Brewery Problem

Brewery brews ale and beer.

- Production limited by supply of corn, hops and barley malt
- Recipes for ale and beer require different amounts of resources

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
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Linear Program

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LP in standard form:

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Standard Form LPs

Original LP

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Standard Form

Add a slack variable to every constraint.

$$\begin{array}{lrcrcccccl}
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 & 4a & + & 4b & & & + & s_h & & = 160 \\
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Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

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standard maximization form

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

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$$\begin{array}{lrcrcccccl} \max & 13a & + & 23b & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & = 480 \\ & 4a & + & 4b & & & + & s_h & & = 160 \\ & 35a & + & 20b & & & & & + \; s_m & = 1190 \\ & a & , & b & , & s_c & , & s_h & , \; s_m & \geq 0 \end{array}$$

Standard Form LPs

There are different standard forms:

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Observations:

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Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP?
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Input size:

- n number of variables, m constraints, L number of bits to encode the input

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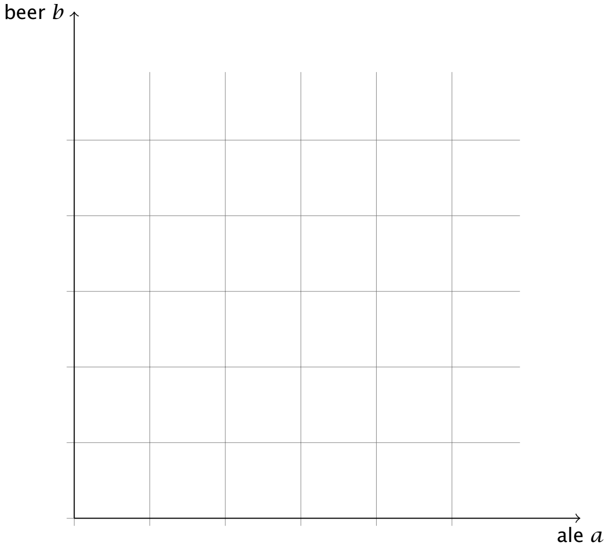

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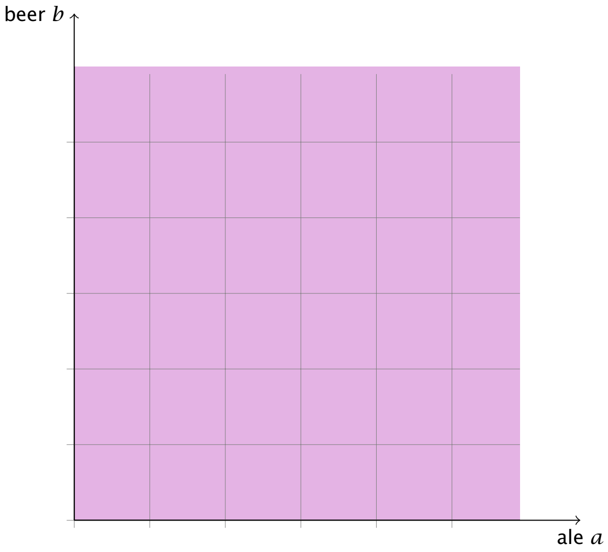

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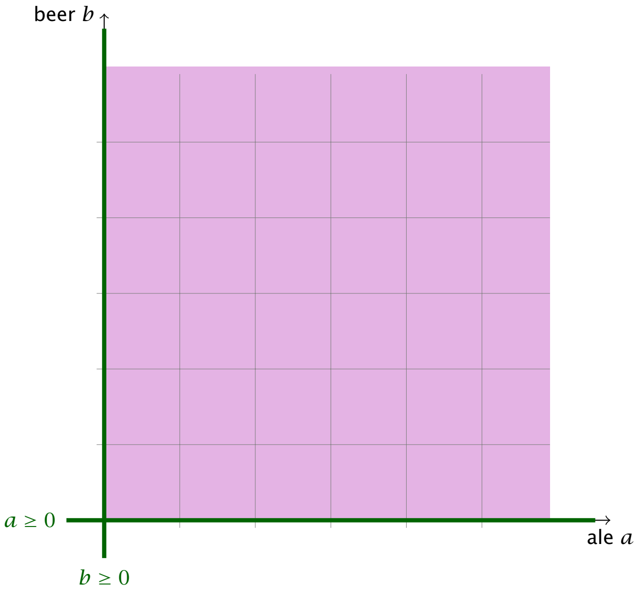

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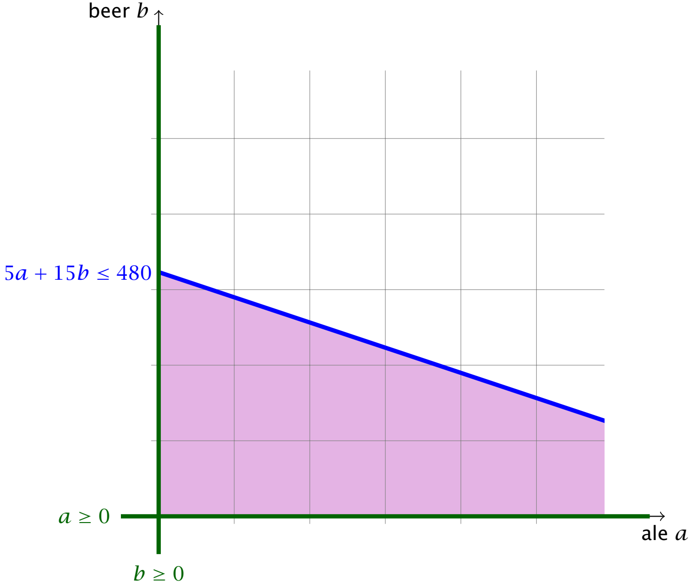

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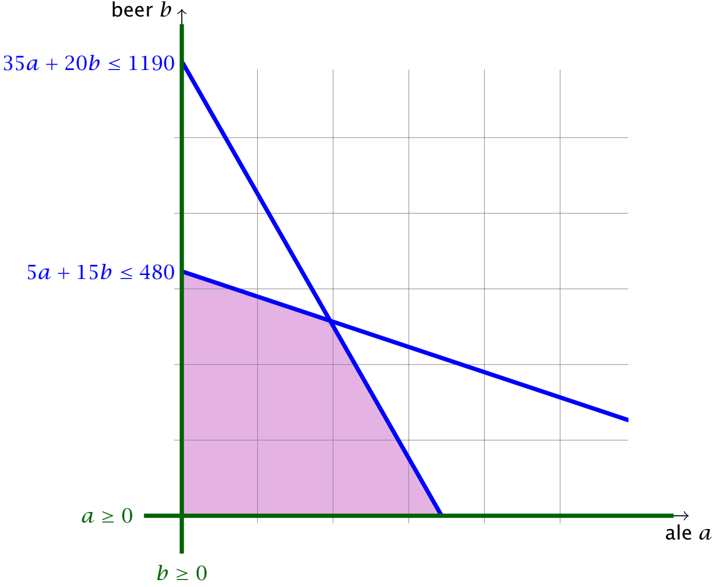

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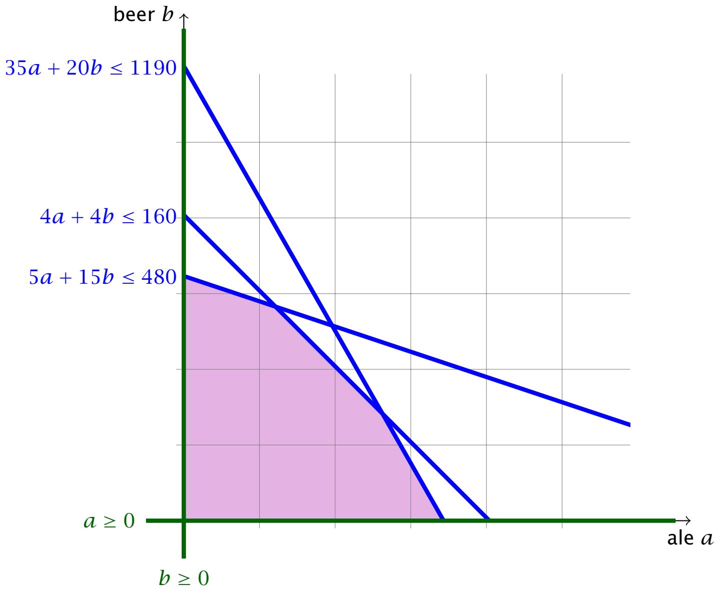

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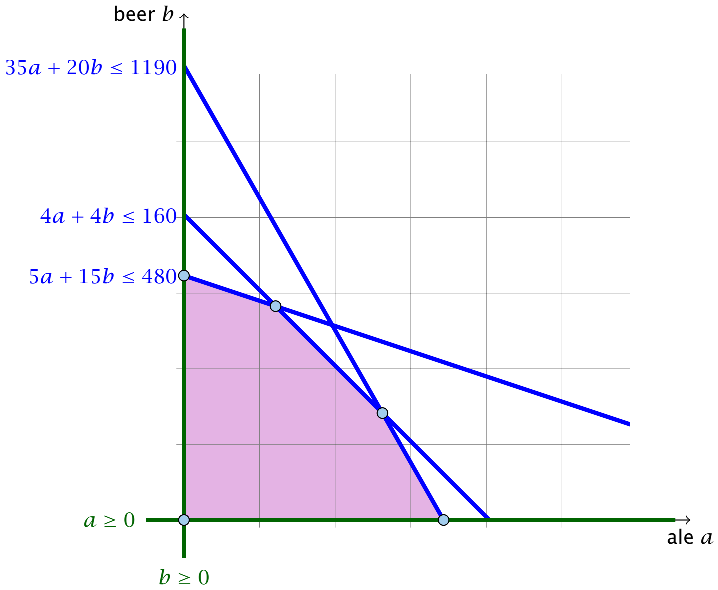

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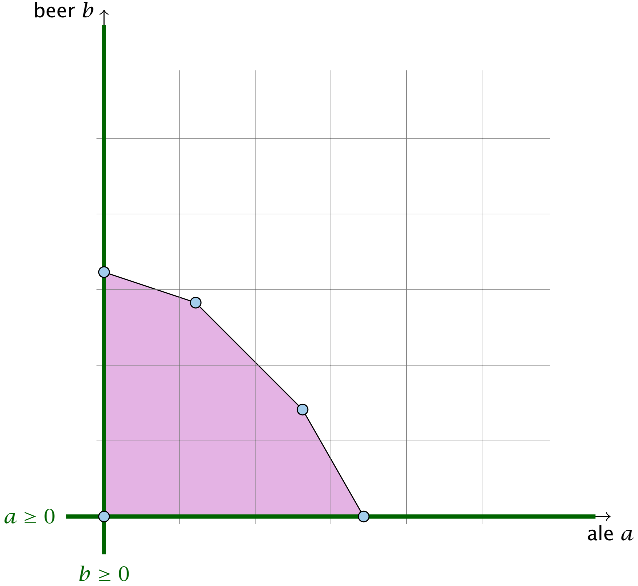

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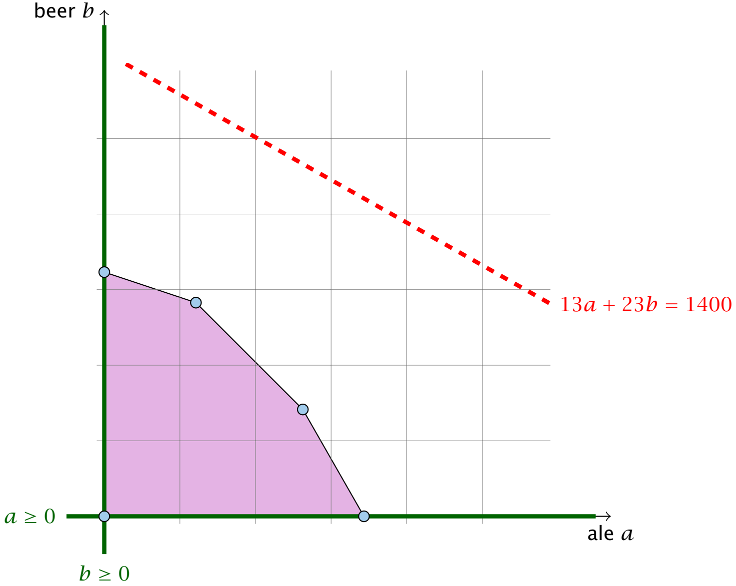

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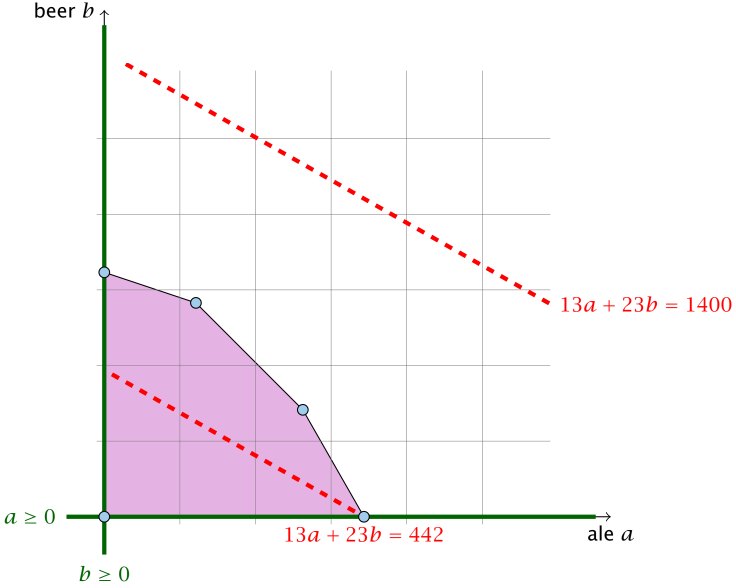

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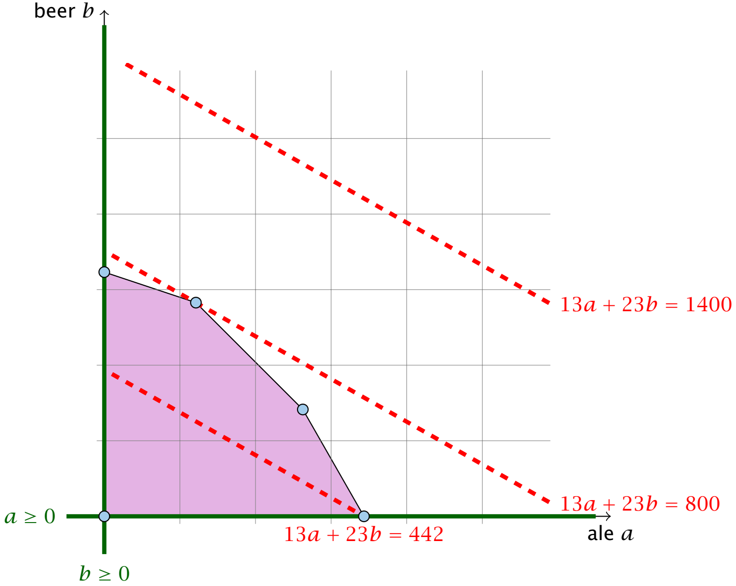

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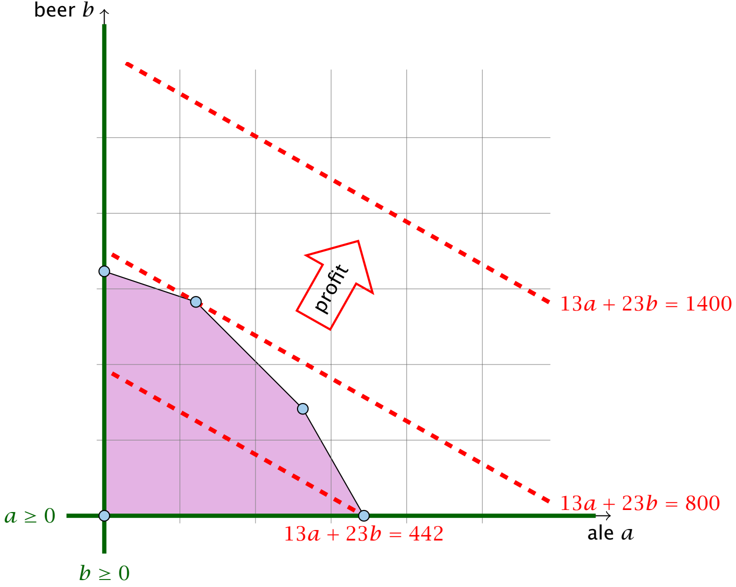

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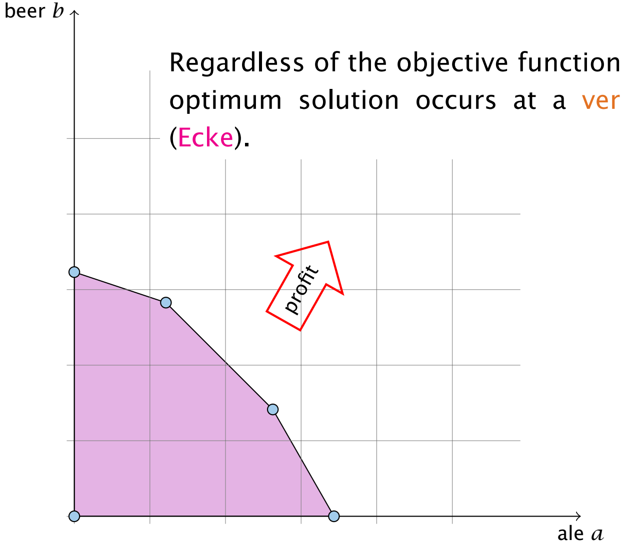

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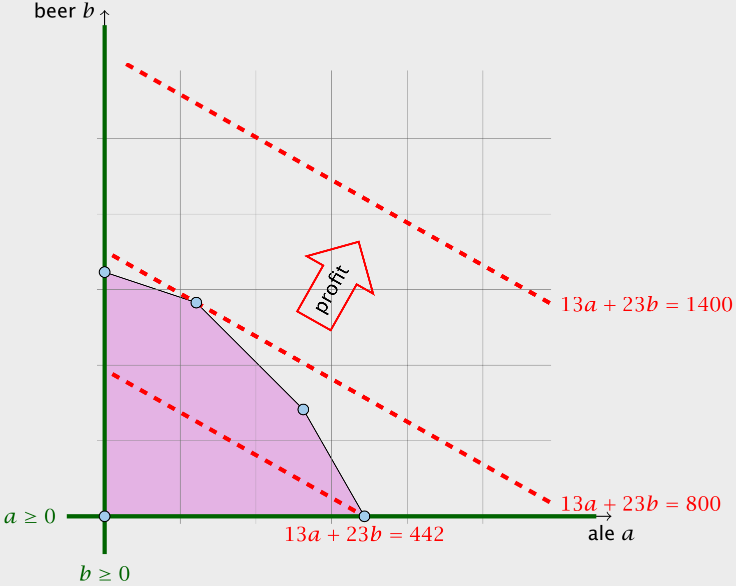

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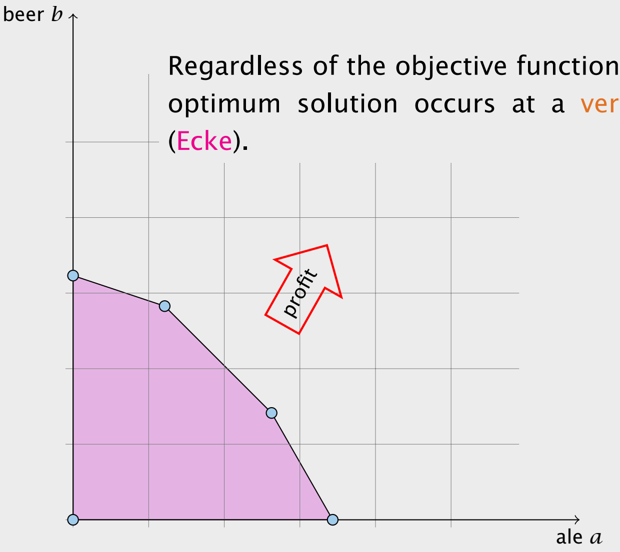

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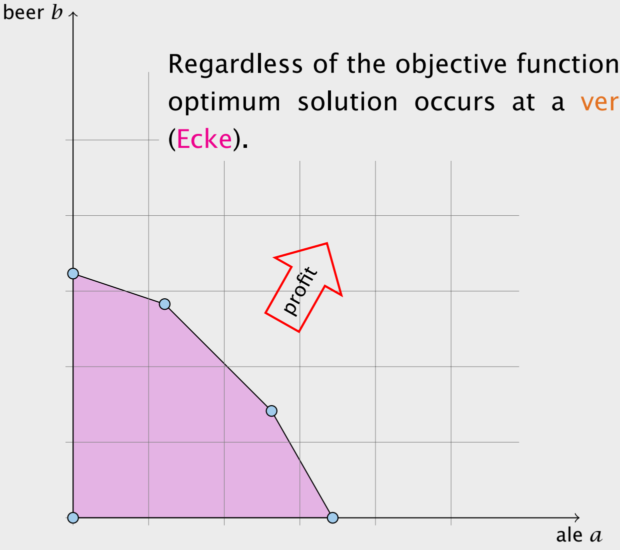

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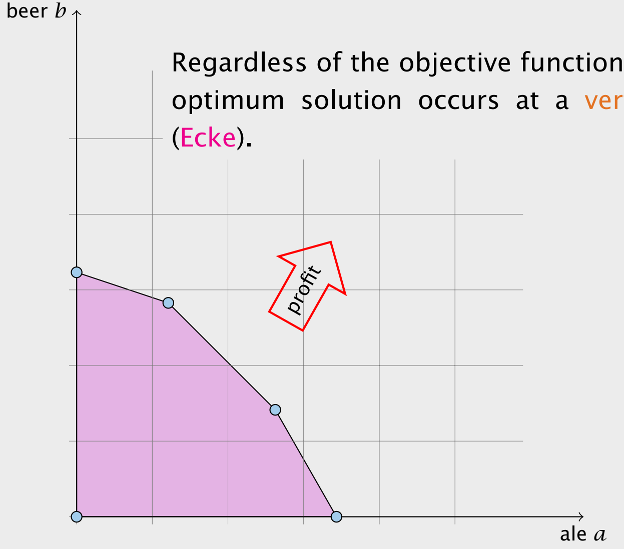

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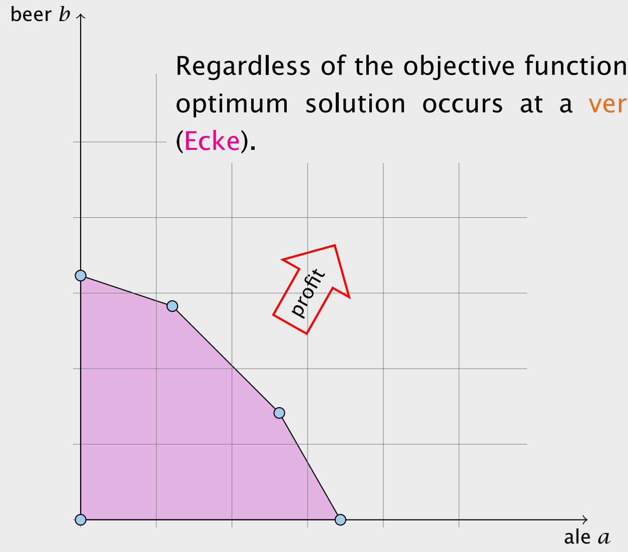

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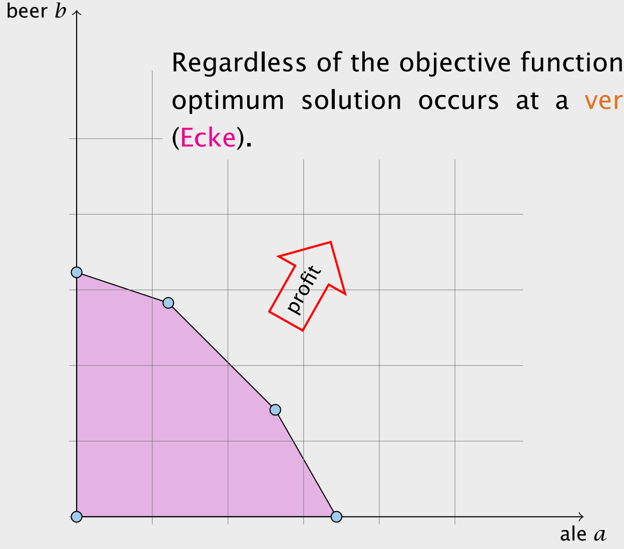

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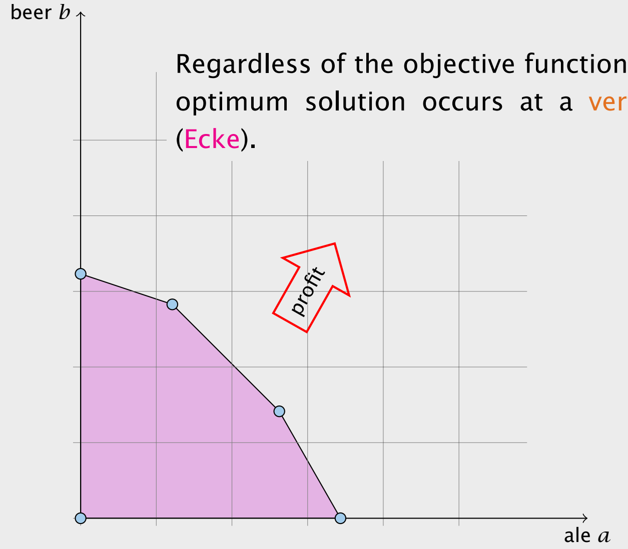

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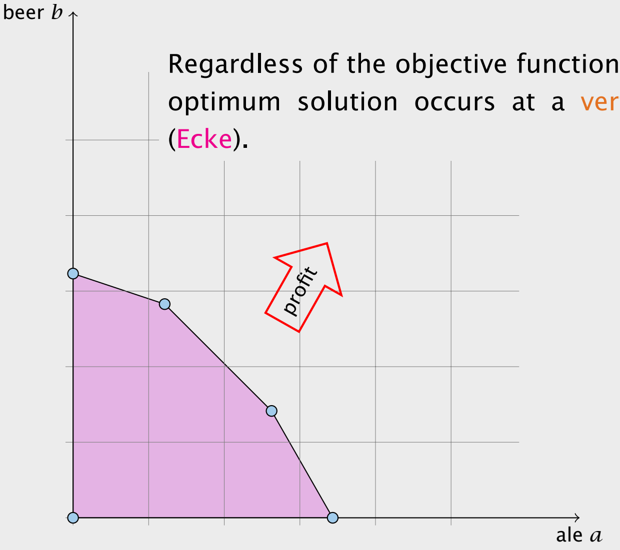

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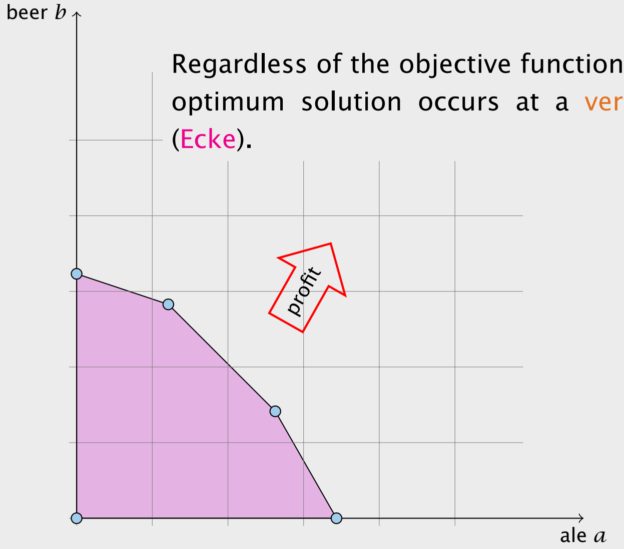

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Given vectors/points $x_1, \dots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- linear combination if $\lambda_i \in \mathbb{R}$.
- affine combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- convex combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
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Definition 3

A set $X \subseteq \mathbb{R}^n$ is called

- a linear subspace if it is closed under linear combinations.
- an affine subspace if it is closed under affine combinations.
- convex if it is closed under convex combinations.
- a convex cone if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

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Definition 4

Given a set $X \subseteq \mathbb{R}^n$.

- $\mathrm{span}(X)$ is the set of all linear combinations of X (linear hull, span)
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$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

Lemma 6

If $P \subseteq \mathbb{R}^n$, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ convex then also

$$Q = \{x \in P \mid f(x) \le t\}$$

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Definition 7
The dimension $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

Definition 8
The dimension $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension of its affine hull $\mathrm{aff}(X)$.

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A set $H \subseteq \mathbb{R}^n$ is a hyperplane if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

Definition 10

A set $H' \subseteq \mathbb{R}^n$ is a (closed) halfspace if $H = \{x \mid a^T x \leq b\}$, for $a \neq 0$.

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The dimension $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension of its affine hull $\text{aff}(X)$.

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A set $H \subseteq \mathbb{R}^n$ is a hyperplane if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

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A polyhedron is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of finitely many half-spaces $\{H(a_1, b_1), \ldots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \le b_i\} \quad .$$

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A polyhedron P is bounded if there exists B s.t. $\|x\|_2 \le B$ for all $x \in P$.

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is a supporting hyperplane of P if $\max\{a^T x \mid x \in P\} = b$.

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Let $P \subseteq \mathbb{R}^n$. F is a face of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H.

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- a face v is a vertex of P if $\{v\}$ is a face of P.
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Equivalent definition for vertex:

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Given polyhedron P. A point $x \in P$ is a vertex if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

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If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

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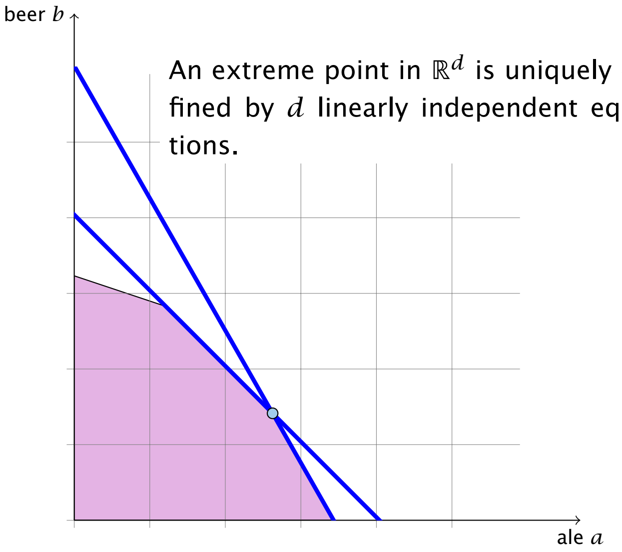
Algebraic View



An extreme point in $\mathbb{R}^d$ is uniquely defined by d linearly independent equations.

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Notation

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

Theorem 22

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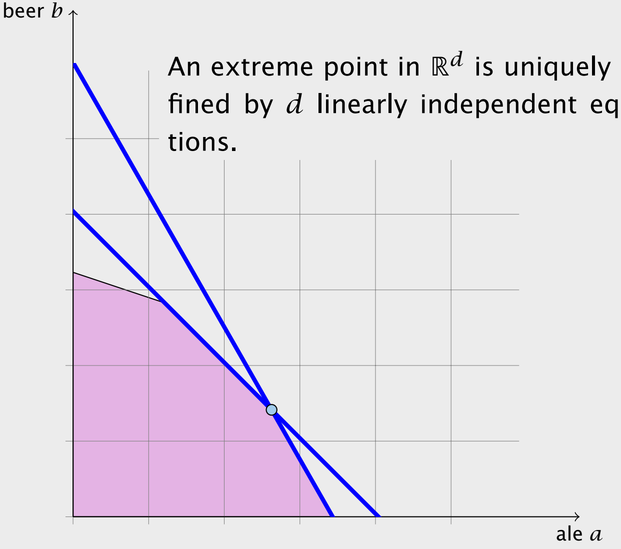

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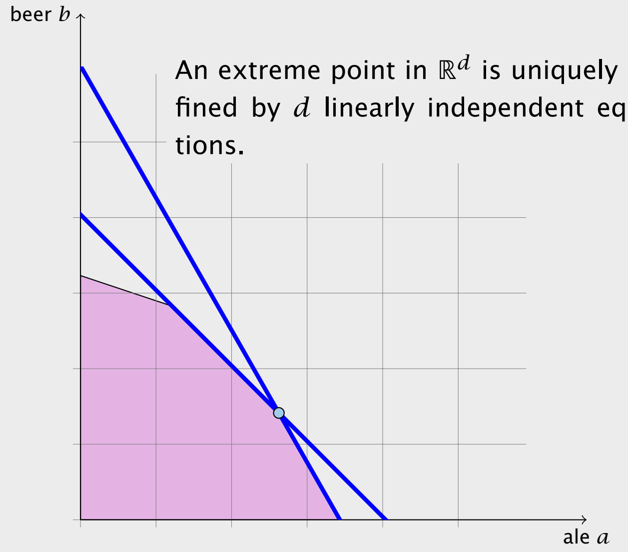
Algebraic View



An extreme point in $\mathbb{R}^d$ is uniquely defined by d linearly independent equations.

Theorem 22

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Proof ($\Leftarrow$)

- assume x is not extreme point
- there exists direction d s.t. $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- define $B' = \{j \mid d_j \neq 0\}$
- $A_{B'}$ has linearly dependent columns as $Ad = 0$
- $d_j = 0$ for all j with $x_j = 0$ as $x \pm d \geq 0$
- Hence, $B' \subseteq B$, $A_{B'}$ is sub-matrix of A_B

Notation

Suppose $B \subseteq \{1 \ldots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

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Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P.

- define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
- then $c^T x = 0$ and $c^T y \leq 0$ for $y \in P$
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- $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
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Let $P = \{x \mid Ax = b, x \geq 0\}$. *For* $x \in P$, *define* $B = \{j \mid x_j > 0\}$. *Then* x *is extreme point* ***iff*** A_B *has linearly independent columns.*

Proof (⇒)

- assume A_B has linearly dependent columns
- there exists $d \neq 0$ such that $A_B d = 0$
- extend d to $\mathbb{R}^n$ by adding 0-components
- now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$
- for sufficiently small λ we have $x \pm \lambda d \in P$
- hence, x is not extreme point

Observation

For an LP we can assume wlog. that the matrix A has full row-rank. This means $\operatorname{rank}(A) = m$.

Theorem 23

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P.

- define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
- then $c^Tx = 0$ and $c^Ty \leq 0$ for $y \in P$
- assume $c^Ty = 0$; then $y_j = 0$ for all $j \notin B$
- $b = Ay = A_By_B = Ax = A_Bx_B$ gives that $A_B(x_B - y_B) = 0$;
- this means that $x_B = y_B$ since A_B has linearly independent columns
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- assume that $\text{rank}(A) < m$
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$$A_1 = \sum_{i=2}^{m} \lambda_i \cdot A_i \text{, for suitable } \lambda_i$$

C1 if now $b_1 = \sum_{i=2}^{m} \lambda_i \cdot b_i$ then for all x with $A_i x = b_i$ we also have $A_1 x = b_1$; hence the first constraint is superfluous

C2 if $b_1 \neq \sum_{i=2}^{m} \lambda_i \cdot b_i$ then the LP is infeasible, since for all x that fulfill constraints $A_2, \ldots, A_m$ we have

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Theorem 24

Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- A_B is non-singular
- $x_B = A_B^{-1} b \geq 0$
- $x_N = 0$

where $N = \{1, \dots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\mathrm{rank}(A) = m$.

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Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and $\mathrm{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a basic feasible solution (gültige Basislösung) if in addition $x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \dots, n\}$ with $\mathrm{rank}(A_B) = m$ and $|B| = m$.

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Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- A_B is non-singular
- $x_B = A_B^{-1} b \geq 0$
- $x_N = 0$

where $N = \{1, \dots, n\} \setminus B$.

Proof
Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\operatorname{rank}(A) = m$.

Basic Feasible Solutions

A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i's are zero. The corresponding non-negativity constraint is fulfilled with equality.

Fact:
In a BFS at least n constraints are fulfilled with equality.

Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and $\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a basic **feasible** solution (gültige Basislösung) if in addition $x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \dots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$.

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Basic Feasible Solutions

Definition 25
For a general LP ($\max\{c^Tx \mid Ax \le b\}$) with n variables a point x is a basic feasible solution if x is feasible and there exist n (linearly independent) constraints that are tight.

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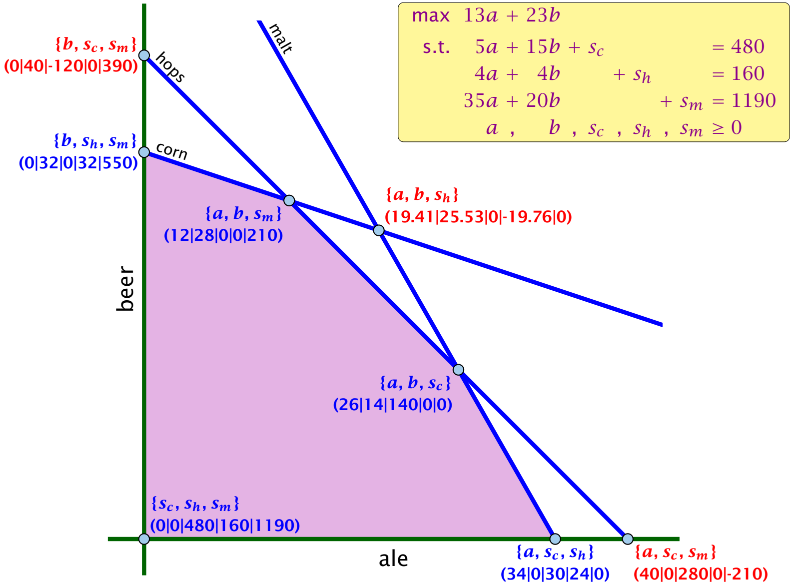
Algebraic View



$$\begin{array}{llllllll}
\max & 13a + 23b & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
 & 4a + 4b & & + s_h & & = & 160 \\
 & 35a + 20b & & & + s_m & = & 1190 \\
 & a \;,\; b & ,\; s_c & ,\; s_h & ,\; s_m & \geq & 0
\end{array}$$

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Fundamental Questions

Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^Tx \geq \alpha$?

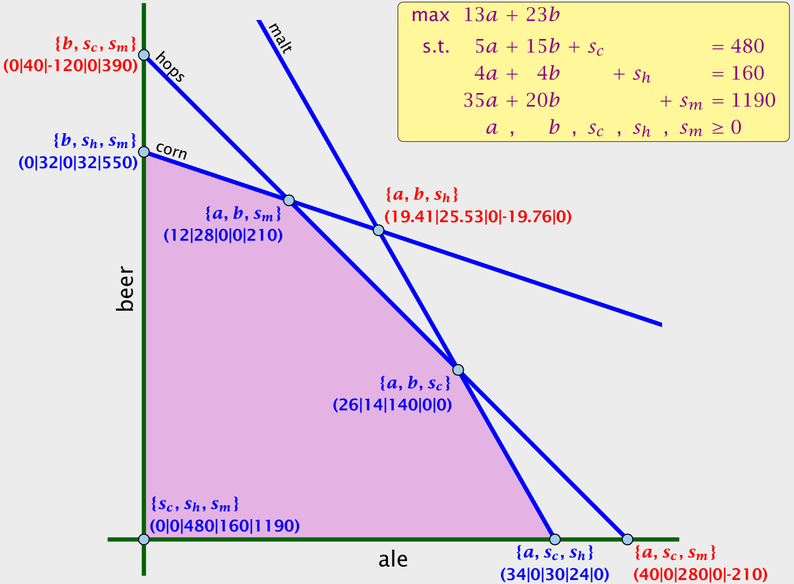
Questions:

- Is LP in NP? yes!
- Is LP in co-NP?
- Is LP in P?

Proof:

- Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

Algebraic View



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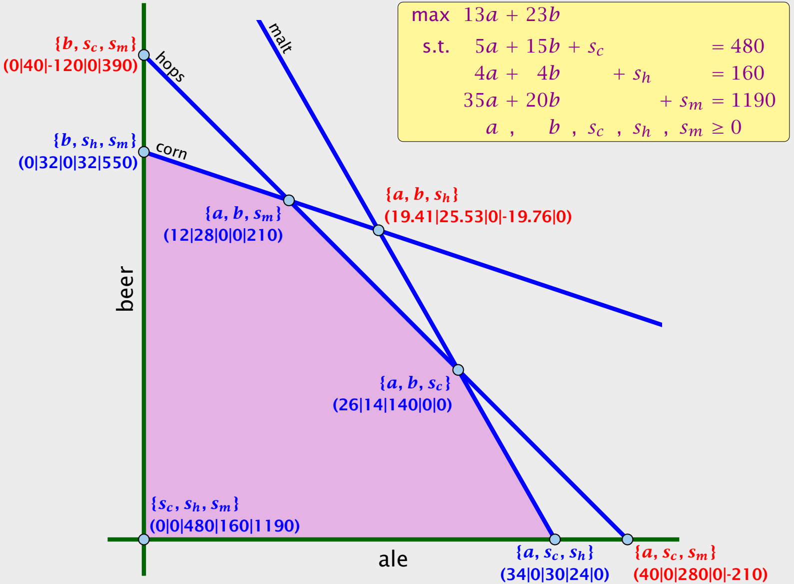

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Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \mathrm{poly}(n, m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

What happens if LP is unbounded?

Fundamental Questions

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