Complexity

LP Feasibility Problem (LP feasibility)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?

Note that allowing A, b to contain rational numbers does not make a difference, as we can multiply every number by a suitable large constant so that everything becomes integral but the feasible region does not change.

The Bit Model

Input size

▶ The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

▶ Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

$$\langle M \rangle := \sum_{i,j} \lceil \log_2(|m_{ij}|) + 1 \rceil$$

▶ In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.

▶ Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

- In the following we sometimes refer to $L := \langle A \rangle + \langle b \rangle$ as the input size (even though the real input size is something in $\Theta(\langle A \rangle + \langle b \rangle)$).
- In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in L).

Suppose that $Ax = b$; $x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

$$x_B = A_B^{-1} b$$

and all other entries in x are 0.

> In the following we show that this x has small encoding length and we give an explicit bound on this length. So far we have only been handwaving and have said that we can compute x via Gaussian elimination and it will be short...

Size of a Basic Feasible Solution

Lemma 2

*Let $M \in \mathbb{Z}^{m \times m}$ be an invertible matrix and let $b \in \mathbb{Z}^m$. Further
define $L = \langle M \rangle + \langle b \rangle + n \log_2 n$. Then a solution to $Mx = b$ has
rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \leq 2^L$ and
$|D| \leq 2^L$.*

Proof:

Cramers rules says that we can compute x_j as

$$x_j = \frac{\det(M_j)}{\det(M)}$$

where M_j is the matrix obtained from M by replacing the j-th
column by the vector b.

Bounding the Determinant

Let $X = A_B$. Then

$$|\det(X)| = \left| \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{1 \le i \le n} X_{i\pi(i)} \right|$$

$$\le \sum_{\pi \in S_n} \prod_{1 \le i \le n} |X_{i\pi(i)}|$$

$$\le n! \cdot 2^{\langle A \rangle + \langle b \rangle} \le 2^L .$$

Analogously for $\det(M_j)$.

Reducing LP-solving to LP decision.

Given an LP $\max\{c^T x \mid Ax = b; x \geq 0\}$ do a binary search for the optimum solution

(Add constraint $c^T x - \delta = M; \delta \geq 0$ or ($c^T x \geq M$). Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2 \left(\frac{2n2^{2L'}}{1/2^{L'}} \right) = \mathcal{O}(L') \ ,$$

as the range of the search is at most $-n2^{2L'}, \ldots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \geq \frac{1}{2^{L'}}$.

Here we use $L' = \langle A \rangle + \langle b \rangle + \langle c \rangle + n \log_2 n$ (it also includes the encoding size of c).

How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

We can add a constraint $c^T x \geq M_{\max} + 1$ and check for feasibility.

Ellipsoid Method

- ▶ Let K be a convex set.

- ▶ Maintain ellipsoid E that is guaranteed to contain K provided that K is non-empty.

- ▶ If center $z \in K$ STOP.

- ▶ Otw. find a hyperplane separating K from z (e.g. a violated constraint in the LP).

- ▶ Shift hyperplane to contain node z. H denotes half-space that contains K.

- ▶ Compute (smallest) ellipsoid E' that contains $E \cap H$.

- ▶ REPEAT
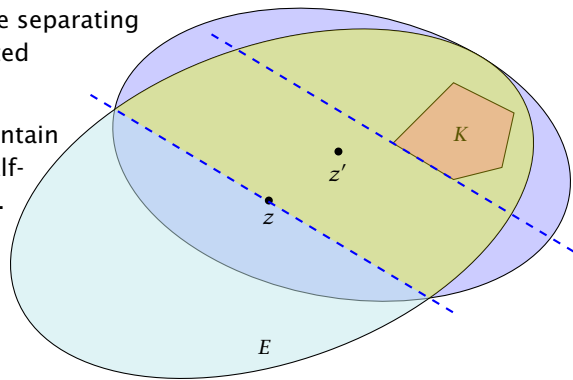
Issues/Questions:

- ▶ How do you choose the first Ellipsoid? What is its volume?
- ▶ How do you measure progress? By how much does the volume decrease in each iteration?
- ▶ When can you stop? What is the minimum volume of a non-empty polytop?

Definition 3

A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an affine transformation.

Definition 4

A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$B(c, r) = \{x \mid (x - c)^T(x - c) \leq r^2\}$$
$$= \{x \mid \sum_i (x - c)_i^2 / r^2 \leq 1\}$$

$B(0, 1)$ is called the unit ball.

Definition 5

An affine transformation of the unit ball is called an ellipsoid.

From $f(x) = Lx + t$ follows $x = L^{-1}(f(x) - t)$.

$$
\begin{aligned}
f(B(0,1)) &= \{f(x) \mid x \in B(0,1)\} \\
&= \{y \in \mathbb{R}^n \mid L^{-1}(y - t) \in B(0,1)\} \\
&= \{y \in \mathbb{R}^n \mid (y - t)^T L^{-1T} L^{-1}(y - t) \le 1\} \\
&= \{y \in \mathbb{R}^n \mid (y - t)^T Q^{-1}(y - t) \le 1\}
\end{aligned}
$$

where $Q = LL^T$ is an invertible matrix.

How to Compute the New Ellipsoid

- ► Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.

- ► Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1.

- ► Compute the new center $\hat{c}'$ and the new matrix $\hat{Q}'$ for this simplified setting.

- ► Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E.
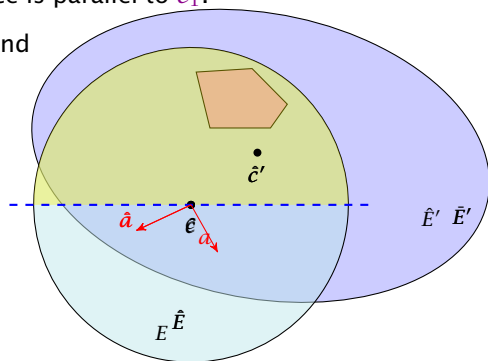
The Easy Case
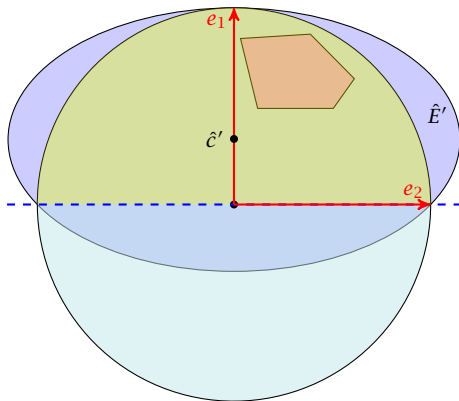
- ▶ The new center lies on axis x_1. Hence, $\hat{c}' = te_1$ for $t > 0$.
- ▶ The vectors $e_1, e_2, \dots$ have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$.

The Easy Case

- To obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is axis-parallel.

- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.

- The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \ldots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

The Easy Case

▸ As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix $\hat{Q}'^{-1}$ is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- $(e_1 - \hat{c}')^T \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $(1-t)^2 = a^2$.

The Easy Case

▶ For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1}(e_i - \hat{c}') = 1$ looks like (here $i = 2$)

$$
\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1
$$

▶ This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$
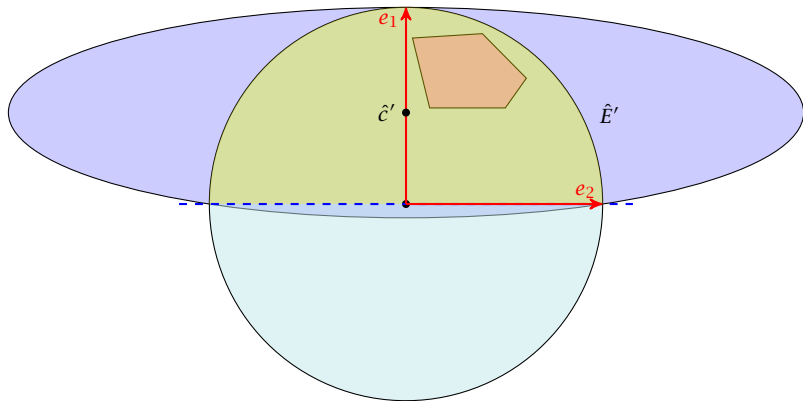\frac{1}{b^2} = 1 - \frac{t^2}{a^2} = 1 - \frac{t^2}{(1-t)^2} = \frac{1-2t}{(1-t)^2}
$$

Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1 - t}{\sqrt{1 - 2t}}$$

The Easy Case

We still have many choices for t:

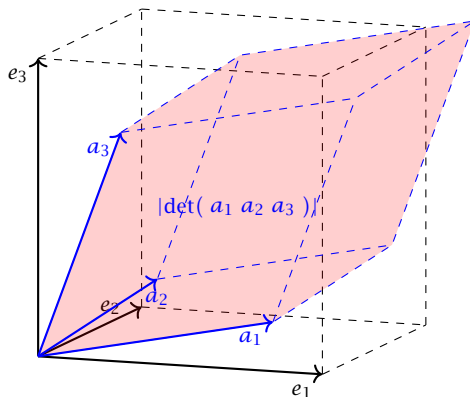Choose t such that the volume of $\hat{E}'$ is minimal!!!

The Easy Case

We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 6

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\operatorname{vol}(L(K)) = |\det(L)| \cdot \operatorname{vol}(K) \ .$$

n-dimensional volume

The Easy Case

▸ We want to choose t such that the volume of $\hat{E}'$ is minimal.

$$\text{vol}(\hat{E}') = \text{vol}(B(0,1)) \cdot |\det(\hat{L}')| \ ,$$

▸ Recall that

$$\hat{L}' = \begin{pmatrix} a & 0 & \ldots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & b \end{pmatrix}$$

▸ Note that a and b in the above equations depend on t, by the previous equations.

The Easy Case

$$\begin{aligned}
\mathrm{vol}(\hat{E}') &= \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')| \\
&= \mathrm{vol}(B(0,1)) \cdot ab^{n-1} \\
&= \mathrm{vol}(B(0,1)) \cdot (1-t) \cdot \left(\frac{1-t}{\sqrt{1-2t}} \right)^{n-1} \\
&= \mathrm{vol}(B(0,1)) \cdot \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}
\end{aligned}$$

We use the shortcut $\Phi := \mathrm{vol}(B(0,1))$.

The Easy Case

$$\frac{d\operatorname{vol}(\hat{E}')}{dt} = \frac{d}{dt}\left(\Phi \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

$$= \frac{\Phi}{N^2} \cdot \Bigg(\underbrace{(-1)\cdot n(1-t)^{n-1}}_{\text{derivative of numerator}} \cdot \overbrace{(\sqrt{1-2t})^{n-1}}^{1-2t}$$

$N = \text{denominator}$

$$\underbrace{-(n-1)(\sqrt{1-2t})^{n-2}}_{\text{outer derivative}} \cdot \underbrace{\frac{1}{2\sqrt{1-2t}}\cdot(-2)}_{\text{inner derivative}} \cdot \overbrace{(1-t)^n}^{1-t} \Bigg)$$

$$= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1}$$

$$\cdot \Big((n-1)(1-t) - n(1-2t) \Big)$$

$$= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \Big((n+1)t - 1 \Big)$$

The Easy Case

▶ We obtain the minimum for $t = \frac{1}{n+1}$.

▶ For this value we obtain

$$a = 1 - t = \frac{n}{n+1} \text{ and } b = \frac{1-t}{\sqrt{1-2t}} = \frac{n}{\sqrt{n^2-1}}$$

To see the equation for b, observe that

$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{(1-\frac{1}{n+1})^2}{1-\frac{2}{n+1}} = \frac{(\frac{n}{n+1})^2}{\frac{n-1}{n+1}} = \frac{n^2}{n^2-1}$$

The Easy Case

Let $y_n = \frac{\text{vol}(\hat{E}')}{\text{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

$$
\begin{aligned}
y_n^2 &= \left(\frac{n}{n+1}\right)^2 \left(\frac{n^2}{n^2-1}\right)^{n-1} \\
&= \left(1 - \frac{1}{n+1}\right)^2 \left(1 + \frac{1}{(n-1)(n+1)}\right)^{n-1} \\
&\le e^{-2\frac{1}{n+1}} \cdot e^{\frac{1}{n+1}} \\
&= e^{-\frac{1}{n+1}}
\end{aligned}
$$

where we used $(1+x)^a \le e^{ax}$ for $x \in \mathbb{R}$ and $a > 0$.

This gives $y_n \le e^{-\frac{1}{2(n+1)}}$.

How to Compute the New Ellipsoid

▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.

▶ Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1.

▶ Compute the new center $\hat{c}'$ and the new matrix $\hat{Q}'$ for this simplified setting.

▶ Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E.
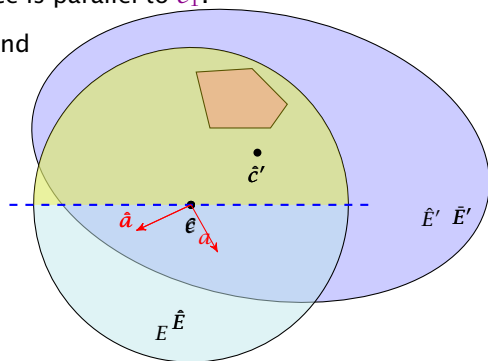
Our progress is the same:

$$e^{-\frac{1}{2(n+1)}} \geq \frac{\text{vol}(\hat{E}')}{\text{vol}(B(0,1))} = \frac{\text{vol}(\hat{E}')}{\text{vol}(\hat{E})} = \frac{\text{vol}(R(\hat{E}'))}{\text{vol}(R(\hat{E}))}$$

$$= \frac{\text{vol}(\bar{E}')}{\text{vol}(\bar{E})} = \frac{\text{vol}(f(\bar{E}'))}{\text{vol}(f(\bar{E}))} = \frac{\text{vol}(E')}{\text{vol}(E)}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \le 0\}$;

$$
\begin{aligned}
f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x - c) \le 0\} \\
&= \{f^{-1}(f(y)) \mid a^T(f(y) - c) \le 0\} \\
&= \{y \mid a^T(f(y) - c) \le 0\} \\
&= \{y \mid a^T(Ly + c - c) \le 0\} \\
&= \{y \mid (a^T L)y \le 0\}
\end{aligned}
$$

This means $\bar{a} = L^T a$.

> The center $\bar{c}$ is of course at the origin.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1}\frac{L^T a}{\|L^T a\|}$$

$$
\begin{aligned}
c' = f(\bar{c}') &= L \cdot \bar{c}' + c \\
&= -\frac{1}{n+1} L \frac{L^T a}{\|L^T a\|} + c \\
&= c - \frac{1}{n+1}\frac{Q a}{\sqrt{a^T Q a}}
\end{aligned}
$$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2 - 1}\Big(I - \frac{2}{n+1}e_1 e_1^T\Big)$$

> Note that $e_1 e_1^T$ is a matrix M that has $M_{11} = 1$ and all other entries equal to 0.

because for $a^2 = {n^2}/{(n+1)^2}$ and $b^2 = {n^2}/{n^2-1}$

$$b^2 - b^2 \frac{2}{n+1} = \frac{n^2}{n^2 - 1} - \frac{2n^2}{(n-1)(n+1)^2}$$

$$= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2$$

9 The Ellipsoid Algorithm

$$\bar{E}' = R(\hat{E}')$$

$$= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\}$$

$$= \{y \mid (R^{-1}y)^T \hat{Q}'^{-1} R^{-1}y \le 1\}$$

$$= \{y \mid y^T (R^T)^{-1} \hat{Q}'^{-1} R^{-1}y \le 1\}$$

$$= \{y \mid y^T \underbrace{(R\hat{Q}'R^T)}_{\tilde{Q}'}^{-1} y \le 1\}$$

9 The Ellipsoid Algorithm

Hence,

$$\bar{Q}' = R\hat{Q}'R^T$$

$$= R \cdot \frac{n^2}{n^2 - 1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big) \cdot R^T$$

$$= \frac{n^2}{n^2 - 1}\Big(R \cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big)$$

$$= \frac{n^2}{n^2 - 1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{\|L^T a\|^2}\Big)$$

Here we used the equation for Re_1 proved before, and the fact that $RR^T = I$, which holds for any rotation matrix. To see this observe that the length of a rotated vector x should not change, i.e.,

$$x^T I x = (Rx)^T(Rx) = x^T(R^T R)x$$

which means $x^T(I - R^T R)x = 0$ for every vector x. It is easy to see that this can only be fulfilled if $I - R^T R = 0$.

9 The Ellipsoid Algorithm

$$
\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\
&= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{L \bar{Q}' L^T}_{Q'})^{-1} y \le 1\}
\end{aligned}
$$

Hence,

$$Q' = L\bar{Q}'L^T$$

$$= L \cdot \frac{n^2}{n^2 - 1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{a^T Q a}\Big) \cdot L^T$$

$$= \frac{n^2}{n^2 - 1}\Big(Q - \frac{2}{n+1}\frac{Q a a^T Q}{a^T Q a}\Big)$$

Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

1: **input:** point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
2: **output:** point $x \in K$ or "K is empty"
3: $Q \leftarrow$???
4: **repeat**
5:     **if** $c \in K$ **then return** c
6:     **else**
7:         choose a violated hyperplane a
8:         $c \leftarrow c - \dfrac{1}{n+1} \dfrac{Qa}{\sqrt{a^T Q a}}$
9:         $Q \leftarrow \dfrac{n^2}{n^2 - 1} \Big(Q - \dfrac{2}{n+1} \dfrac{Q a a^T Q}{a^T Q a} \Big)$
10:     **endif**
11: **until** ???
12: **return** "K is empty"

Repeat: Size of basic solutions

Lemma 7

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a bounded polyhedron. Let $\langle a_{\max} \rangle$ be the maximum encoding length of an entry in A, b. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \leq 2^{2n\langle a_{\max} \rangle + 2n \log_2 n}$.

In the following we use $\delta := 2^{2n\langle a_{\max} \rangle + 2n \log_2 n}$.

Note that here we have $P = \{x \mid Ax \leq b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

Repeat: Size of basic solutions

Proof:

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$, b, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices $\bar{A}_B$ and $\bar{M}_j$ (matrix obt. when replacing the j-th column of $\bar{A}_B$ by b) can become at most

$$\det(\bar{A}_B), \det(\bar{M}_j) \leq \|\vec{\ell}_{\max}\|^{2n}$$
$$\leq (\sqrt{2n} \cdot 2^{\langle a_{\max}\rangle})^{2n} \leq 2^{2n\langle a_{\max}\rangle + 2n\log_2 n} \; ,$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from $\bar{A}$.

This holds because columns from I_m selected when going from $\bar{A}$ to $\bar{A}_B$ do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that $\bar{A}$ consists of contribute.

How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \leq \delta$.

Hence, P is contained in the cube $-\delta \leq x_i \leq \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \operatorname{vol}(B(0, 1)) \leq (n\delta)^n \operatorname{vol}(B(0, 1))$.

When can we terminate?

Let $P := \{x \mid Ax \le b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop. Let $\langle a_{\max} \rangle$ be the encoding length of the largest entry in A or b.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \le b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

where $\lambda = \delta^2 + 1$.

Lemma 8

P_λ is feasible if and only if P is feasible.

$\Longleftarrow$: obvious!

$\Longrightarrow$:

Consider the polyhedrons

$$\bar{P} = \left\{ x \mid \begin{bmatrix} A & -A & I_m \end{bmatrix} x = b; x \geq 0 \right\}$$

and

$$\bar{P}_\lambda = \left\{ x \mid \begin{bmatrix} A & -A & I_m \end{bmatrix} x = b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0 \right\} .$$

P is feasible if and only if $\bar{P}$ is feasible, and P_λ feasible if and only if $\bar{P}_\lambda$ feasible.

$\bar{P}_\lambda$ is bounded since P_λ and P are bounded.

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$.

$\bar{P}_\lambda$ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1}b + \frac{1}{\lambda}\bar{A}_B^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1}b)_i < 0 \le (\bar{A}_B^{-1}b)_i + \frac{1}{\lambda}(\bar{A}_B^{-1}\vec{1})_i$$

By Cramers rule we get

$$(\bar{A}_B^{-1}b)_i < 0 \quad \implies \quad (\bar{A}_B^{-1}b)_i \le -\frac{1}{\det(\bar{A}_B)}$$

and

$$(\bar{A}_B^{-1}\vec{1})_i \le \det(\bar{M}_j) \ ,$$

where $\bar{M}_j$ is obtained by replacing the j-th column of $\bar{A}_B$ by $\vec{1}$.

However, we showed that the determinants of $\bar{A}_B$ and $\bar{M}_j$ can become at most δ.

Since, we chose $\lambda = \delta^2 + 1$ this gives a contradiction.

Lemma 9

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n\mathrm{vol}(B(0,1)) = \frac{1}{\delta^{3n}}\mathrm{vol}(B(0,1))$.

Proof:

If P_λ feasible then also P. Let x be feasible for P.
This means $Ax \le b$.

Let $\vec{\ell}$ with $\|\vec{\ell}\| \le r$. Then

$$(A(x + \vec{\ell}))_i = (Ax)_i + (A\vec{\ell})_i \le b_i + \vec{a}_i^T \vec{\ell}$$
$$\le b_i + \|\vec{a}_i\| \cdot \|\vec{\ell}\| \le b_i + \sqrt{n} \cdot 2^{\langle a_{\max} \rangle} \cdot r$$
$$\le b_i + \frac{\sqrt{n} \cdot 2^{\langle a_{\max} \rangle}}{\delta^3} \le b_i + \frac{1}{\delta^2 + 1} \le b_i + \frac{1}{\lambda}$$

Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

How many iterations do we need until the volume becomes too small?

$$e^{-\frac{i}{2(n+1)}} \cdot \text{vol}(B(0,R)) < \text{vol}(B(0,r))$$

Hence,

$$\begin{aligned} i &> 2(n+1) \ln\Big(\frac{\text{vol}(B(0,R))}{\text{vol}(B(0,r))}\Big) \\ &= 2(n+1) \ln\Big(n^n \delta^n \cdot \delta^{3n}\Big) \\ &= 8n(n+1)\ln(\delta) + 2(n+1)n\ln(n) \\ &= \mathcal{O}(\text{poly}(n, \langle a_{\max} \rangle)) \end{aligned}$$

Algorithm 1 ellipsoid-algorithm

1: **input:** point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
2:         with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
3: **output:** point $x \in K$ or "K is empty"
4: $Q \leftarrow \text{diag}(R^2, \ldots, R^2)$ // i.e., $L = \text{diag}(R, \ldots, R)$
5: **repeat**
6:      **if** $c \in K$ **then return** c
7:      **else**
8:          choose a violated hyperplane a
9:          $c \leftarrow c - \dfrac{1}{n+1} \dfrac{Qa}{\sqrt{a^T Q a}}$
10:          $Q \leftarrow \dfrac{n^2}{n^2-1} \Big(Q - \dfrac{2}{n+1} \dfrac{Q a a^T Q}{a^T Q a} \Big)$
11:      **endif**
12: **until** $\det(Q) \leq r^{2n}$ // i.e., $\det(L) \leq r^n$
13: **return** "K is empty"

Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- ▶ certifies that $x \in K$,
- ▶ or finds a hyperplane separating x from K.

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- ▶ a guarantee that a ball of radius r is contained in K,
- ▶ an initial ball $B(c, R)$ with radius R that contains K,
- ▶ a separation oracle for K.

The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.