Duality

How do we get an upper bound to a maximization LP?

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a, b & \ge 0 \end{array}$$

Note that a lower bound is easy to derive. Every choice of $a, b \ge 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \ge 0$) such that $\sum_i y_i a_{ij} \ge c_j$ then $\sum_i y_i b_i$ will be an upper bound.

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Definition 2
Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

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5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
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This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

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$$= \min\left\{b^Ty' \;\middle|\; A^Ty' \geq c\right\}$$

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^Tx \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^Ty \mid A^Ty \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T(A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

Proof

Primal:

$$\begin{aligned} &\max\{c^T x \mid Ax = b, x \ge 0\} \\ &\quad = \max\{c^T x \mid Ax \le b, -Ax \le -b, x \ge 0\} \\ &\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\} \end{aligned}$$

Dual:

$$\begin{aligned} &\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \ge c, y \ge 0\} \\ &\quad = \min\left\{ [b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0 \right\} \\ &\quad = \min\left\{ b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0 \right\} \\ &\quad = \min\left\{ b^T y' \,\middle|\, A^T y' \ge c \right\} \end{aligned}$$

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5.3 Strong Duality

$P = \max\{c^T x \mid Ax \leq b, x \geq 0\}$
n_A: number of variables, m_A: number of constraints

We can put the non-negativity constraints into A (which gives us unrestricted variables): $\bar{P} = \max\{c^T x \mid \bar{A}x \leq \bar{b}\}$
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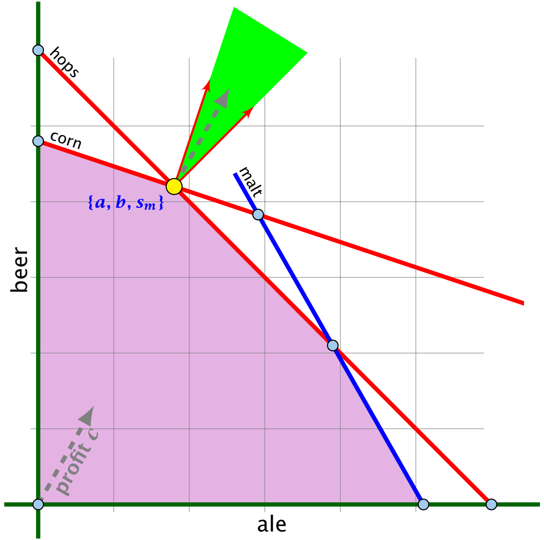

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Theorem 5 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

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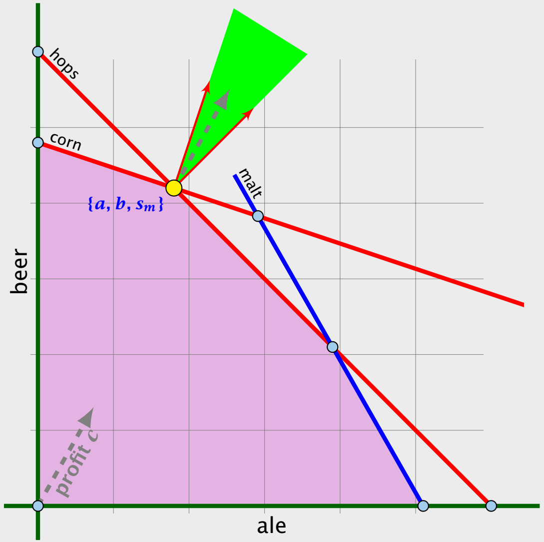

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Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.

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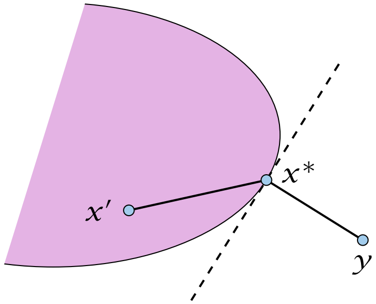
Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y. Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \le 0$.



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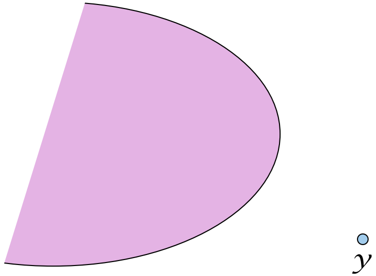

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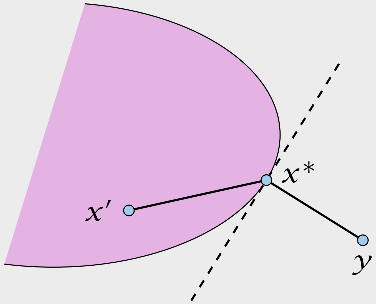

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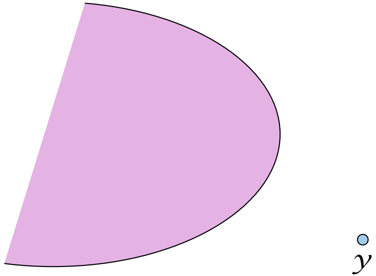

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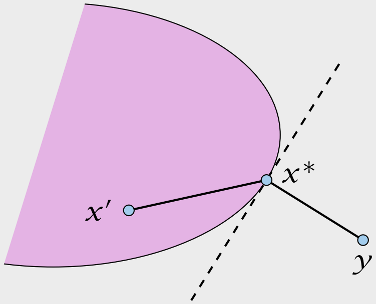

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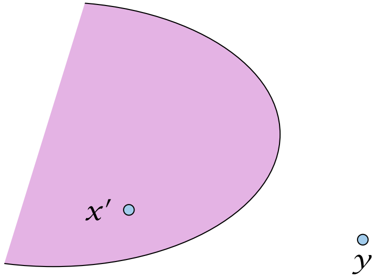

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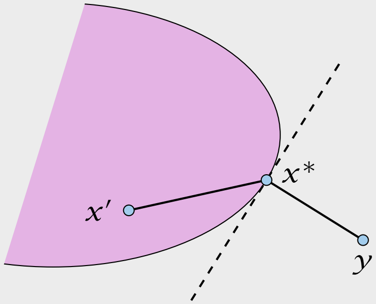

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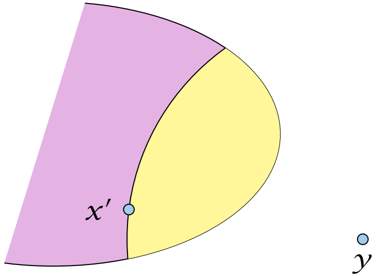

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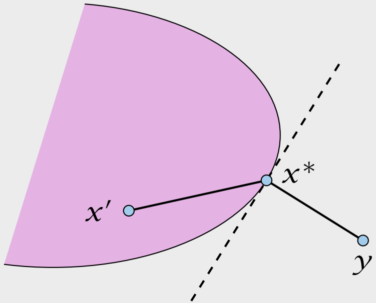

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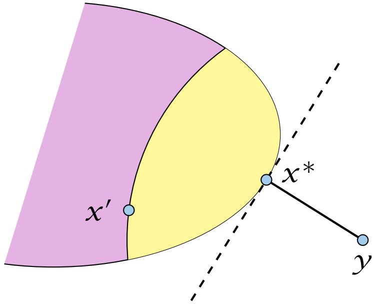

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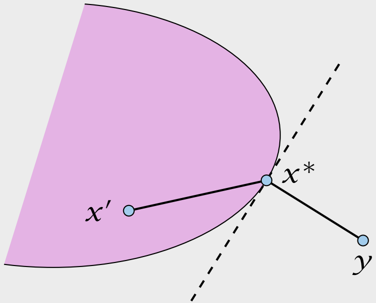
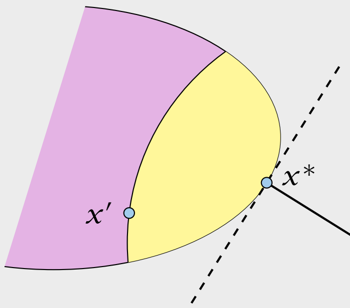
Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y. Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \leq 0$.



Proof of the Projection Lemma (continued)

Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
- We want to apply Weierstrass but X may not be bounded.
- $X \neq \emptyset$. Hence, there exists $x' \in X$.
- Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- Applying Weierstrass gives the existence.



Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \le \|y - x\|^2$ for all $x \in X$.

Proof of the Projection Lemma

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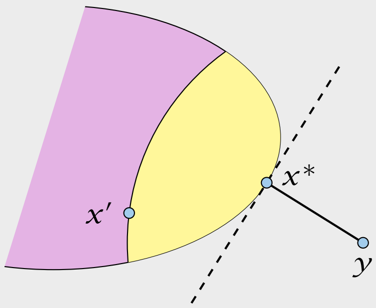

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \le \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \le \epsilon \le 1$.

Proof of the Projection Lemma

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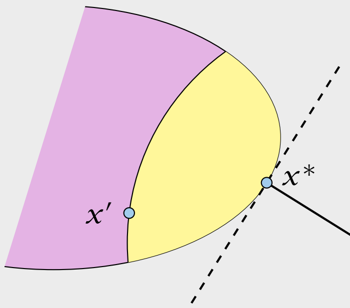

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$\|y - x^*\|^2$

Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
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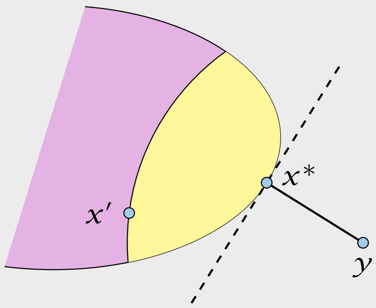

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$$\|y - x^*\|^2 \le \|y - x^* - \epsilon(x - x^*)\|^2$$

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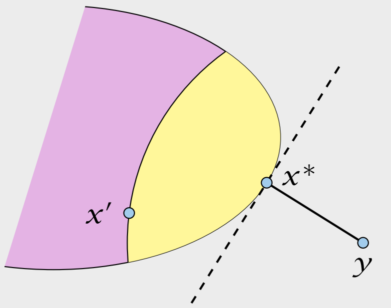

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2\|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
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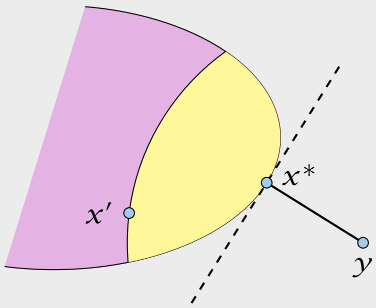

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Hence, $(y - x^*)^T(x - x^*) \le \frac{1}{2}\epsilon\|x - x^*\|^2$.

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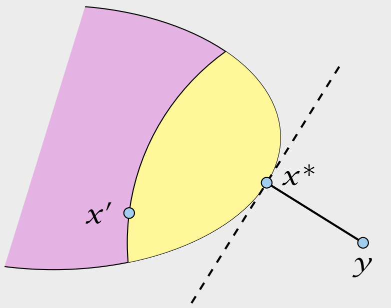

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Letting $\epsilon \to 0$ gives the result.

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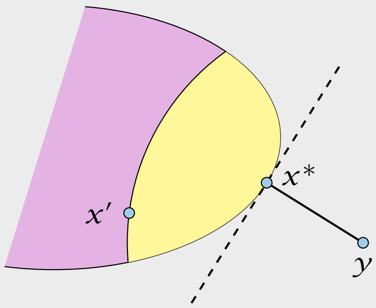

Theorem 8 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)

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Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to y in X.
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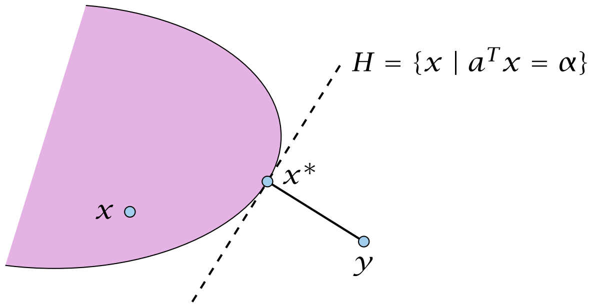

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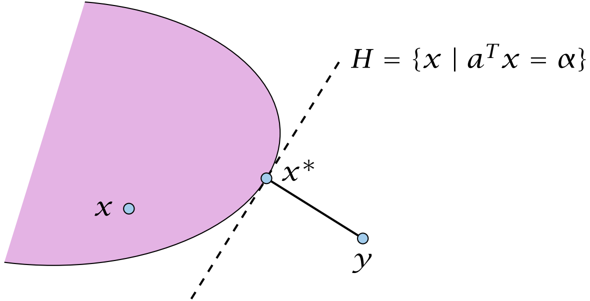

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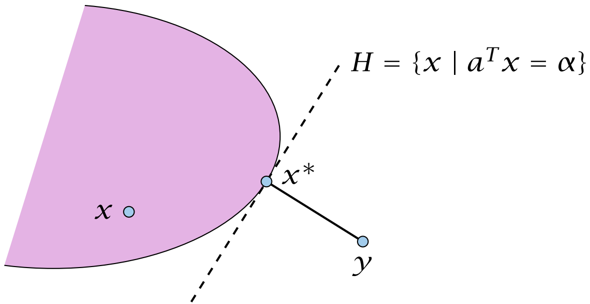

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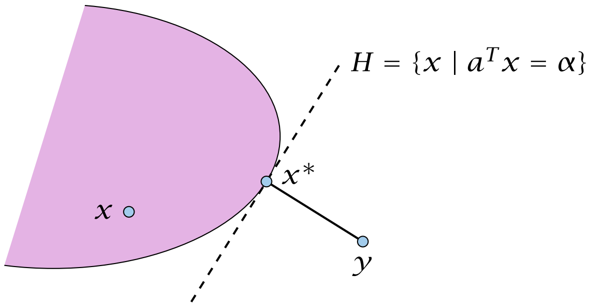

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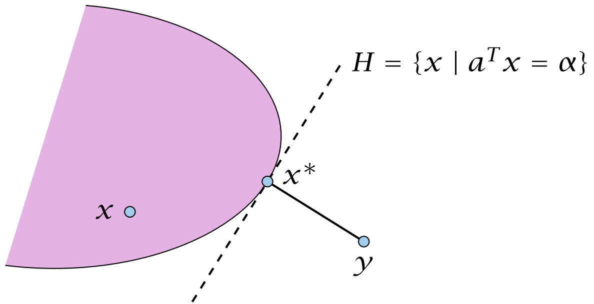

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Lemma 9 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ *with* $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^T y \geq 0$, $b^T y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^T b = y^T A x \geq 0$$

Hence, at most one of the statements can hold.

Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to y in X.
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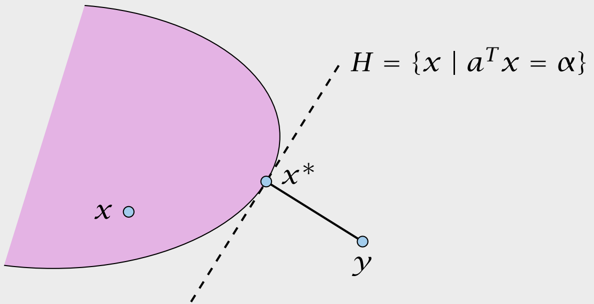

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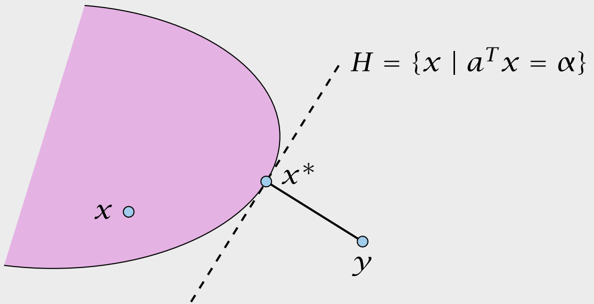

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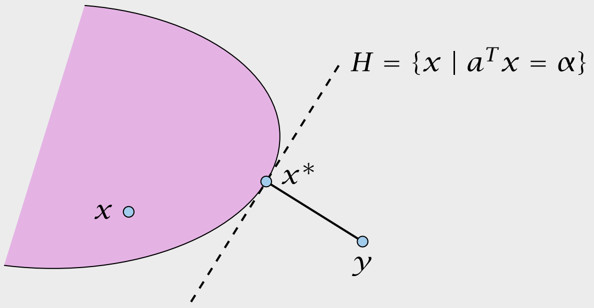

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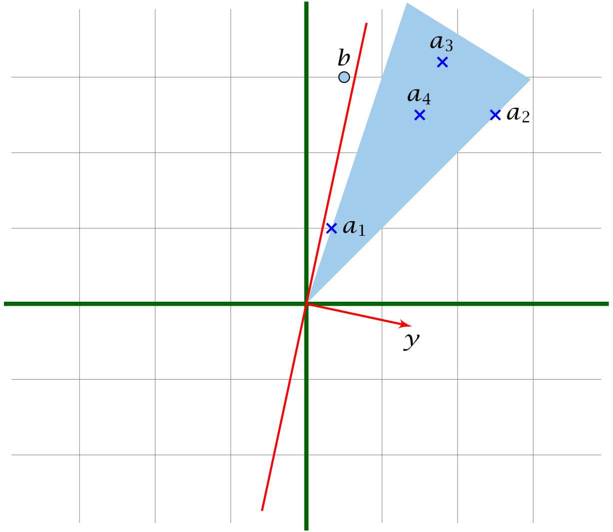

If b is not in the cone generated by the columns of A, there exists a hyperplane y that separates b from the cone.

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$$0 > y^T b = y^T Ax \geq 0$$

Hence, at most one of the statements can hold.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

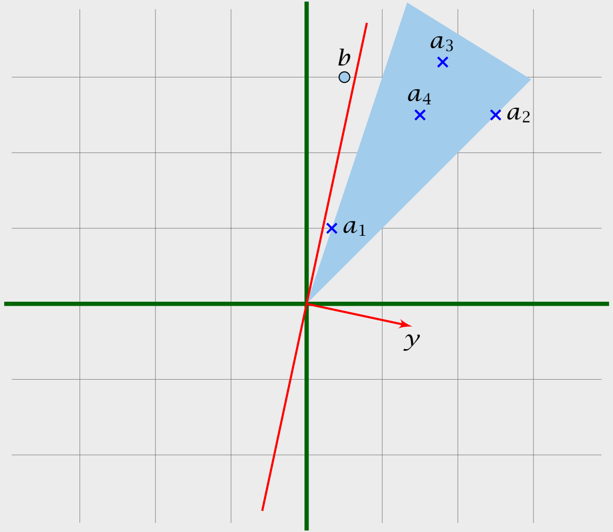
We want to show that there is y with $A^T y \geq 0$, $b^T y < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

Farkas Lemma



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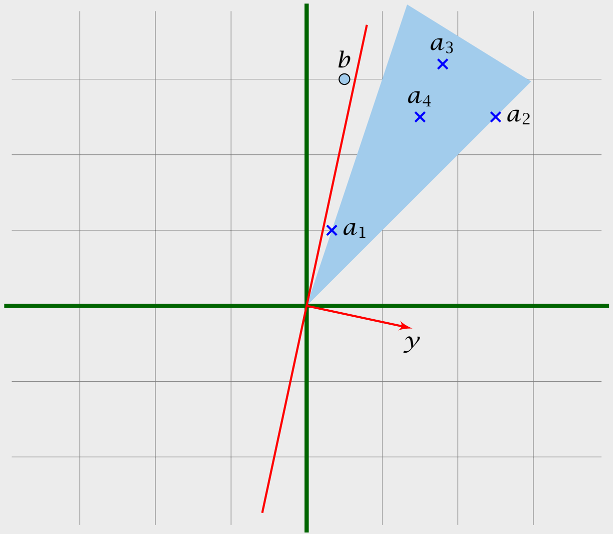
We want to show that there is y with $A^Ty \ge 0$, $b^Ty < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^Tb < \alpha$ and $y^Ts \ge \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \le 0 \Rightarrow y^Tb < 0$

$y^TAx \ge \alpha$ for all $x \ge 0$. Hence, $y^TA \ge 0$ as we can choose x arbitrarily large.

Farkas Lemma



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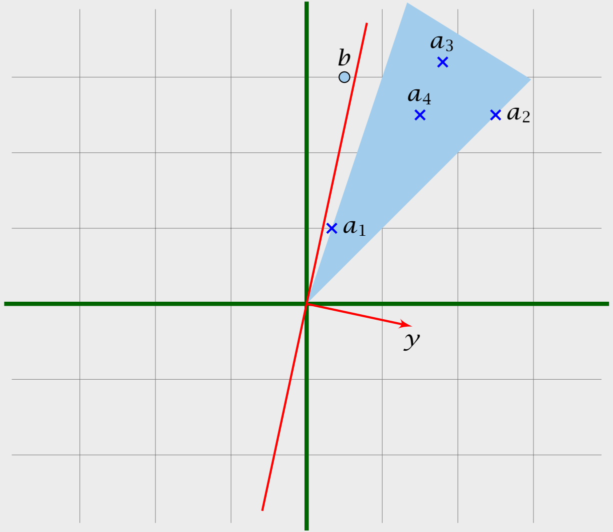

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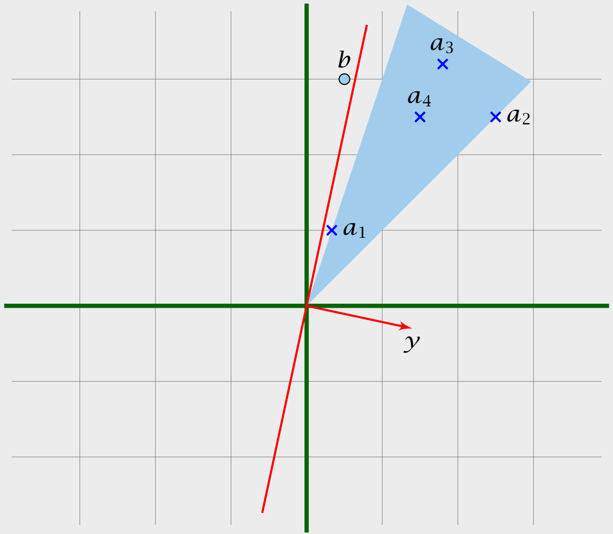

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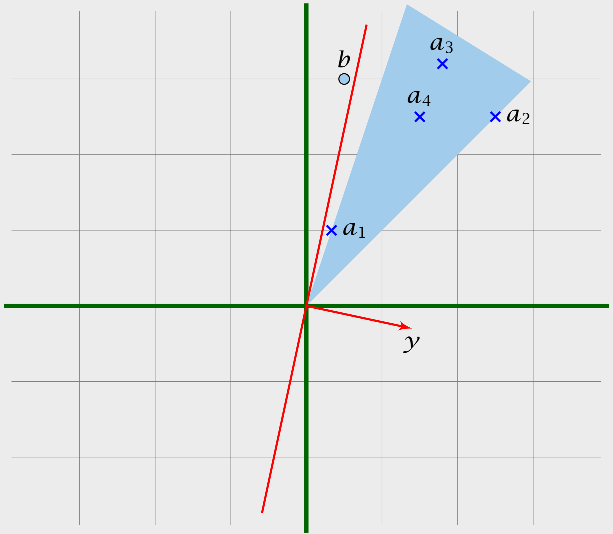

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Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

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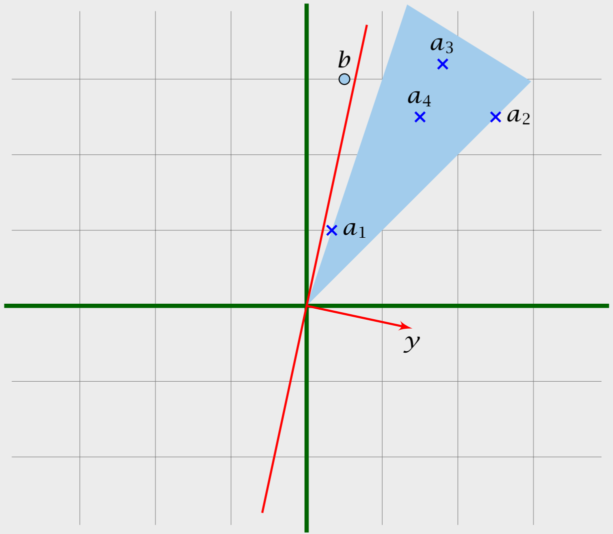

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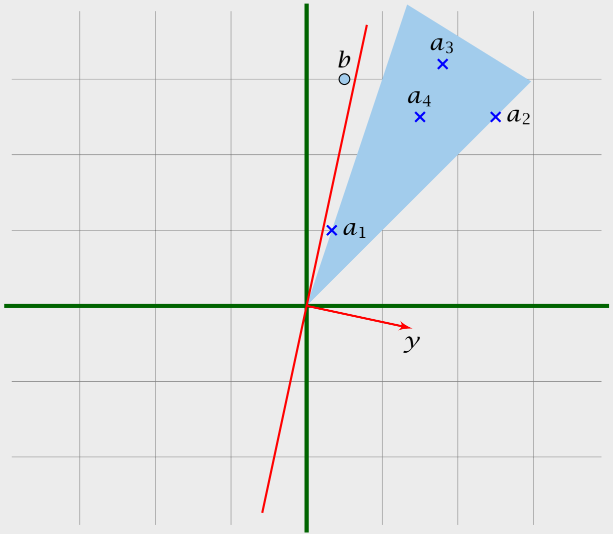

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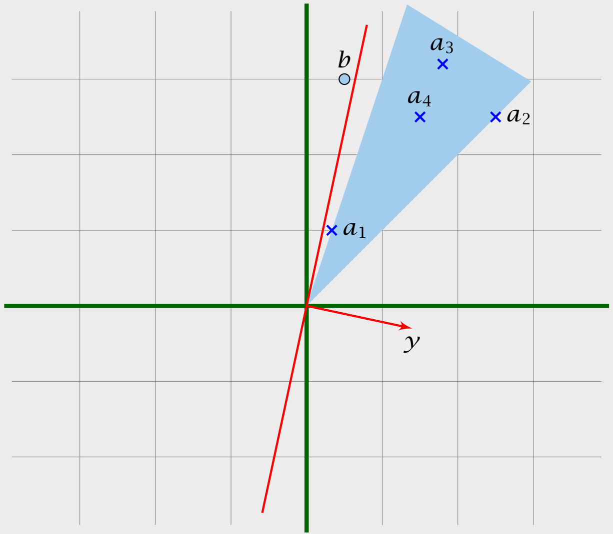

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Lemma 10 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ *with* $Ax \le b$, $x \ge 0$
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1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \ge 0$, $s \ge 0$
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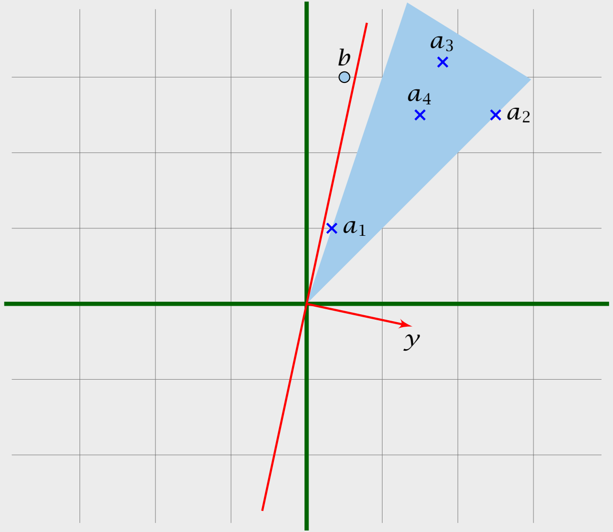

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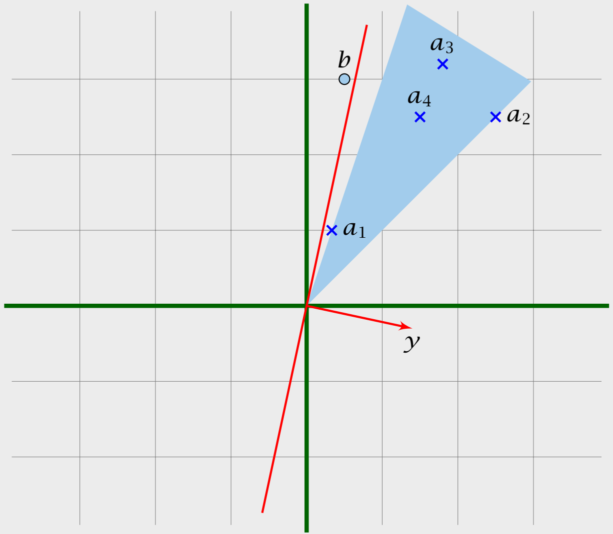

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P: $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$

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Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

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From the definition of α we know that the first system is infeasible; hence the second must be feasible.

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Definition 12 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

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- Is LP in co-NP? yes!
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Complementary Slackness

Lemma 13

Assume a linear program $P = \max\{c^T x \mid Ax \le b; x \ge 0\}$ *has solution* x^* *and its dual* $D = \min\{b^T y \mid A^T y \ge c; y \ge 0\}$ *has solution* y^**.*

1. *If* $x_j^* > 0$ *then the* j*-th constraint in* D *is tight.*
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Analogous to the proof of weak duality we obtain

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From the constraint of the dual it follows that $y^T A \ge c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

Complementary Slackness

Lemma 13

Assume a linear program $P = \max\{c^T x \mid Ax \le b; x \ge 0\}$ has solution x^* and its dual $D = \min\{b^T y \mid A^T y \ge c; y \ge 0\}$ has solution y^*.

1. *If $x_j^* > 0$ then the j-th constraint in D is tight.*
2. *If the j-th constraint in D is not tight than $x_j^* = 0$.*
3. *If $y_i^* > 0$ then the i-th constraint in P is tight.*
4. *If the i-th constraint in P is not tight than $y_i^* = 0$.*

If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$
\begin{array}{llcll}
\text{max} & 13a & + & 23b & \\
\text{s.t.} & 5a & + & 15b & \le 480 \\
 & 4a & + & 4b & \le 160 \\
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 & & & a, b & \ge 0
\end{array}
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- Entrepeneur: buy resources from brewer at minimum cost
  C, H, M: unit price for corn, hops and malt.

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\text{min} & 480C & + & 160H & + & 1190M & \\
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Note that brewer won't sell (at least not all) if e.g. $5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \le y^{*T} A x^* \le b^T y^*$$

Because of strong duality we then get

$$c^T x^* = y^{*T} A x^* = b^T y^*$$

This gives e.g.

$$\sum_j (y^T A - c^T)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^T A \ge c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

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- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ε_C, ε_H, and ε_M, respectively.

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If ϵ is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

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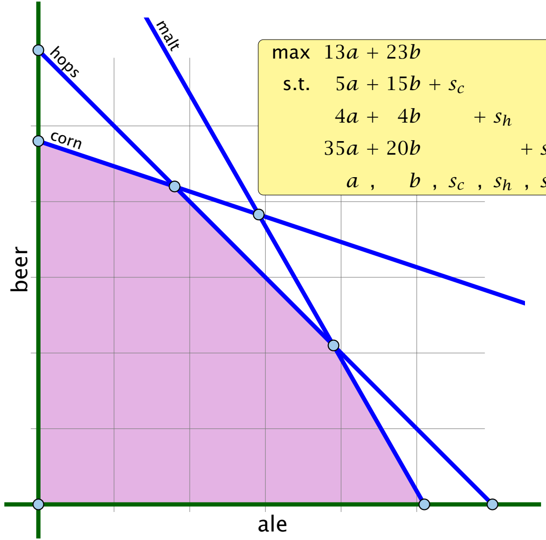

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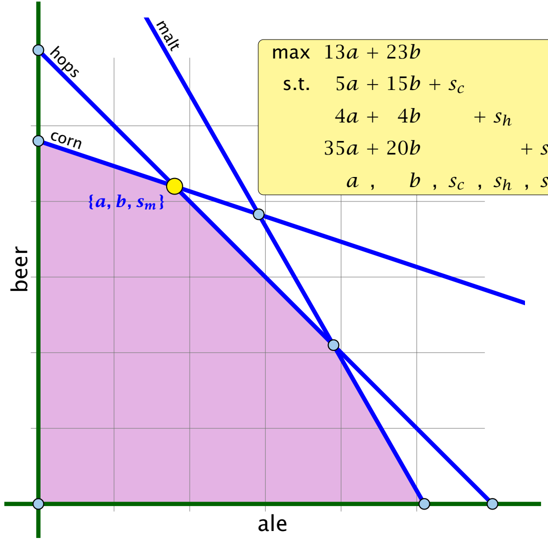

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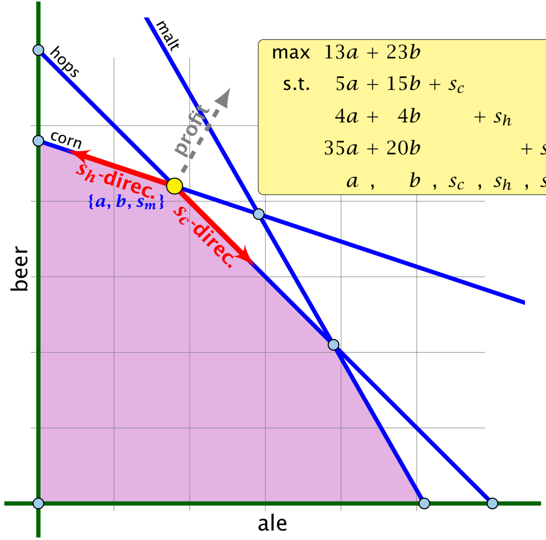

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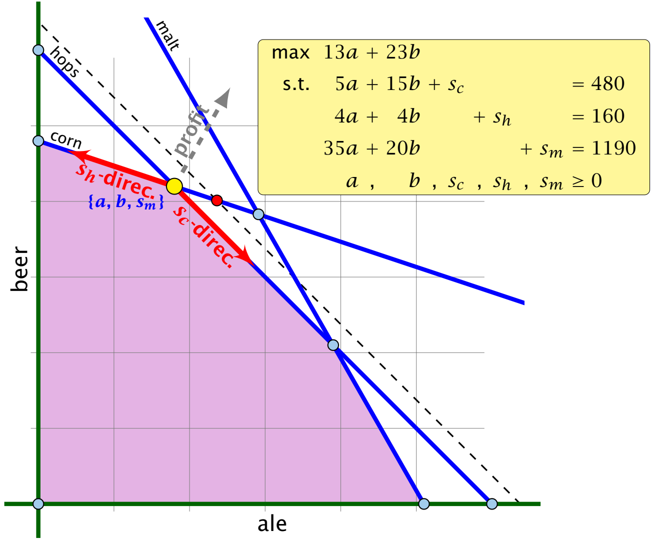

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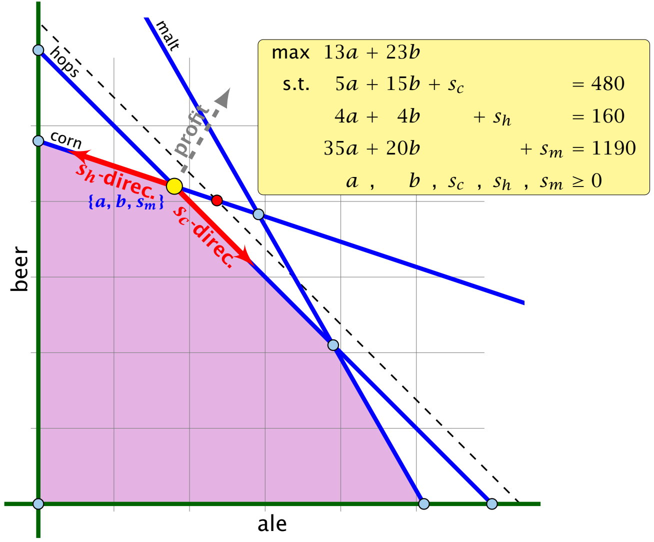

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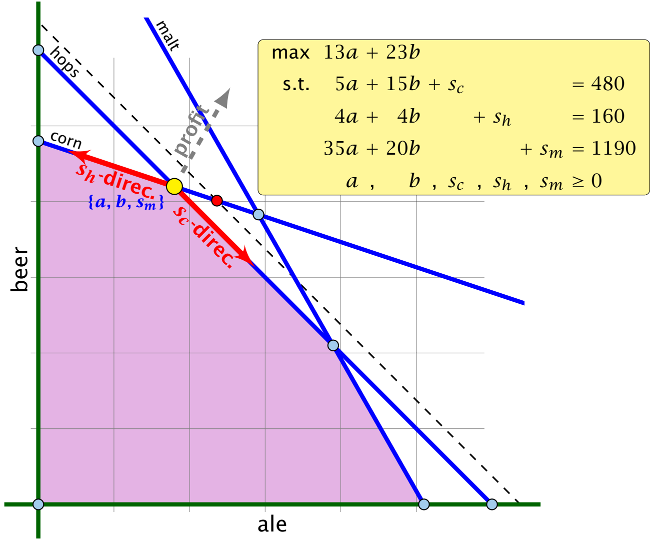

The change in profit when increasing hops by one unit is

$= c_B^T A_B^{-1} e_h$.

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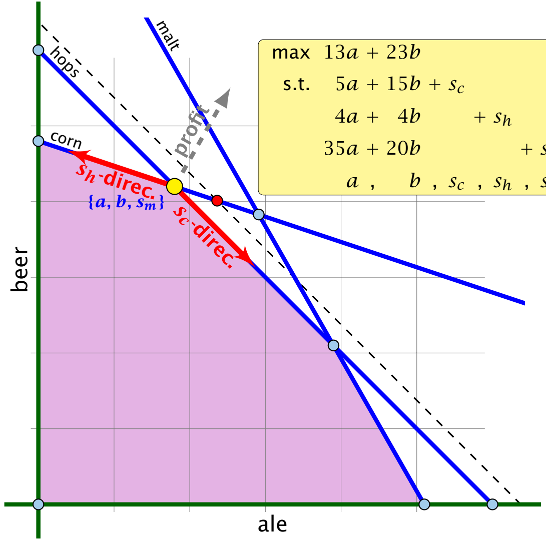

max $13a + 23b$

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Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

Example

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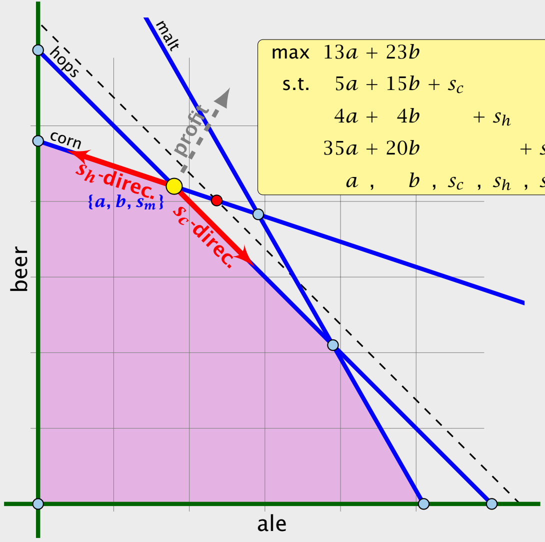

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Flows

Definition 14

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)

$$0 \le f_{xy} \le c_{xy} \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

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The value of an (s,t)-flow f is defined as

$$\operatorname{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \ .$$

Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

Flows

Definition 14

An (s,t)-flow in a (complete) directed graph $G = (V, V\times V, c)$ is a function $f : V\times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)

$$0 \le f_{xy} \le c_{xy} \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$

   (flow conservation constraints)

Flows

Definition 15

The value of an (s,t)-flow f is defined as

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LP-Formulation of Maxflow

max		$\sum_z f_{sz} - \sum_z f_{zs}$			
s.t.	$\forall (z,w) \in V \times V$	f_{zw}	$\le$	c_{zw}	ℓ_{zw}
	$\forall w \ne s,t$	$\sum_z f_{zw} - \sum_z f_{wz}$	$=$	0	p_w
		f_{zw}	$\ge$	0	

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	$\forall w \neq s,t$	$\sum_z f_{zw} - \sum_z f_{wz}$	$=$	0	p_w
		f_{zw}	$\geq$	0	

min		$\sum_{(xy)} c_{xy}\ell_{xy}$		
s.t.	$f_{xy}\ (x,y \neq s,t):$	$1\ell_{xy} - 1p_x + 1p_y$	$\geq$	0
	$f_{sy}\ (y \neq s,t):$	$1\ell_{sy} \quad + 1p_y$	$\geq$	1
	$f_{xs}\ (x \neq s,t):$	$1\ell_{xs} - 1p_x$	$\geq$	-1
	$f_{ty}\ (y \neq s,t):$	$1\ell_{ty} \quad + 1p_y$	$\geq$	0
	$f_{xt}\ (x \neq s,t):$	$1\ell_{xt} - 1p_x$	$\geq$	0
	$f_{st}:$	$1\ell_{st}$	$\geq$	1
	$f_{ts}:$	$1\ell_{ts}$	$\geq$	-1
		ℓ_{xy}	$\geq$	0

Flows

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LP-Formulation of Maxflow

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\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - \quad 1 + 1p_y & \geq & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + \quad 1 & \geq & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - \quad 0 + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + \quad 0 & \geq & 0 \\
 & f_{st}: & 1\ell_{st} - \quad 1 + \quad 0 & \geq & 0 \\
 & f_{ts}: & 1\ell_{ts} - \quad 0 + \quad 1 & \geq & 0 \\
 & & \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \leq & c_{zw} & \ell_{zw} \\
 & \forall w \neq s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
 & & f_{zw} & \geq & 0 &
\end{array}
$$

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
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 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x & \geq & -1 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} \quad + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x & \geq & 0 \\
 & f_{st}: & 1\ell_{st} & \geq & 1 \\
 & f_{ts}: & 1\ell_{ts} & \geq & -1 \\
 & & \ell_{xy} & \geq & 0
\end{array}
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LP-Formulation of Maxflow

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& f_{sy}\ (y \neq s,t): & 1\ell_{sy} - p_s + 1p_y & \geq & 0 \\
& f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + p_s & \geq & 0 \\
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& f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \geq & 0 \\
& f_{st}: & 1\ell_{st} - p_s + p_t & \geq & 0 \\
& f_{ts}: & 1\ell_{ts} - p_t + p_s & \geq & 0 \\
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with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

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\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& f_{sy}\ (y \neq s,t): & 1\ell_{sy} - 1 + 1p_y & \geq & 0 \\
& f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + 1 & \geq & 0 \\
& f_{ty}\ (y \neq s,t): & 1\ell_{ty} - 0 + 1p_y & \geq & 0 \\
& f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + 0 & \geq & 0 \\
& f_{st}: & 1\ell_{st} - 1 + 0 & \geq & 0 \\
& f_{ts}: & 1\ell_{ts} - 0 + 1 & \geq & 0 \\
& & \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & & \ell_{xy} & \geq & 0 \\
 & & p_s & = & 1 \\
 & & p_t & = & 0
\end{array}
$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

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 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \geq & 0 \\
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with $p_t = 0$ and $p_s = 1$.

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.

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