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$P = \max\{c^T x \mid Ax \le b, x \ge 0\}$

n_A: number of variables, m_A: number of constraints

We can put the non-negativity constraints into A (which gives us unrestricted variables): $\bar{P} = \max\{c^T x \mid \bar{A}x \le \bar{b}\}$

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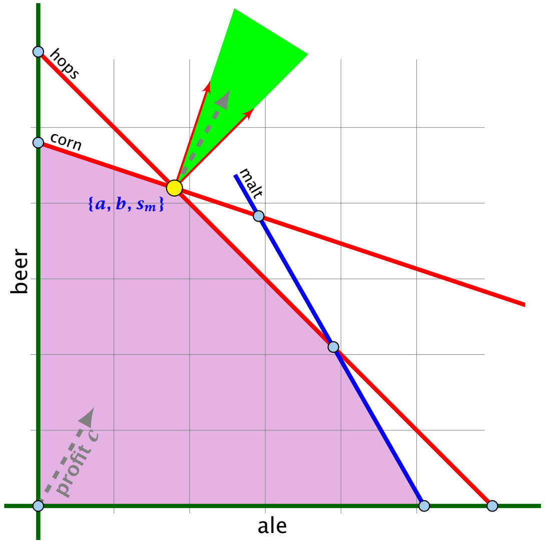

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Theorem 2 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

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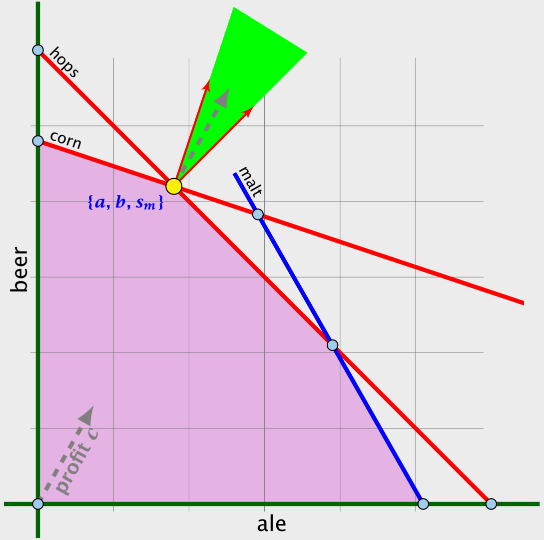

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Lemma 3 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.

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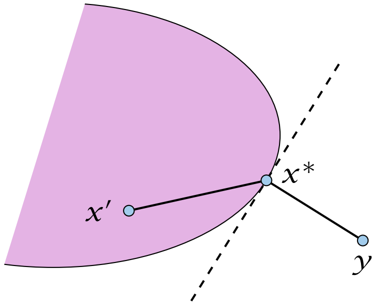

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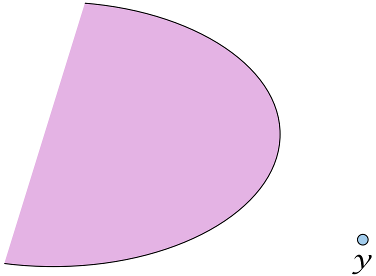

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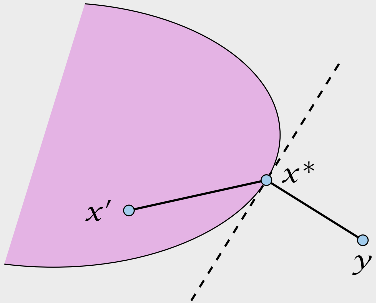

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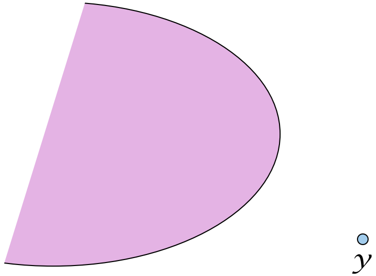

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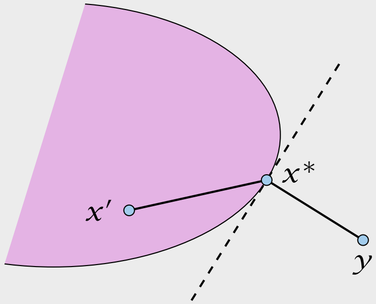

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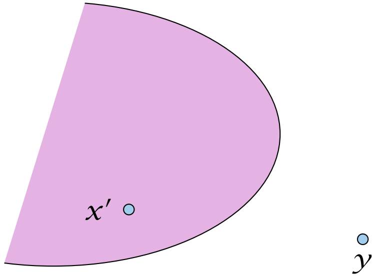

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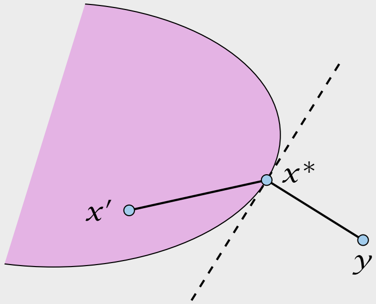

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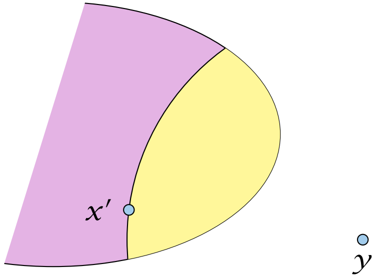

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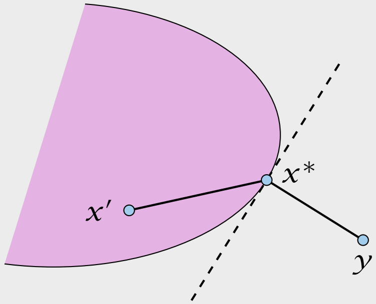

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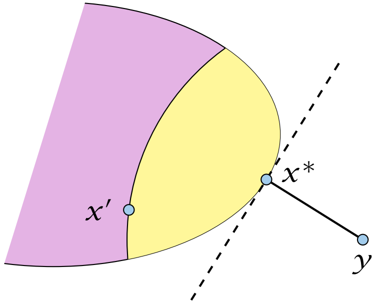

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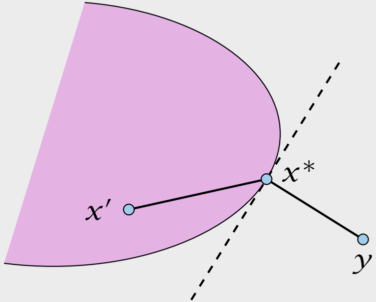

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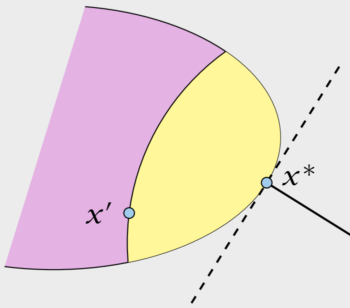

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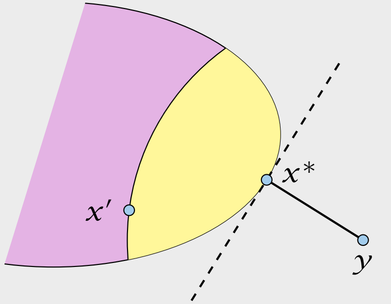

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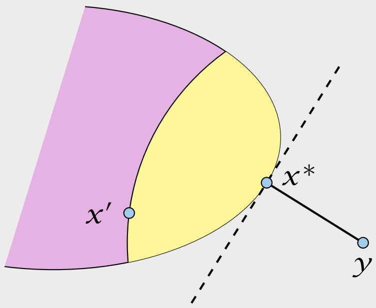

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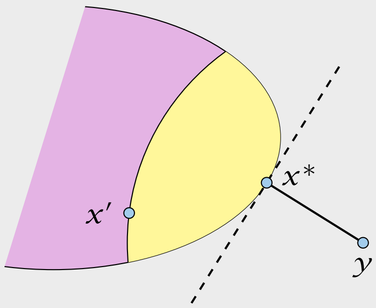

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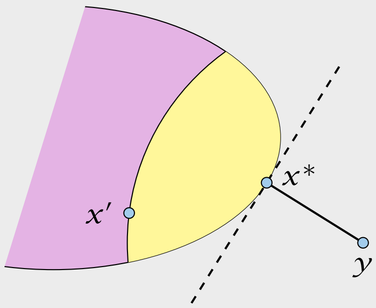

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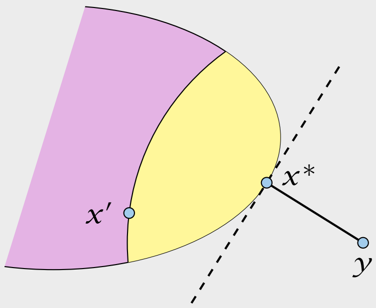

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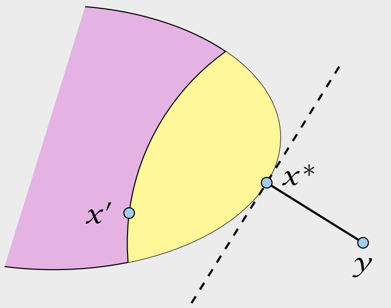

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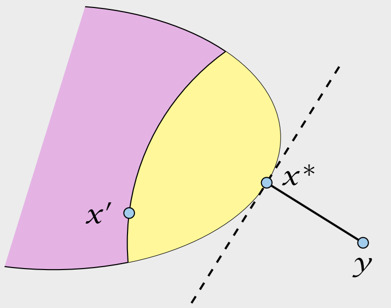

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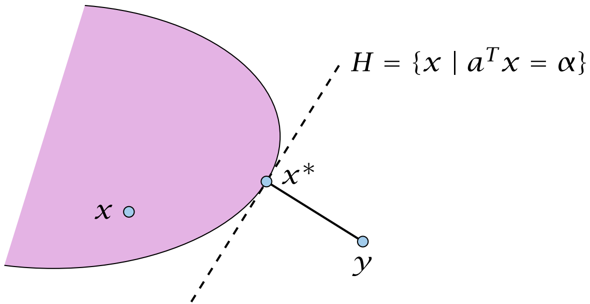

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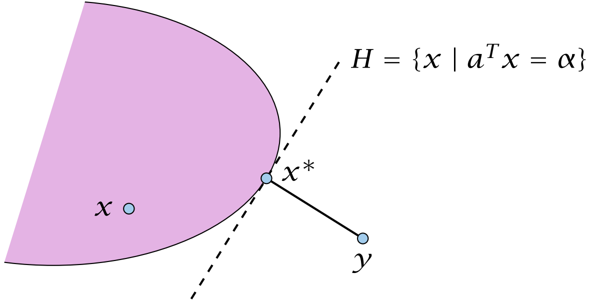

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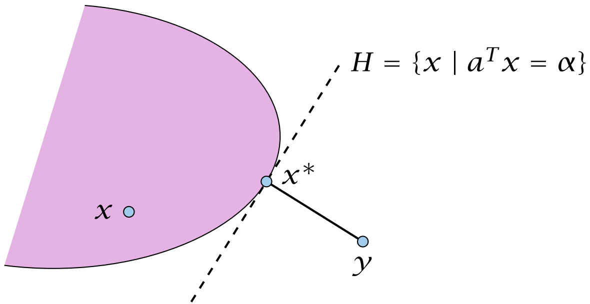

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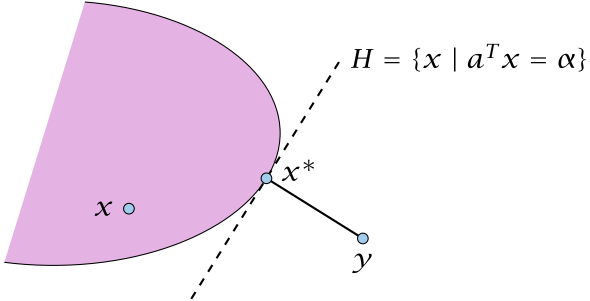

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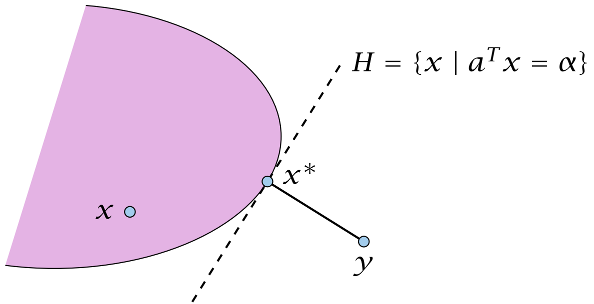

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Lemma 6 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^T y \geq 0$, $b^T y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^T b = y^T A x \geq 0$$

Hence, at most one of the statements can hold.

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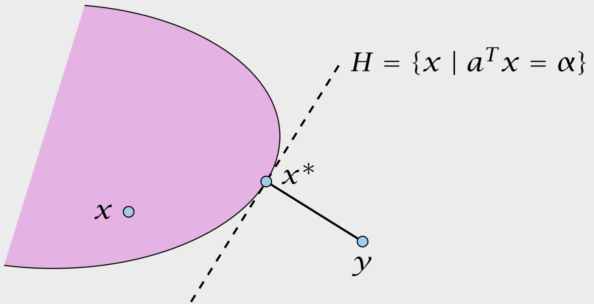

Lemma 6 (Farkas Lemma)

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Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^T b = y^T A x \geq 0$$

Hence, at most one of the statements can hold.

Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to y in X.
- By previous lemma $(y - x^*)^T(x - x^*) \leq 0$ for all $x \in X$.
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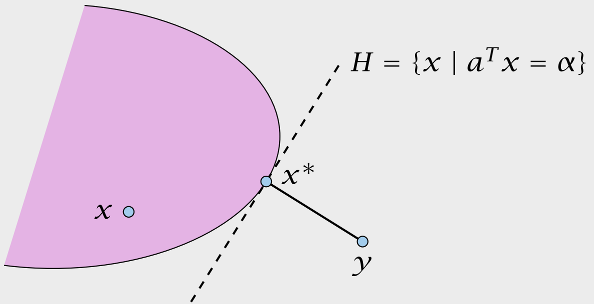

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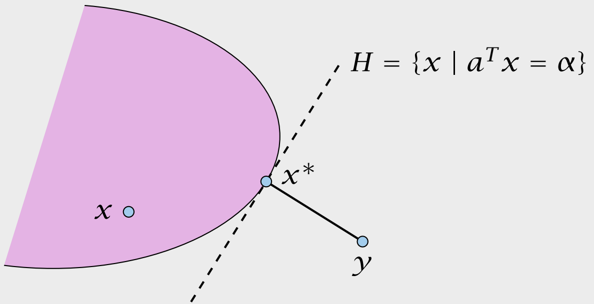

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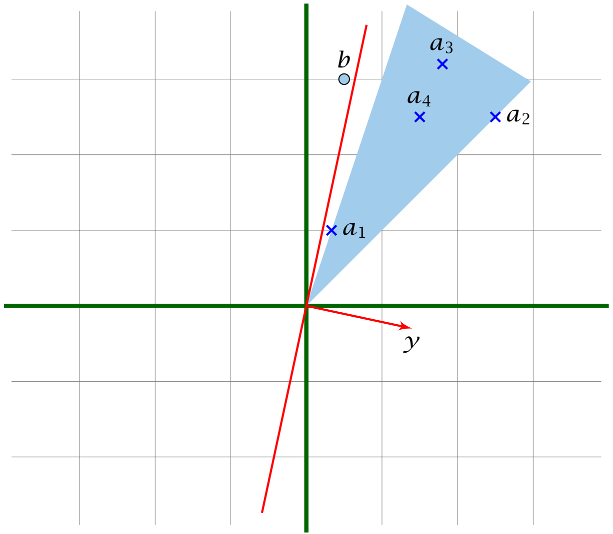

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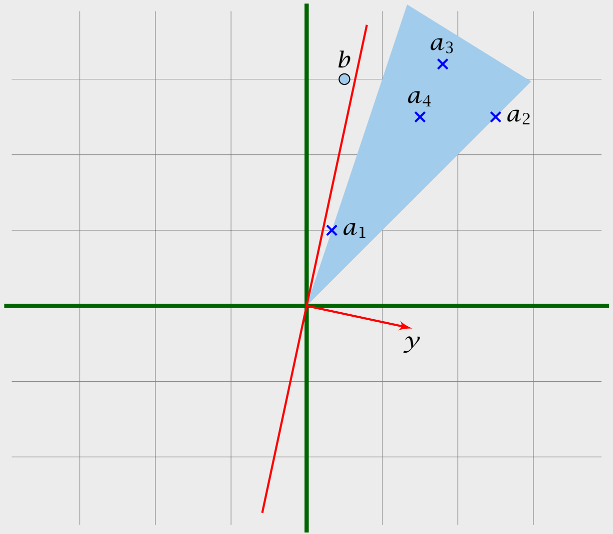

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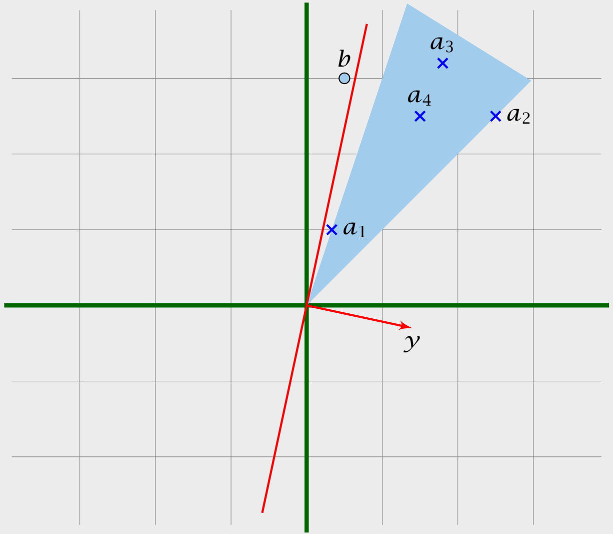

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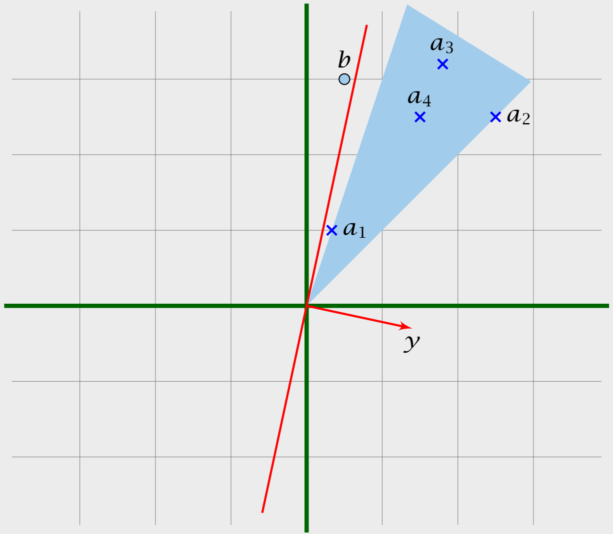

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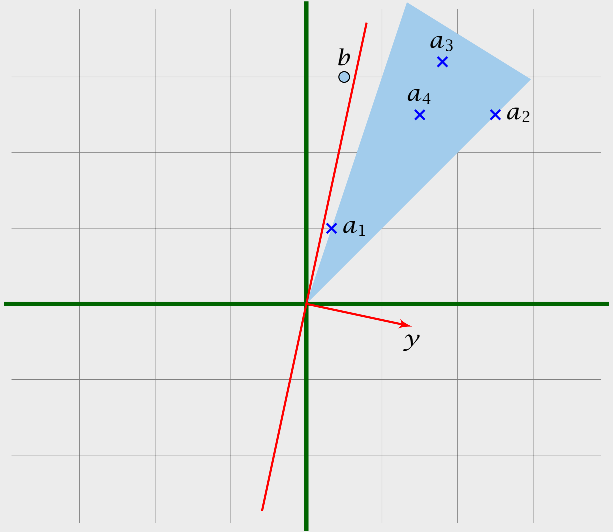

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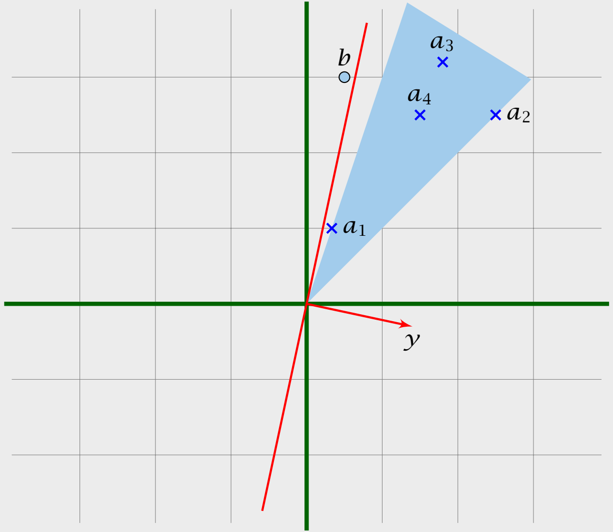

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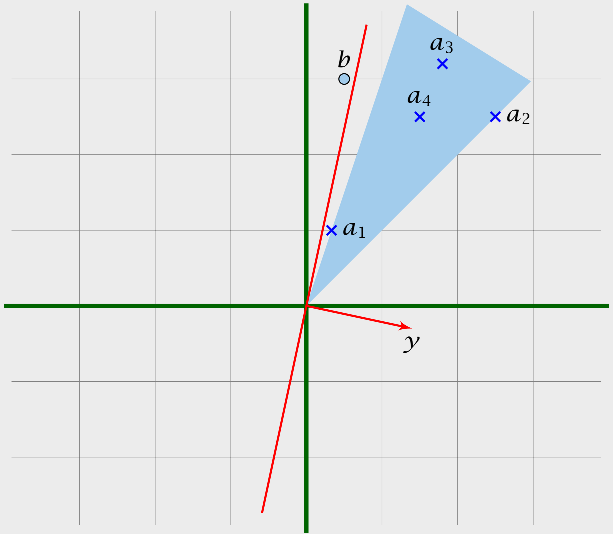

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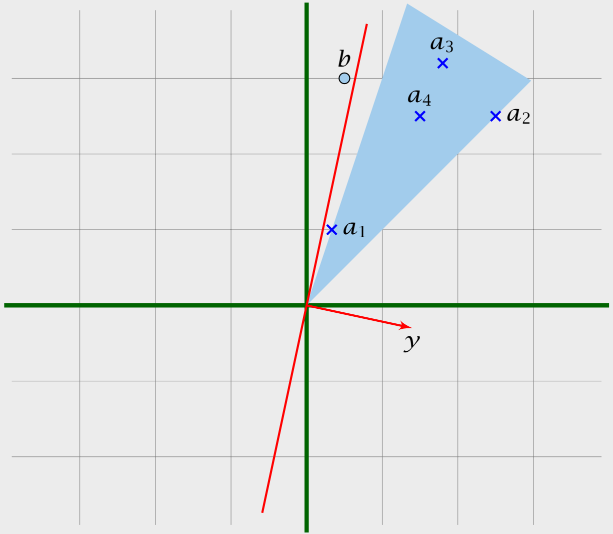

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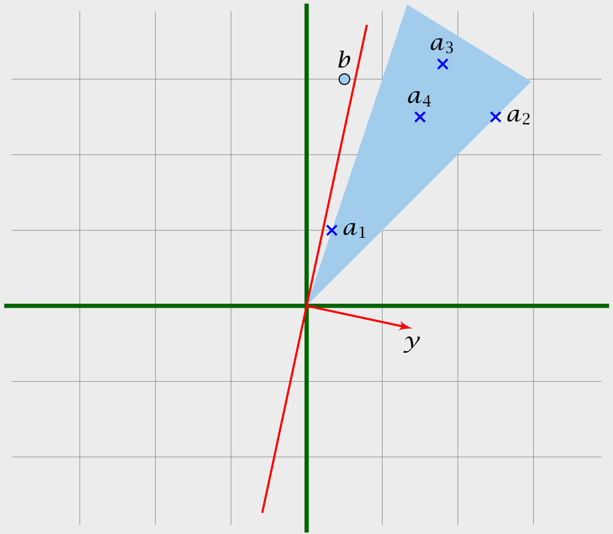

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Lemma 7 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

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Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \geq 0$, $s \geq 0$
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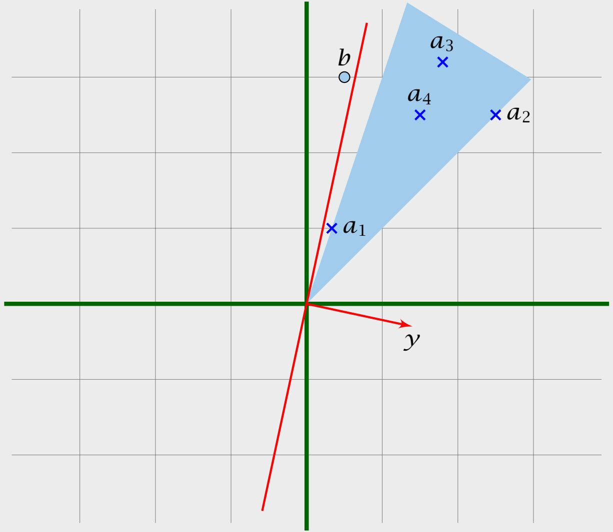

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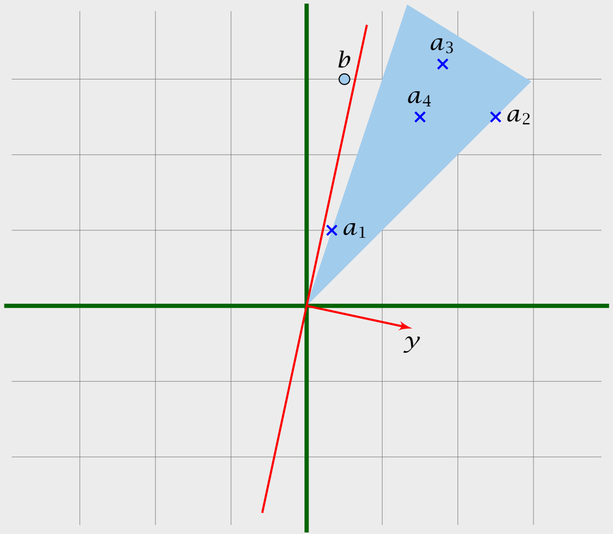

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$P\colon\ z = \max\{c^T x \mid Ax \le b, x \ge 0\}$

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Theorem 8 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

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We show $z < \alpha$ implies $w < \alpha$.

Proof of Strong Duality

P: $z = \max\{c^Tx \mid Ax \le b, x \ge 0\}$

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From the definition of α we know that the first system is infeasible; hence the second must be feasible.

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Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

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is feasible. By Farkas lemma this gives that LP P is infeasible. Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, v with $v > 0$.

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Proof of Strong Duality

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Definition 9 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP? yes!
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- A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost $< \alpha$.

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