between 0.2 and 0.8.

Exercise 10.3 (10 points)
We show how to construct a random permutation π on $[1, n]$, given a black box that outputs numbers independently and uniformly at random from $[1, k]$ where $k \geq n$. If we compute a function $f : [1, n] \to [1, k]$ with $f(i) \neq f(j)$ for $i \neq j$, this yields a permutation: simply output the numbers $[1, n]$ according to the order of the $f(i)$ values. To construct such a function f, do the following for $j = 1, \ldots, n$: choose $f(j)$ by repeatedly obtaining numbers from the black box and setting $f(j)$ to the first number found such that $f(j) \neq f(i)$ for $i < j$.

Prove that this approach gives a permutation chosen uniformly at random from all permutations. Find the expected number of calls to the black box that are needed when $k = n$ and $k = 2n$. For the case $k = 2n$, argue that for each call to the black box, the probability that a value $f(j)$ is assigned to some j is at least $1/2$. Based on this, use a Chernoff bound to bound the probability that the number of calls to the black box is at least $4n$.
Hint: Use the following fact. Given binary random variables V_i and W_i for which it holds that

$$P[V_i = 1] = p_i \geq \tfrac{1}{2} \ \forall i \qquad \textit{and} \qquad P[W_i = 1] = \tfrac{1}{2} \ \forall i$$

and $V = \sum_{i=1}^{k} V_i$ *and* $W = \sum_{i=1}^{k} W_i$*, for some* k*, then for any* $x > 0$*, it holds that*

$$P[V \leq x] \leq P[W \leq x].$$

Use the Chernoff bound for a sum of n *independent binary random variables that have a probability of being equal to* 1 *of* $\frac{1}{2}$*: for any* $0 < a < \mu$

$$P[X \leq \mu - a] \leq e^{-2a^2/n}.$$

Exercise 10.4 (10 points)
Let $X_1, \ldots, X_n$ be independent random variables such that

$$P[X_i = 1 - p_i] = p_i \qquad \text{and} \qquad P[X_i = -p_i] = 1 - p_i$$

Let $X = \sum_{i=1}^{n} X_i$. Prove that

$$P[|X| \geq a] \leq 2e^{-2a^2/n}.$$

Hint: Use the inequality

$$p_i e^{\lambda(1-p_i)} + (1 - p_i)e^{-\lambda p_i} \leq e^{\lambda^2/8}.$$