4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

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Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

4 Simplex Algorithm

$$
\begin{array}{lrcrcrcrcrcl}
\text{max} & 13a & + & 23b \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
 & 4a & + & 4b & & & + & s_h & & & = & 160 \\
 & 35a & + & 20b & & & & & + & s_m & = & 1190 \\
 & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
$$

$$
\begin{array}{lrcrcrcrcrcl}
\text{max } Z \\
 & 13a & + & 23b & & & & & & & - Z = & 0 \\
 & 5a & + & 15b & + & s_c & & & & & = & 480 \\
 & 4a & + & 4b & & & + & s_h & & & = & 160 \\
 & 35a & + & 20b & & & & & + & s_m & = & 1190 \\
 & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$A = B = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

4 Simplex Algorithm

$$
\begin{array}{lrcrcrcrcrl}
\text{max} & 13a & + & 23b & & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = 480 \\
& 4a & + & 4b & & & + & s_h & & & = 160 \\
& 35a & + & 20b & & & & & + & s_m & = 1190 \\
& a & , & b & , & s_c & , & s_h & , & s_m & \geq 0
\end{array}
$$

$$
\begin{array}{lrcrcrcrcrcrl}
\text{max } Z & & & & & & & & & & & & \\
& 13a & + & 23b & & & & & & & - & Z & = 0 \\
& 5a & + & 15b & + & s_c & & & & & & & = 480 \\
& 4a & + & 4b & & & + & s_h & & & & & = 160 \\
& 35a & + & 20b & & & & & + & s_m & & & = 1190 \\
& a & , & b & , & s_c & , & s_h & , & s_m & & & \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$A = B = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Pivoting Step

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcccccl}
13a & + & 23b & & & & & -\ Z & = 0 \\
5a & + & 15b & + s_c & & & & & = 480 \\
4a & + & 4b & & + s_h & & & & = 160 \\
35a & + & 20b & & & + s_m & & & = 1190 \\
a & , & b & , s_c & , s_h & , s_m & & & \geq 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

choose variable to bring into the basis

chosen variable should have positive coefficient in objective function

apply test to find out by how much the variable can be increased

pivot on row found by min ratio test

the existing basis variable in this row leaves the basis

Pivoting Step

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15b + s_c \qquad\qquad\quad = 480 \\
\quad 4a + 4b \quad + s_h \qquad\quad = 160 \\
\quad 35a + 20b \qquad + s_m \quad = 1190 \\
\quad a, b, s_c, s_h, s_m \qquad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

Pivoting Step

$$
\begin{aligned}
&\max Z\\
&13a + 23b - Z = 0\\
&5a + 15b + s_c = 480\\
&4a + 4b + s_h = 160\\
&35a + 20b + s_m = 1190\\
&a\,,\ b\,,\ s_c\,,\ s_h\,,\ s_m \geq 0
\end{aligned}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
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- choose variable to bring into the basis
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Pivoting Step

$$\begin{array}{l} \max Z \\ \begin{array}{rcrcccccl} 13a & + & 23b & & & & & - Z & = 0 \\ 5a & + & 15b & + s_c & & & & & = 480 \\ 4a & + & 4b & & + s_h & & & & = 160 \\ 35a & + & 20b & & & + s_m & & & = 1190 \\ a & , & b & , s_c & , s_h & , s_m & & & \geq 0 \end{array} \end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
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- choose variable to bring into the basis
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Pivoting Step

$$
\begin{aligned}
&\max Z\\
&13a + 23b - Z = 0\\
&5a + 15b + s_c = 480\\
&4a + 4b + s_h = 160\\
&35a + 20b + s_m = 1190\\
&a, b, s_c, s_h, s_m \geq 0
\end{aligned}
$$

$$
\begin{aligned}
&\text{basis} = \{s_c, s_h, s_m\}\\
&a = b = 0\\
&Z = 0\\
&s_c = 480\\
&s_h = 160\\
&s_m = 1190
\end{aligned}
$$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
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- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

Pivoting Step

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15b + s_c \qquad\qquad\qquad = 480 \\
\quad 4a + 4b \quad + s_h \qquad\qquad = 160 \\
\quad 35a + 20b \qquad\quad + s_m \qquad = 1190 \\
\quad a \;,\; b \;,\; s_c \;,\; s_h \;,\; s_m \qquad \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
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- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

max Z

$$
\begin{array}{rl}
13a + 23b \qquad\qquad\qquad\qquad - Z &= 0 \\
5a + 15b + s_c \qquad\qquad\qquad &= 480 \\
4a + 4b \quad + s_h \qquad\quad &= 160 \\
35a + 20b \qquad\quad + s_m &= 1190 \\
a \;,\; b \;,\; s_c \;,\; s_h \;,\; s_m &\geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

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$$\max Z$$

$$\begin{aligned}
13a + 23\boldsymbol{b} \qquad\qquad\qquad\qquad - Z &= 0 \\
5a + 15\boldsymbol{b} + s_c \qquad\qquad\qquad &= 480 \\
4a + 4\boldsymbol{b} \qquad + s_h \qquad\qquad &= 160 \\
35a + 20\boldsymbol{b} \qquad\qquad + s_m \quad &= 1190 \\
a\ ,\ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \quad &\geq 0
\end{aligned}$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- Choose variable with coefficient > 0 as entering variable.

$$
\begin{array}{l}
\max\ Z \\
\quad 13a + 23\boldsymbol{b} \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15\boldsymbol{b} + s_c \qquad\qquad\quad = 480 \\
\quad 4a + 4\boldsymbol{b} \qquad + s_h \qquad\quad = 160 \\
\quad 35a + 20\boldsymbol{b} \qquad\qquad + s_m \quad = 1190 \\
\quad a\ ,\ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0
\end{array}
$$

$$
\begin{aligned}
&\text{basis} = \{s_c, s_h, s_m\} \\
&a = b = 0 \\
&Z = 0 \\
&s_c = 480 \\
&s_h = 160 \\
&s_m = 1190
\end{aligned}
$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23\boldsymbol{b} \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15\boldsymbol{b} + s_c \qquad\qquad\qquad = 480 \\
\quad 4a + 4\boldsymbol{b} \qquad + s_h \qquad\qquad = 160 \\
\quad 35a + 20\boldsymbol{b} \qquad\qquad + s_m \qquad = 1190 \\
\quad a, \ \boldsymbol{b}, \ s_c, \ s_h, \ s_m \qquad\qquad \geq 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.

$$\begin{array}{lcl}
\text{max } Z & & \\
13a + 23b & -Z & = 0 \\
5a + 15\mathbf{b} + s_c & & = 480 \\
4a + 4b + s_h & & = 160 \\
35a + 20b + s_m & & = 1190 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & & \geq 0
\end{array}$$

$$\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.

$$
\begin{array}{l}
\max Z \\
\quad 13a + 23b \qquad\qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15\mathbf{b} + \mathbf{s_c} \qquad\qquad\qquad = 480 \\
\quad 4a + 4b \qquad + s_h \qquad\qquad = 160 \\
\quad 35a + 20b \qquad\qquad + s_m \quad = 1190 \\
\quad a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$\max Z$$

$$\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15b & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23\boldsymbol{b} & & & & & & & - & Z & = & 0 \\
5a & + & 15\boldsymbol{b} & + & s_c & & & & & & & = & 480 \\
4a & + & 4\boldsymbol{b} & & & + & s_h & & & & & = & 160 \\
35a & + & 20\boldsymbol{b} & & & & & + & s_m & & & = & 1190 \\
a & , & \boldsymbol{b} & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
13a + 23\boldsymbol{b} \qquad\qquad\qquad - Z = 0 \\
5a + 15\boldsymbol{b} + s_c \qquad\qquad = 480 \\
4a + 4\boldsymbol{b} \qquad + s_h \qquad = 160 \\
35a + 20\boldsymbol{b} \qquad\qquad + s_m = 1190 \\
a, \boldsymbol{b}, s_c, s_h, s_m \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\frac{16}{3}a - \frac{23}{15}s_c \qquad\qquad - Z = -736 \\
\frac{1}{3}a + b + \frac{1}{15}s_c \qquad\qquad = 32 \\
\frac{8}{3}a - \frac{4}{15}s_c + s_h \qquad = 32 \\
\frac{85}{3}a - \frac{4}{3}s_c \qquad + s_m = 550 \\
a, b, s_c, s_h, s_m \geq 0
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

max Z

$$
\begin{array}{rcl}
\frac{16}{3}a \quad - \frac{23}{15}s_c \quad - Z & = & -736 \\
\frac{1}{3}a + b + \frac{1}{15}s_c & = & 32 \\
\frac{8}{3}a \quad - \frac{4}{15}s_c + s_h & = & 32 \\
\frac{85}{3}a \quad - \frac{4}{3}s_c \quad + s_m & = & 550 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq & 0
\end{array}
$$

basis = $\{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}\boldsymbol{a} & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}\boldsymbol{a} & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}\boldsymbol{a} & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
\boldsymbol{a} & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{b, s_h, s_m\} \\
a = s_c = 0 \\
Z = 736 \\
\\
b = 32 \\
s_h = 32 \\
s_m = 550
\end{array}
$$

Choose variable a to bring into basis.

$$\max Z$$

$$\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}\boldsymbol{a} & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}\boldsymbol{a} & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}\boldsymbol{a} & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
\boldsymbol{a} & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

$$
\begin{array}{l}
\text{max } Z \\
\begin{array}{rcccccl}
\frac{16}{3}a & & -\frac{23}{15}s_c & & & -Z & = -736 \\
\frac{1}{3}a & + b & + \frac{1}{15}s_c & & & & = 32 \\
\frac{8}{3}\mathbf{a} & & -\frac{4}{15}s_c & + s_h & & & = 32 \\
\frac{85}{3}a & & -\frac{4}{3}s_c & & + s_m & & = 550 \\
a, & b, & s_c, & s_h, & s_m & & \geq 0
\end{array}
\end{array}
$$

basis = $\{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \text{max } Z \\ \frac{16}{3}a - \frac{23}{15}s_c - Z = -736 \\ \frac{1}{3}a + b + \frac{1}{15}s_c = 32 \\ \frac{8}{3}\boldsymbol{a} - \frac{4}{15}s_c + s_h = 32 \\ \frac{85}{3}a - \frac{4}{3}s_c + s_m = 550 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

basis = $\{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/8, {}^{3\cdot 550}/85\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \text{max } Z \\ - s_c - 2s_h - Z = -800 \\ b + \frac{1}{10}s_c - \frac{1}{8}s_h = 28 \\ a - \frac{1}{10}s_c + \frac{3}{8}s_h = 12 \\ \frac{3}{2}s_c - \frac{85}{8}s_h + s_m = 210 \\ a, b, s_c, s_h, s_m \geq 0 \end{array}$$

basis = $\{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$
$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

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Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \ge 0$, $s_h \ge 0$
- hence optimum solution value is at most 800
- the current solution has value 800

4 Simplex Algorithm

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- hence optimum solution value is at most 800
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Solution is optimal:

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Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{rcrcl} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Matrix View

Let our linear program be

$$\begin{array}{rcrcl} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

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The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

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Let our linear program be

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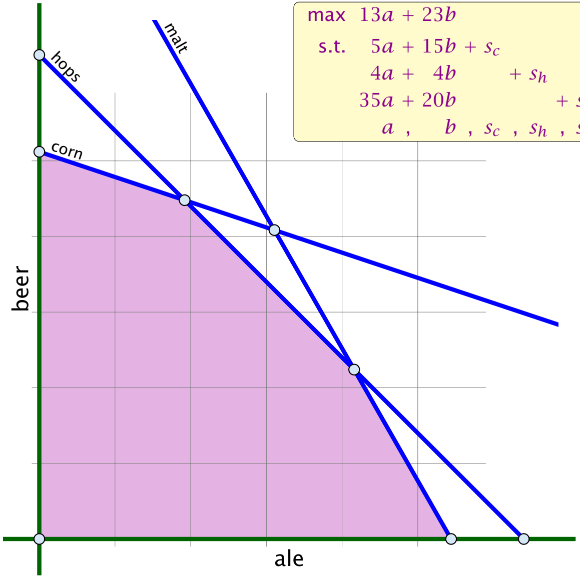

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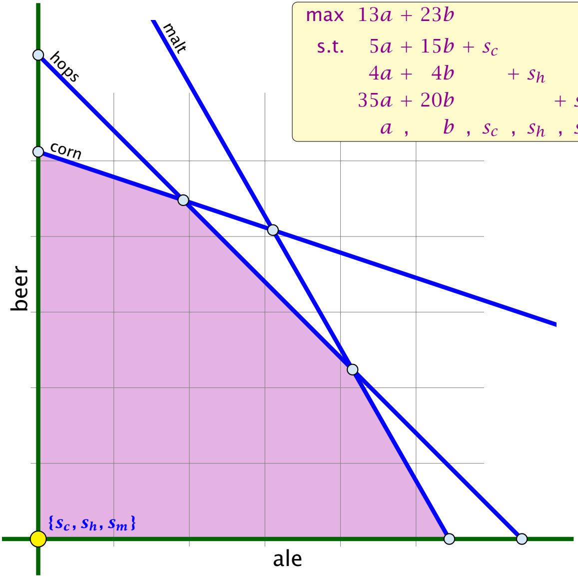

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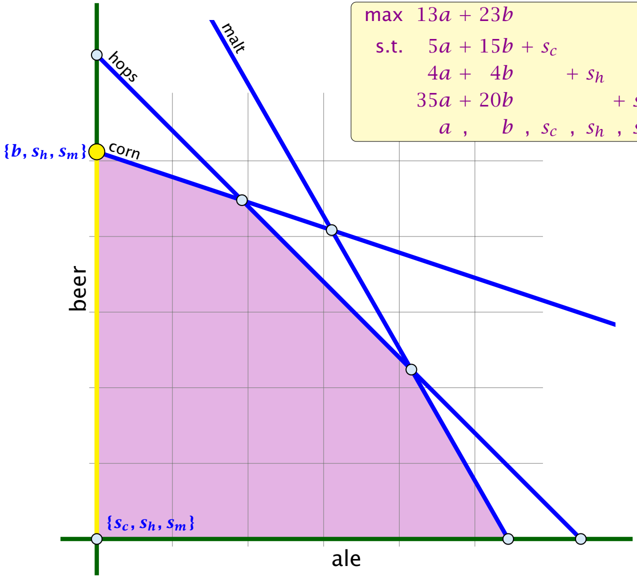

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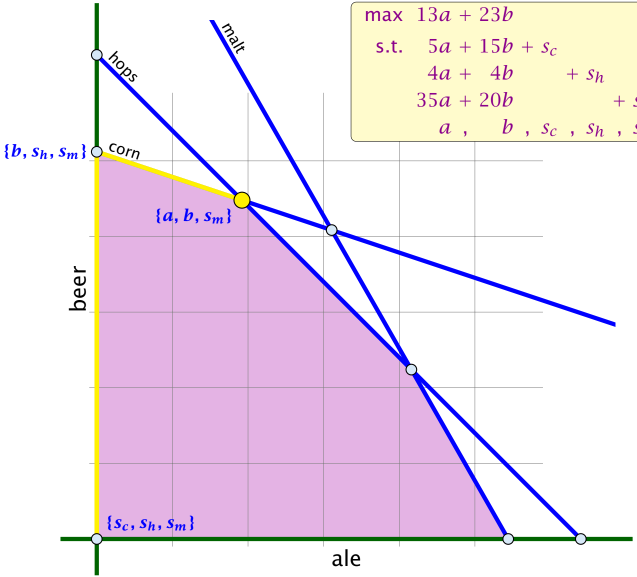

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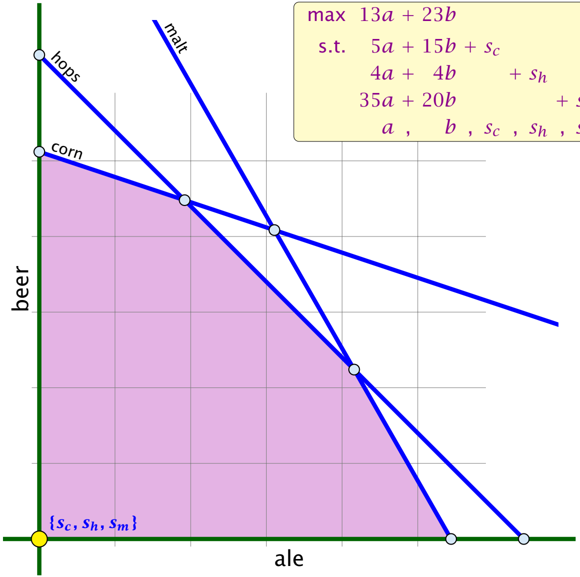

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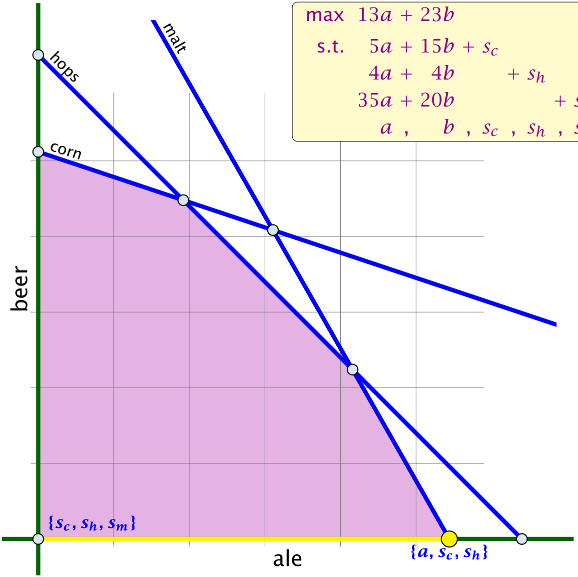

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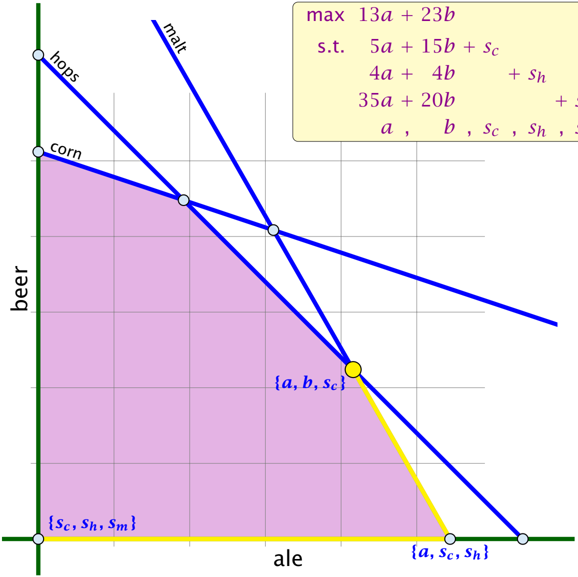

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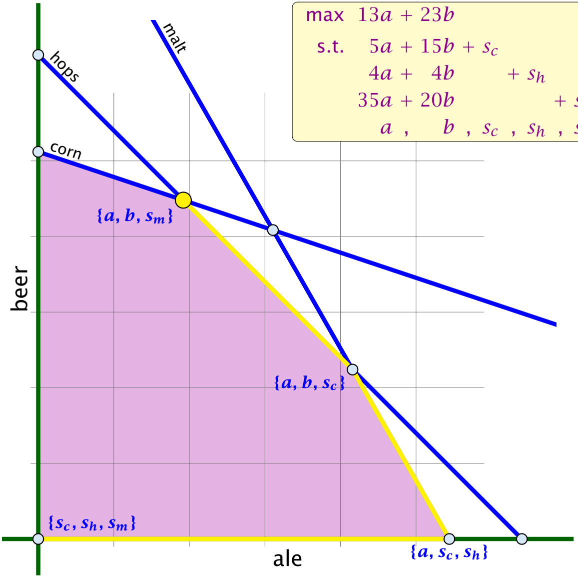

Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
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  - Basis variables change to maintain feasibility.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

- $d_j = 1$ (normalization)
- $d_\ell = 0$, $\ell \notin B$, $\ell \neq j$
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Definition 2 (j-th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1}A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

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Definition 3 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the reduced cost for variable x_j.

Note that this is defined for every j. If $j \in B$ then the above term is 0.

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  Then we can terminate because we know that the solution is optimal.
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The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

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Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is degenerate (also the basis is degenerate).

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A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

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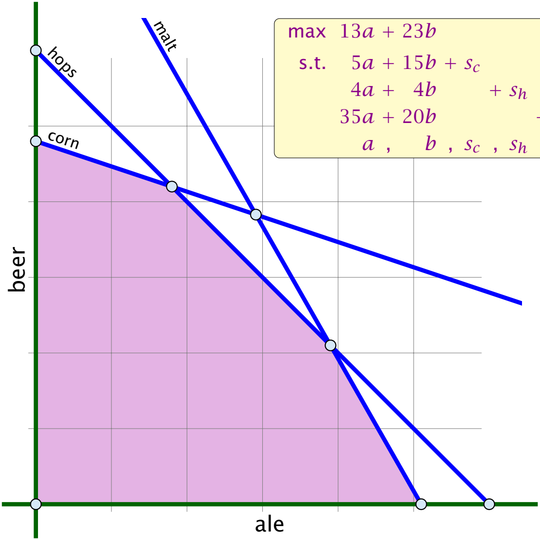

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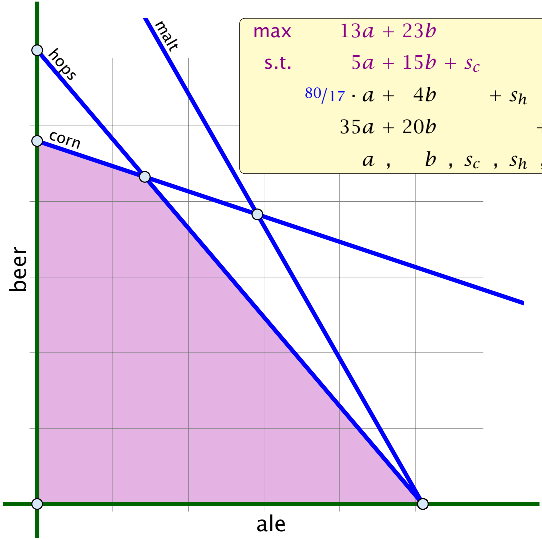
Degenerate Example



$$\begin{array}{lrcrcrcrcrcl}
\text{max} & 13a & + & 23b & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
 & {}^{80}/_{17}\cdot a & + & 4b & & & + & s_h & & & = & 160 \\
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\end{array}$$

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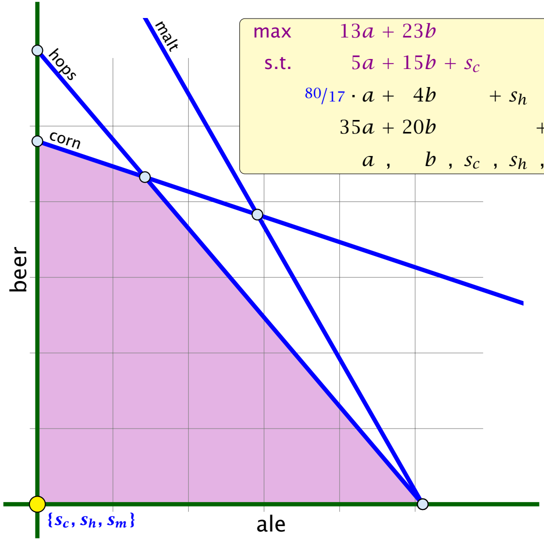

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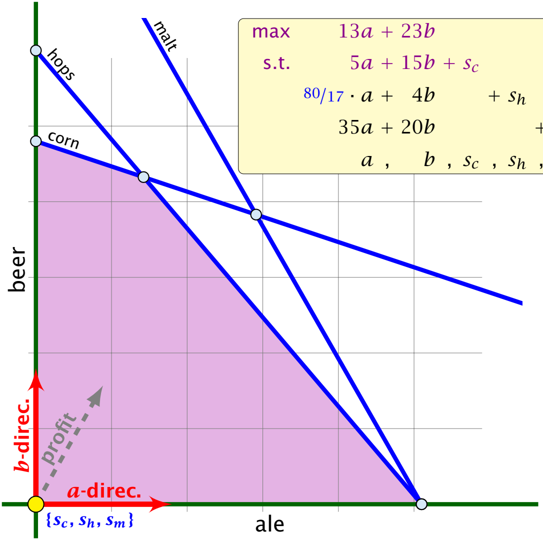

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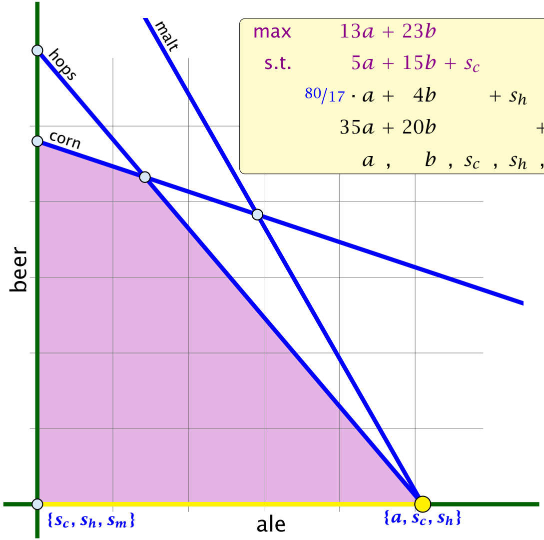

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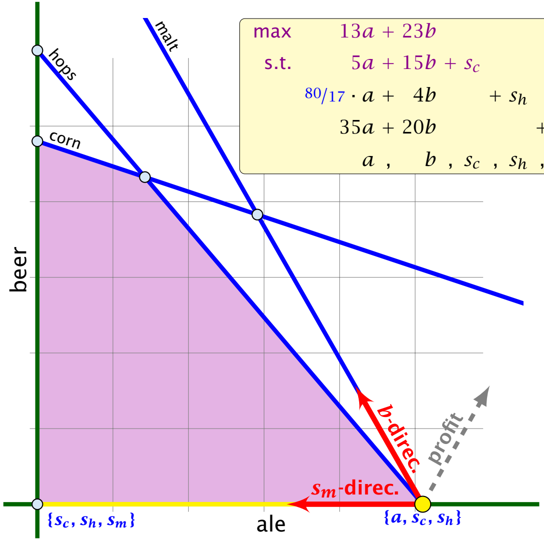

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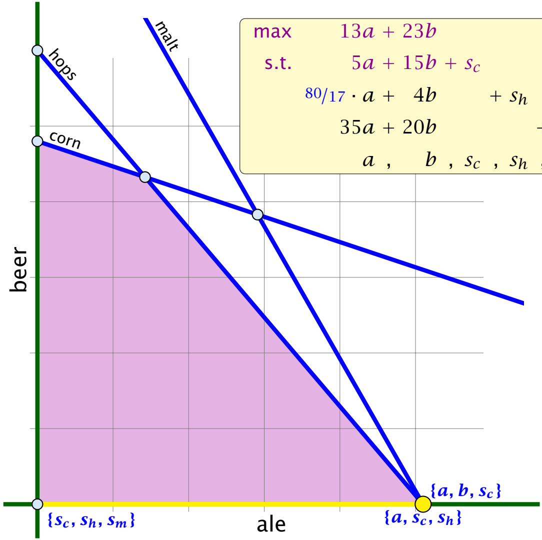

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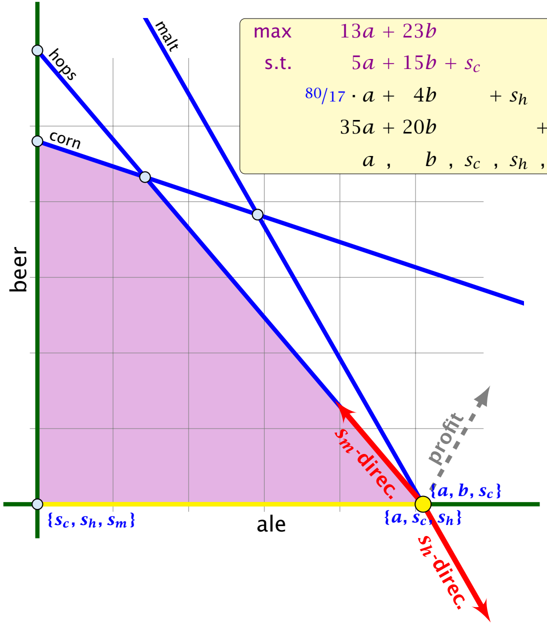

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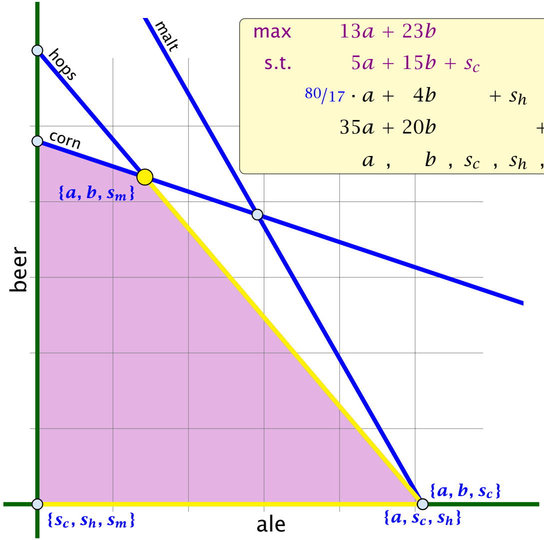

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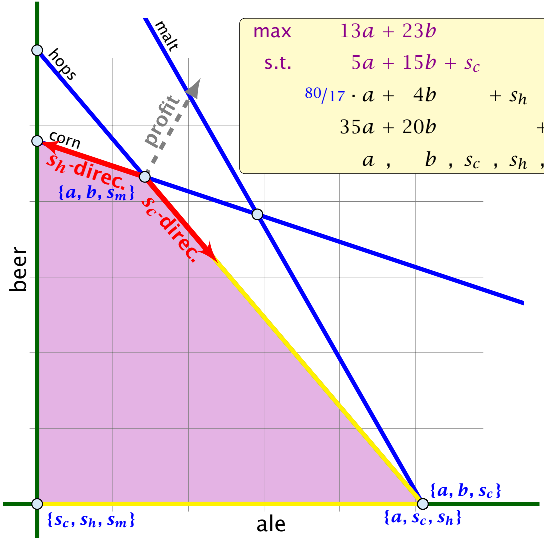

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Summary: How to choose pivot-elements

- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \leq 0$ for all $i \in \{1, \ldots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.
- Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

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Termination

What do we have so far?

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

How do we come up with an initial solution?

- $Ax \le b, x \ge 0$, and $\boldsymbol{b \ge 0}$.
- The standard slack from for this problem is $Ax + Is = b, x \ge 0, s \ge 0$, where s denotes the vector of slack variables.
- Then $s = b$, $x = 0$ is a basic feasible solution (how?).
- We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

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How do we find an initial basic feasible solution for an arbitrary problem?

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1.
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b$, $x \geq 0$, $v \geq 0$ using Simplex. $x = 0$, $v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

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Optimality

Lemma 5
Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.