7 Dictionary

Dictionary:

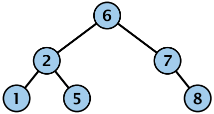
- $\boldsymbol{S.\,\mathrm{insert}(x)}$: Insert an element x.
- $\boldsymbol{S.\,\mathrm{delete}(x)}$: Delete the element pointed to by x.
- $\boldsymbol{S.\,\mathrm{search}(k)}$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

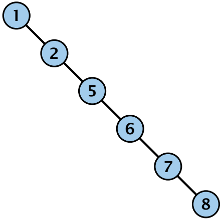
7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:



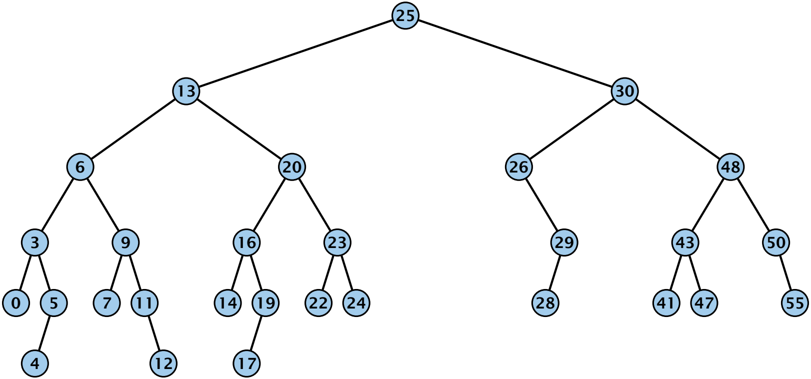


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\mathrm{insert}(x)$
- $T.\mathrm{delete}(x)$
- $T.\mathrm{search}(k)$
- $T.\mathrm{successor}(x)$
- $T.\mathrm{predecessor}(x)$
- $T.\mathrm{minimum}()$
- $T.\mathrm{maximum}()$

Binary Search Trees: Searching



Algorithm 1 TreeSearch(x, k)

1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
2: **if** $k < \text{key}[x]$ **return** TreeSearch$(\text{left}[x], k)$
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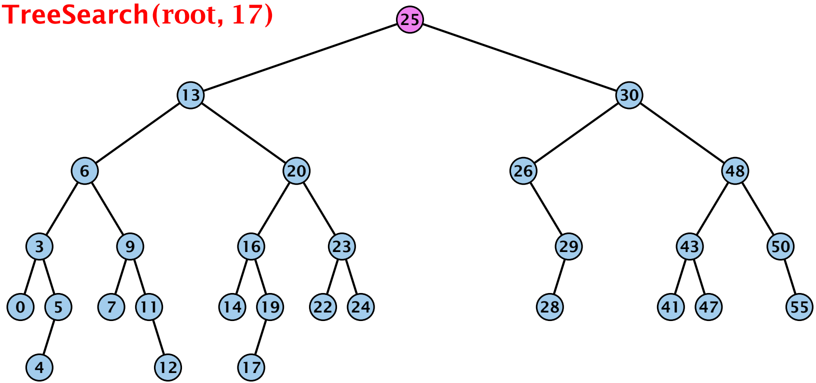
Binary Search Trees: Searching



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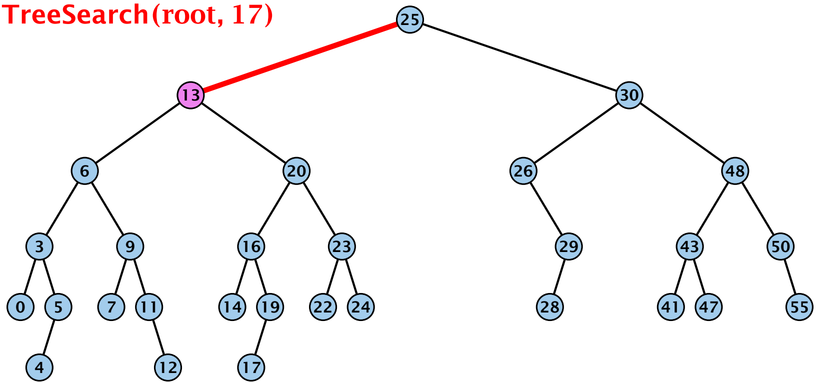

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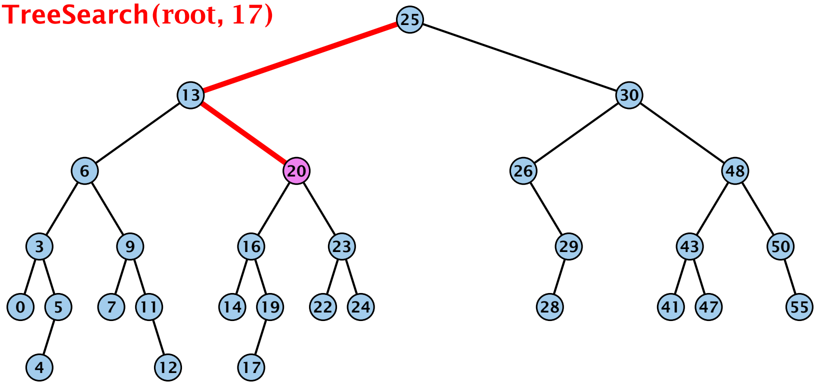
Binary Search Trees: Searching



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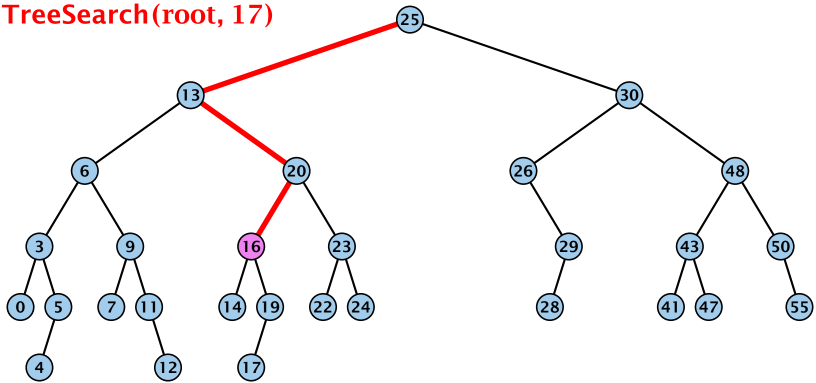

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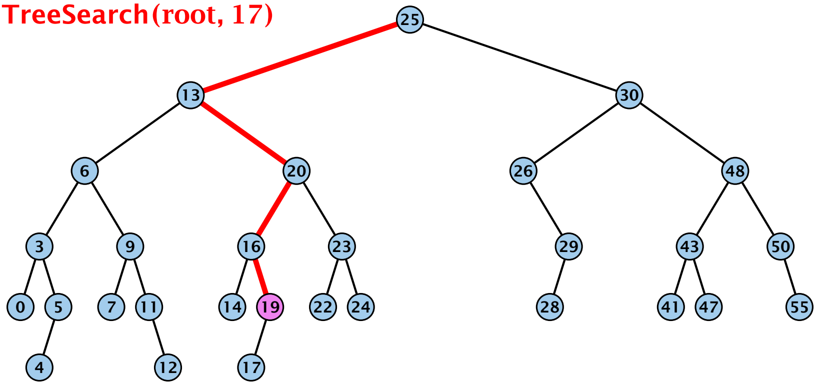

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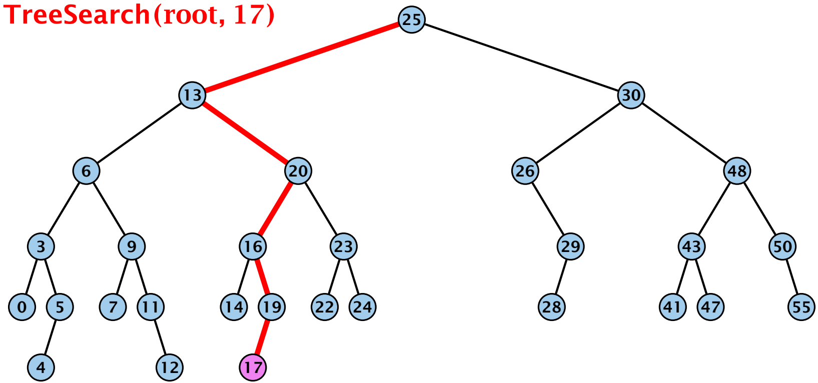
Binary Search Trees: Searching



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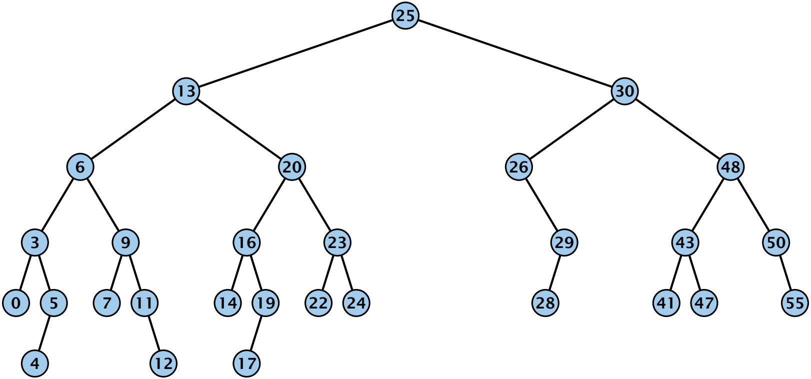
Binary Search Trees: Searching



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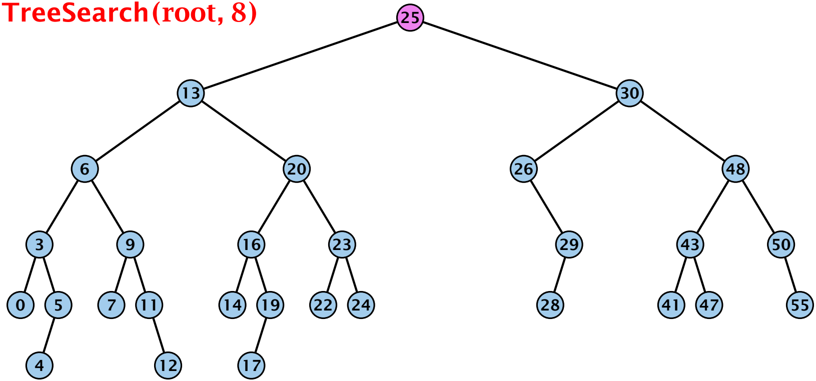
Binary Search Trees: Searching

TreeSearch(root, 8)



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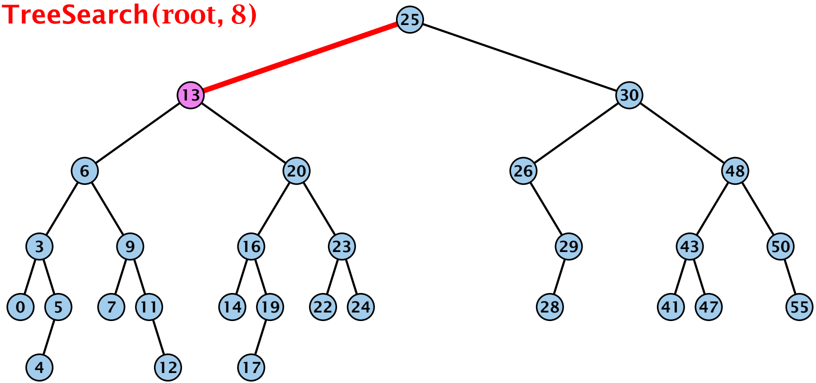

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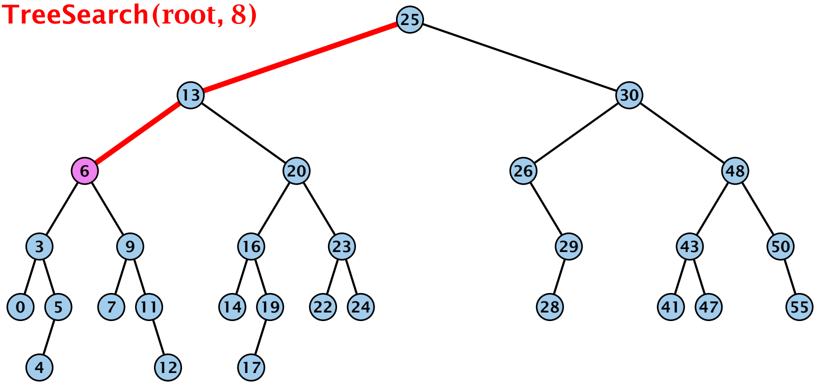
Binary Search Trees: Searching



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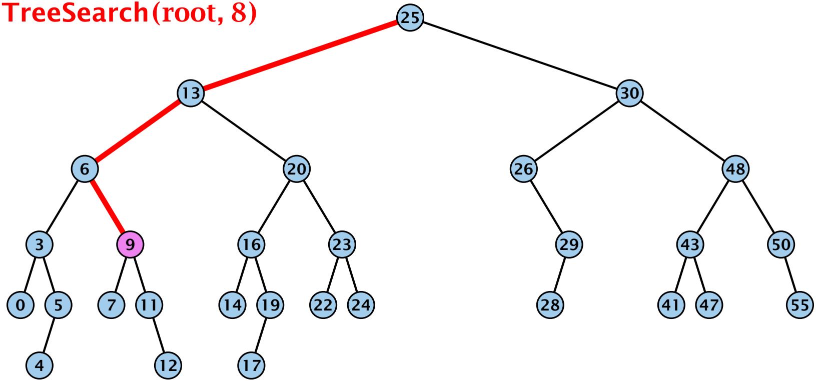
Binary Search Trees: Searching

TreeSearch(root, 8)



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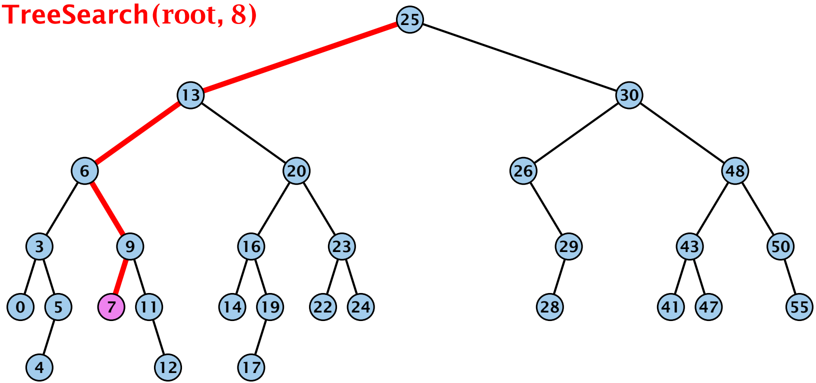

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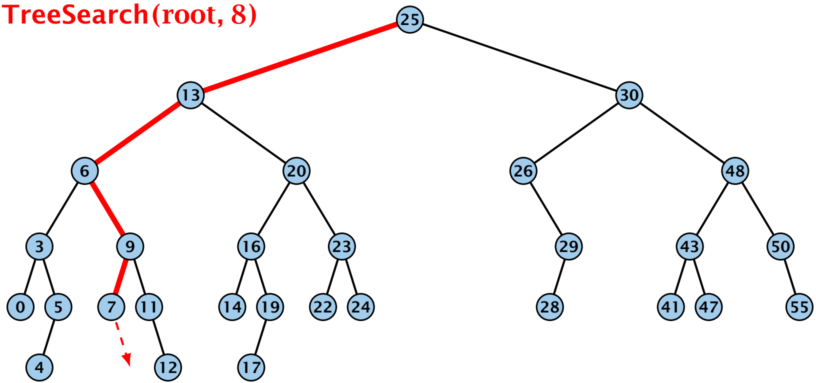

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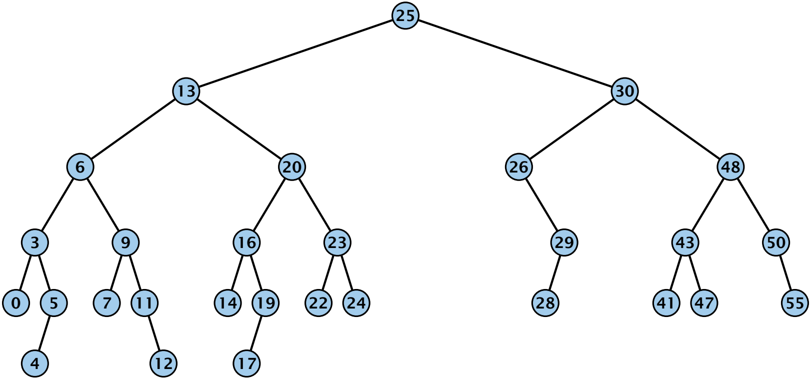
Binary Search Trees: Minimum

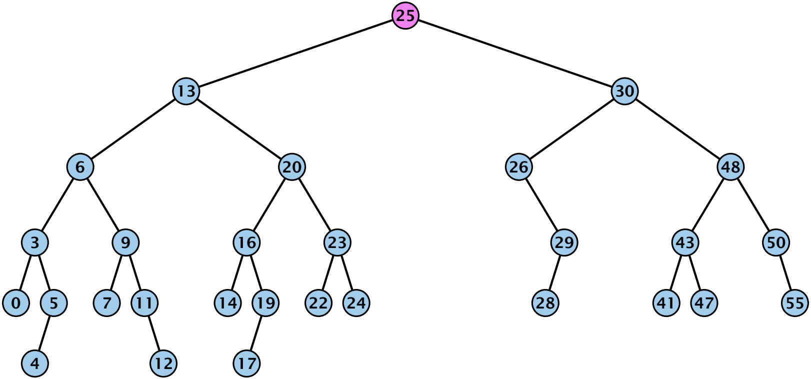


Algorithm 2 TreeMin(x)

1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x

2: **return** TreeMin($\text{left}[x]$)

Binary Search Trees: Minimum



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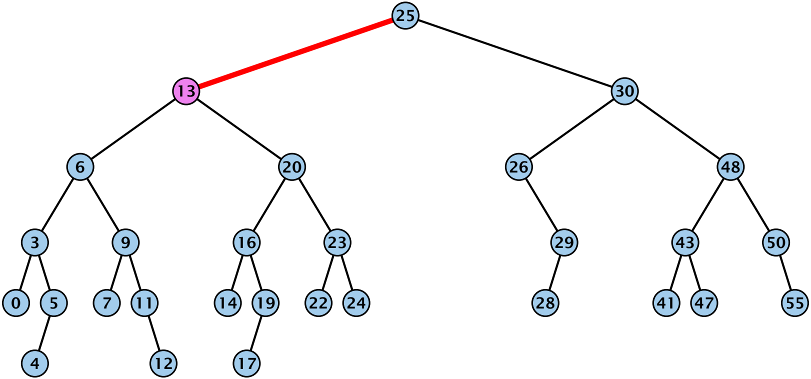
Binary Search Trees: Minimum



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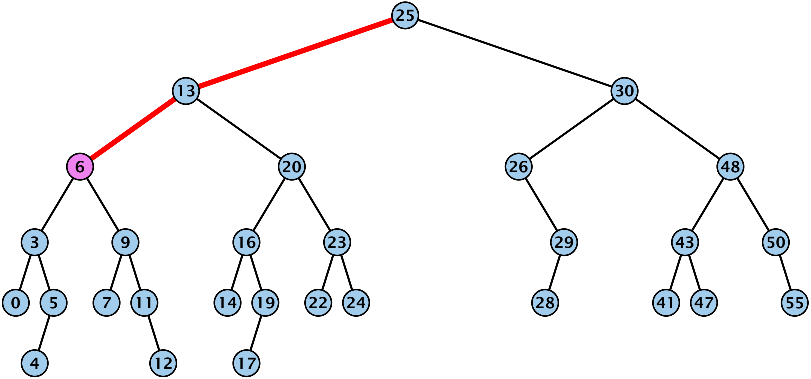
Binary Search Trees: Minimum



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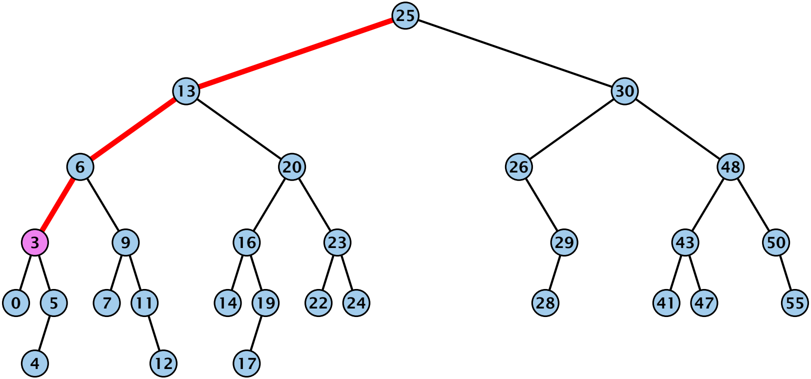

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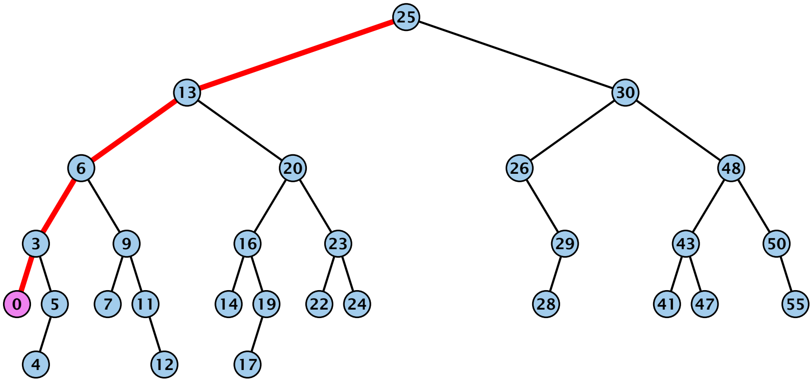

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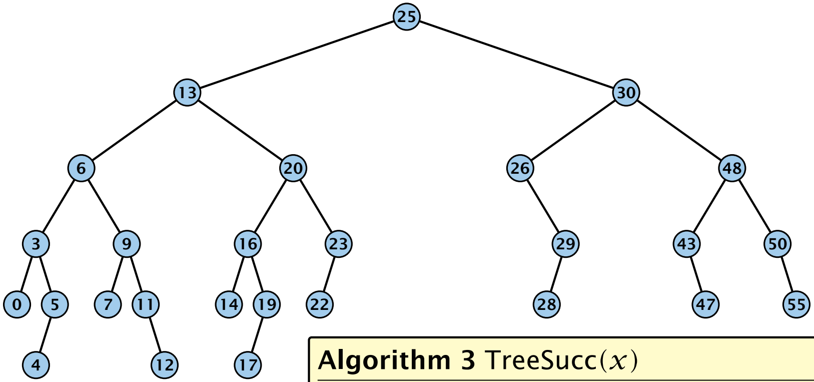
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

if right[x] $\neq$ null **return** TreeMin(right[x])

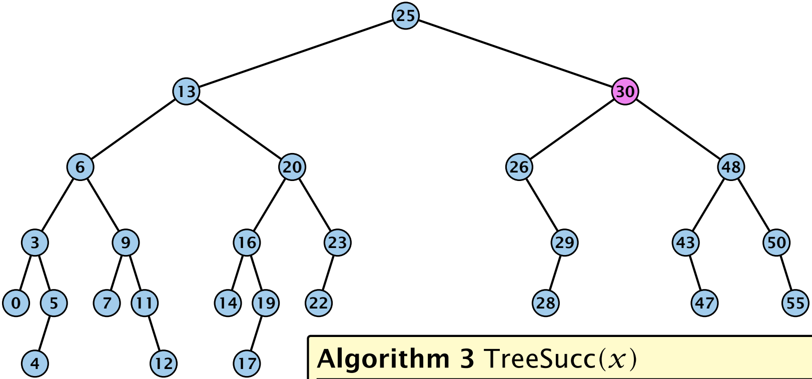
$y \leftarrow$ parent[x]

while $y \neq$ null **and** $x =$ right[y] **do**

  $x \leftarrow y; y \leftarrow$ parent[x]

return y;

Binary Search Trees: Successor



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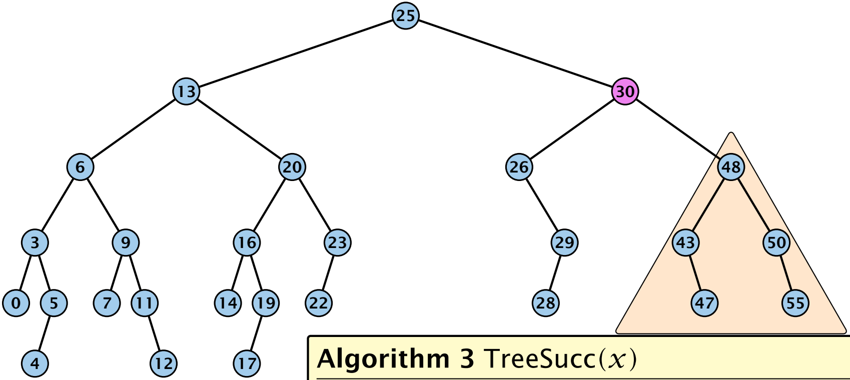

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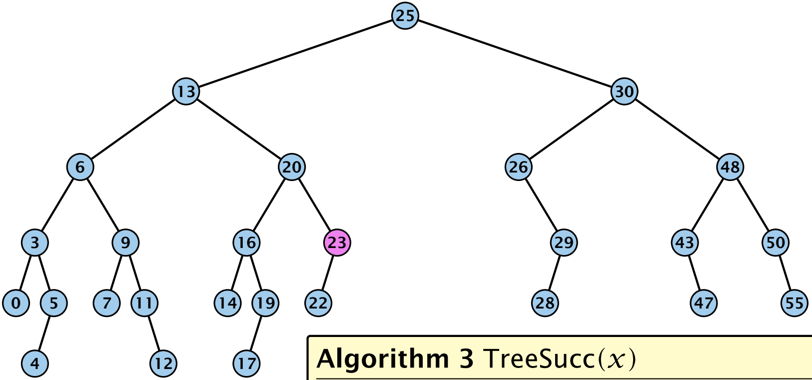

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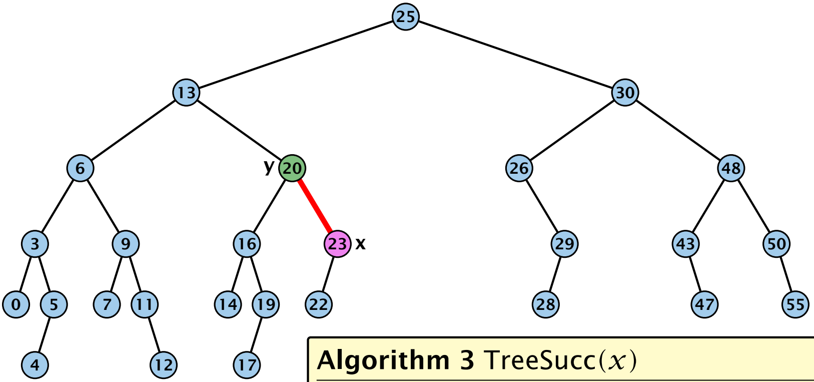

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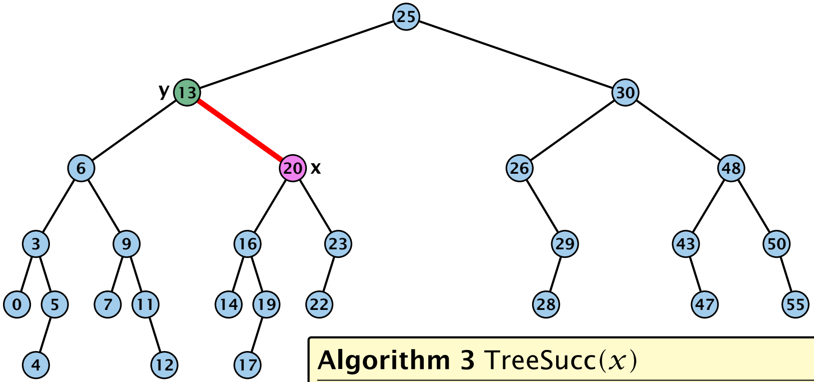

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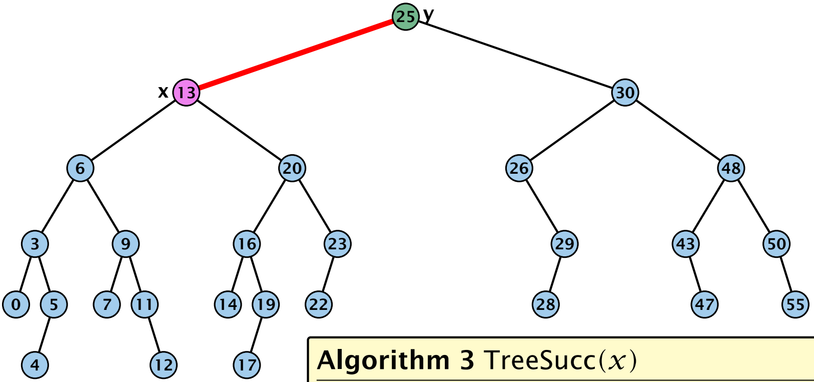

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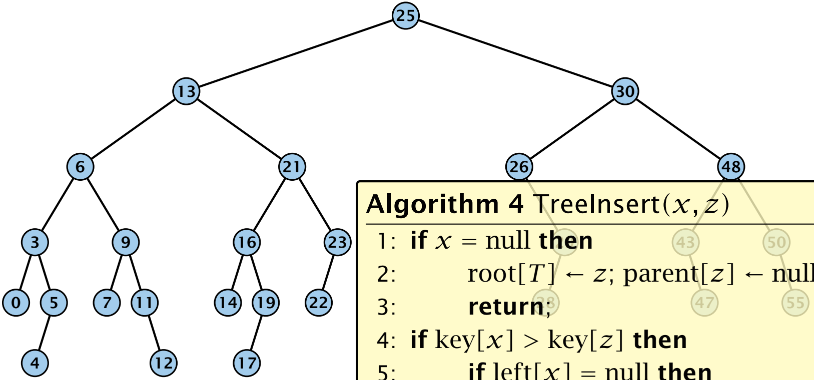
Binary Search Trees: Insert



Algorithm 4 TreeInsert(x, z)

if x = null **then**
  root$[T] \leftarrow z$; parent$[z] \leftarrow$ null;
  return;
if key$[x] >$ key$[z]$ **then**
  if left$[x]$ = null **then**
    left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
  else TreeInsert(left$[x], z$);
else
  if right$[x]$ = null **then**
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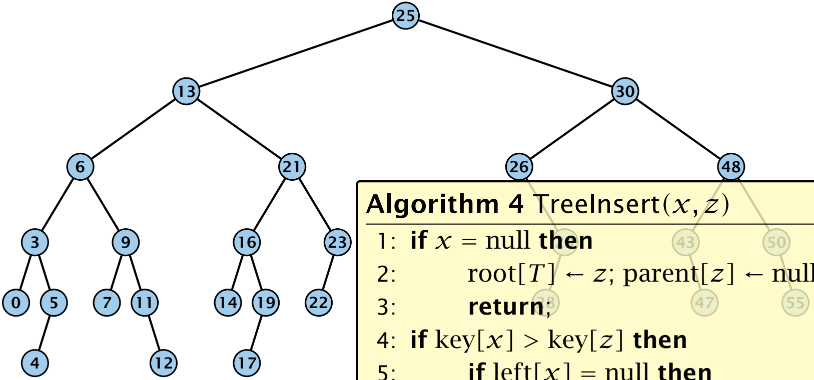
  else TreeInsert(right$[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.



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```
if x = null then
    root[T] ← z; parent[z] ← null;
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if key[x] > key[z] then
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    else TreeInsert(left[x], z);
else
    if right[x] = null then
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Binary Search Trees: Insert

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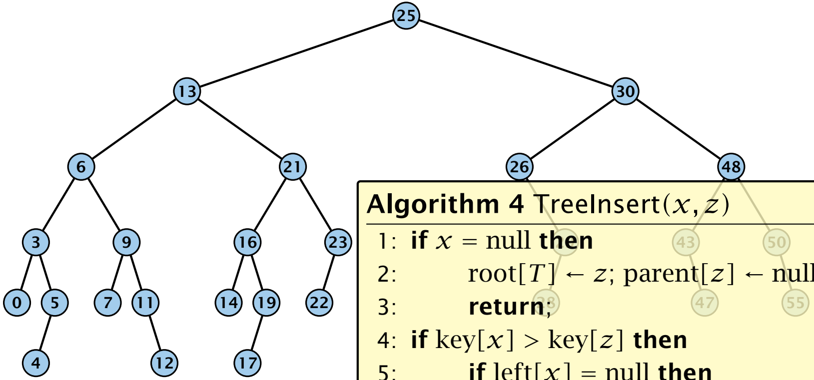

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

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```

Binary Search Trees: Insert

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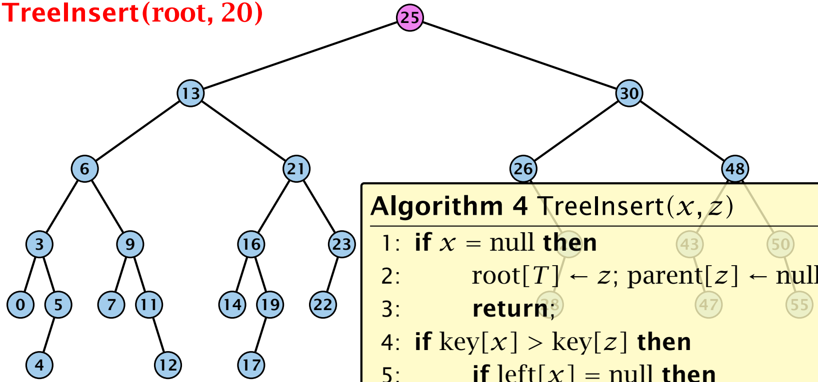
TreeInsert(root, 20)



Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

if $x = \text{null}$ **then**
    $\text{root}[T] \leftarrow z$; $\text{parent}[z] \leftarrow \text{null}$;
    return;
if $\text{key}[x] > \text{key}[z]$ **then**
    if $\text{left}[x] = \text{null}$ **then**
        $\text{left}[x] \leftarrow z$; $\text{parent}[z] \leftarrow x$;
    else TreeInsert$(\text{left}[x], z)$;
else
    if $\text{right}[x] = \text{null}$ **then**
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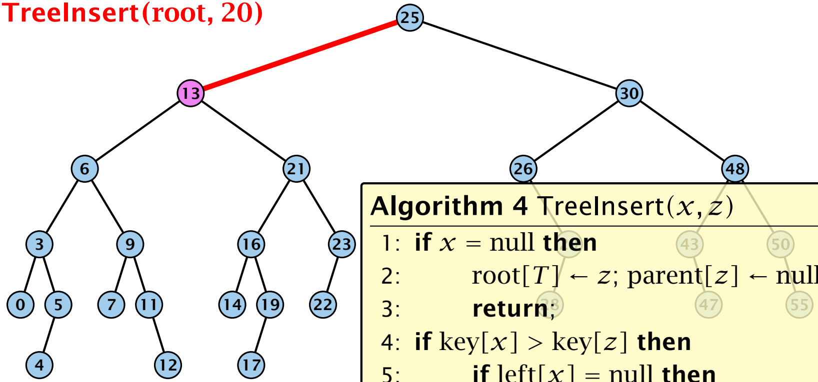
Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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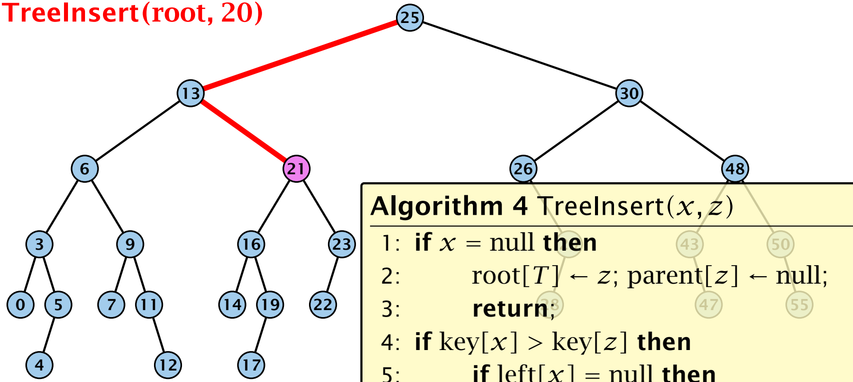
Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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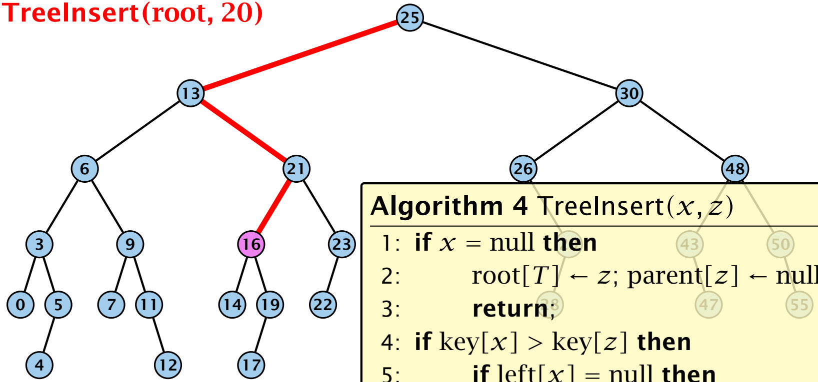
Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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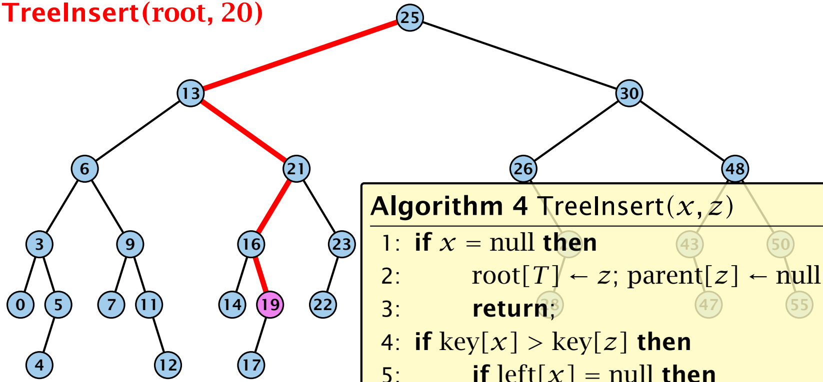
Insert element **not** in the tree.

TreeInsert(root, 20)



Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

```
if x = null then
    root[T] ← z; parent[z] ← null;
    return;
if key[x] > key[z] then
    if left[x] = null then
        left[x] ← z; parent[z] ← x;
    else TreeInsert(left[x], z);
else
    if right[x] = null then
        right[x] ← z; parent[z] ← x;
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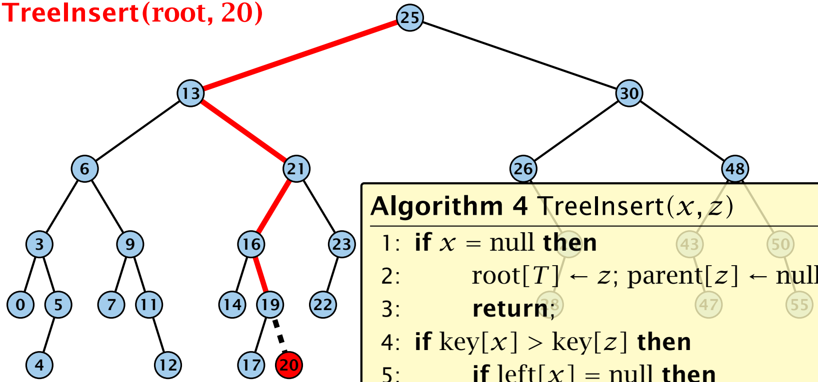
Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)



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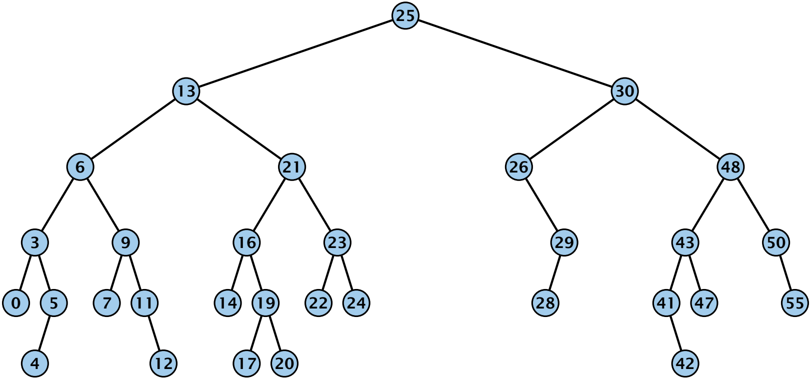
Binary Search Trees: Delete



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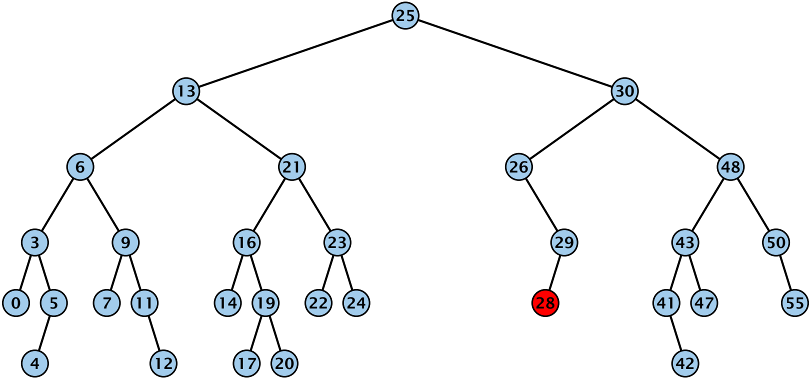
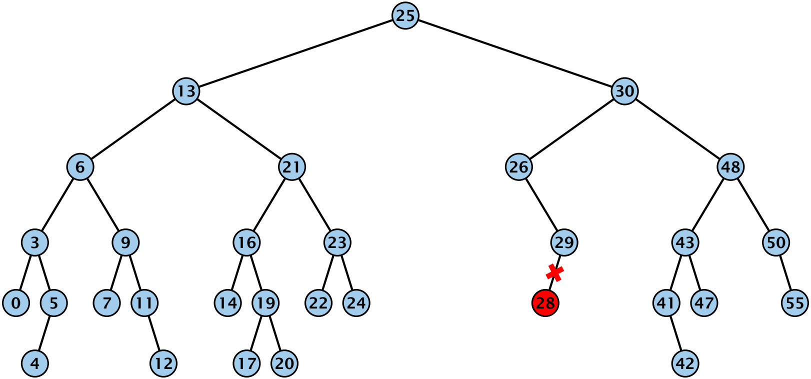


Case 1:

Element does not have any children

- Simply go to the parent and set the corresponding pointer to null.

Binary Search Trees: Delete



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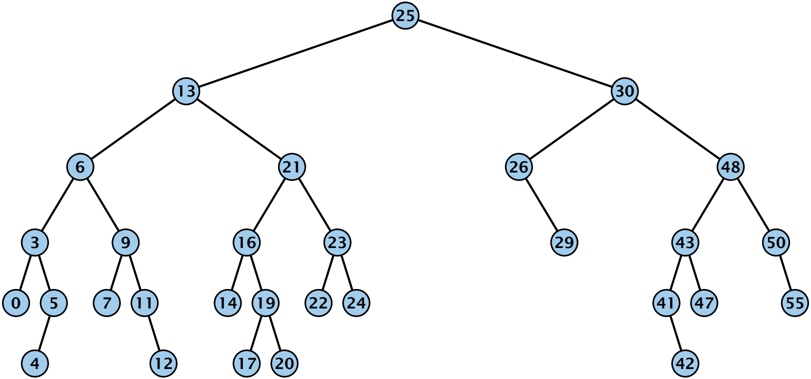

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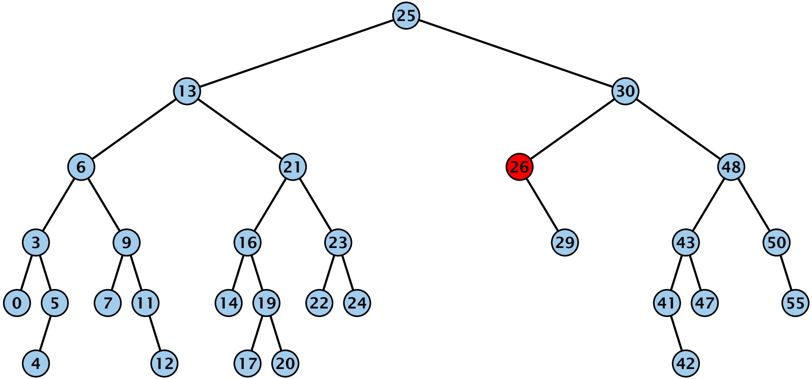
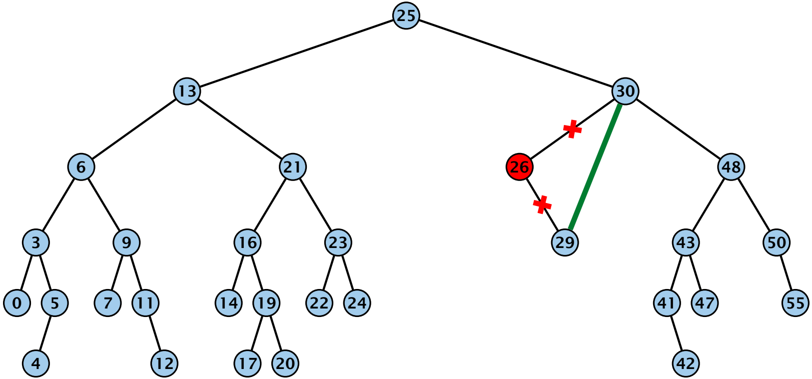


Case 2:

Element has exactly one child

- Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete



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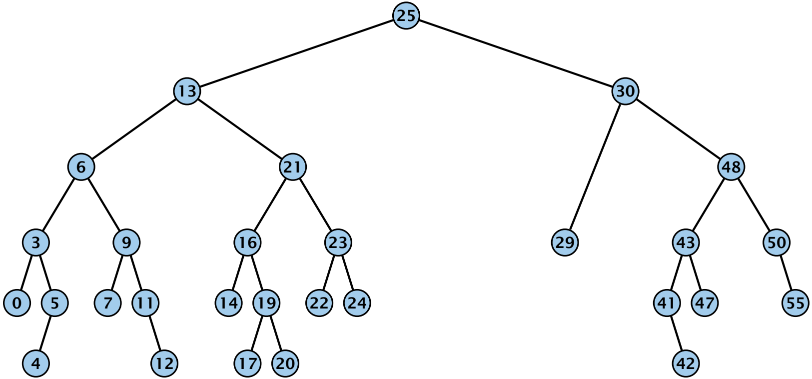

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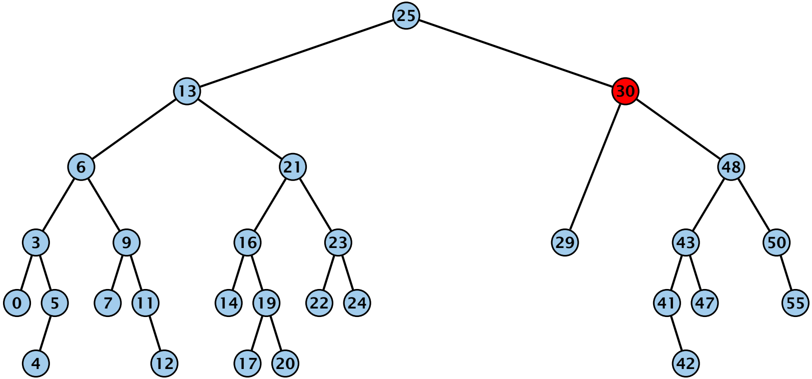

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Element has two children

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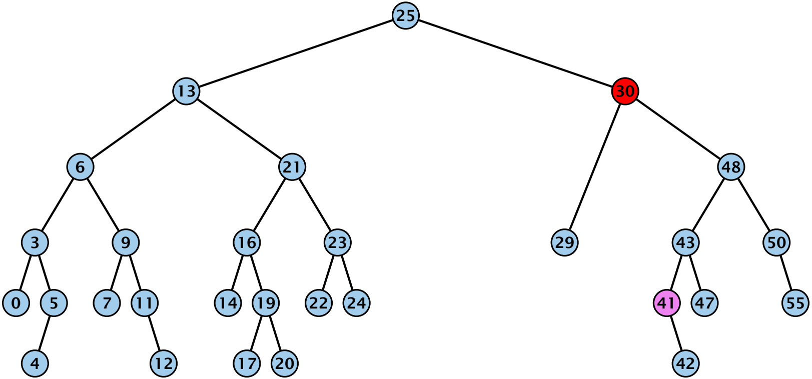
- Replace content of element by content of successor

Binary Search Trees: Delete



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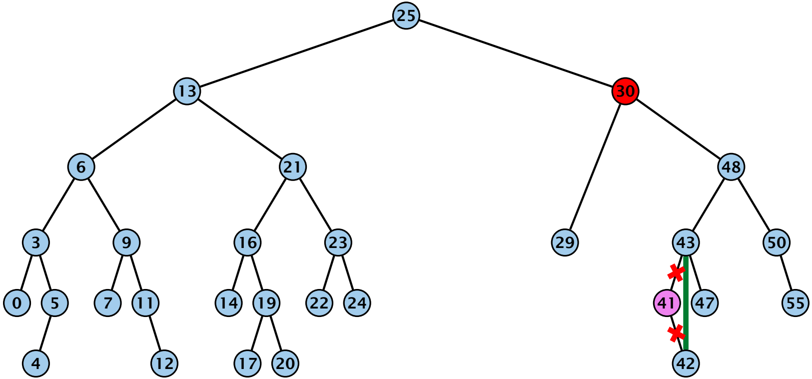

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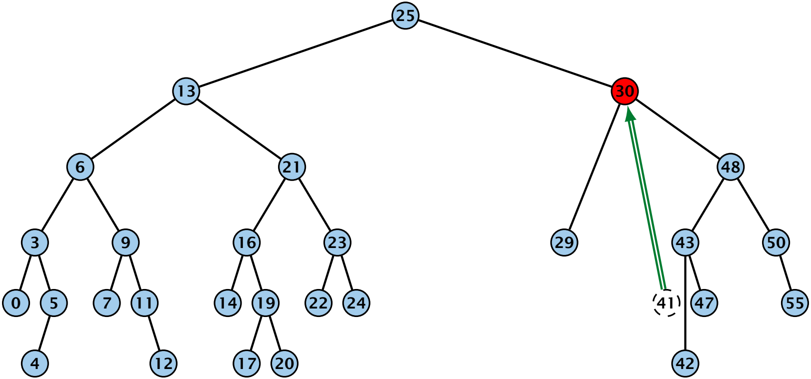

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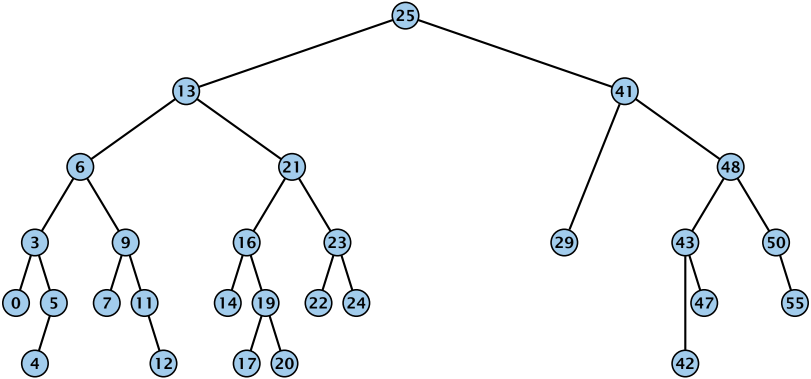

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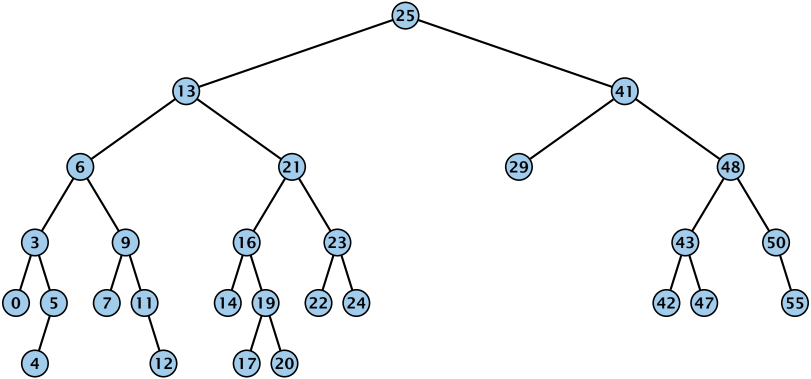

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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

if left[z] $=$ null **or** right[z] $=$ null
  then $y \leftarrow z$ **else** $y \leftarrow$ TreeSucc(z);   select y to splice out
if left[y] $\neq$ null
  then $x \leftarrow$ left[y] **else** $x \leftarrow$ right[y]; x is child of y (or null)
if $x \neq$ null **then** parent[x] $\leftarrow$ parent[y];   parent[x] is correct
if parent[y] $=$ null **then**
  root[T] $\leftarrow x$
else
  if $y =$ left[parent[y]] **then**
    left[parent[y]] $\leftarrow x$
  else
    right[parent[y]] $\leftarrow x$
if $y \neq z$ **then** copy y-data to z

fix pointer to x

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees
With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

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7.2 Red Black Trees

Definition 1

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data

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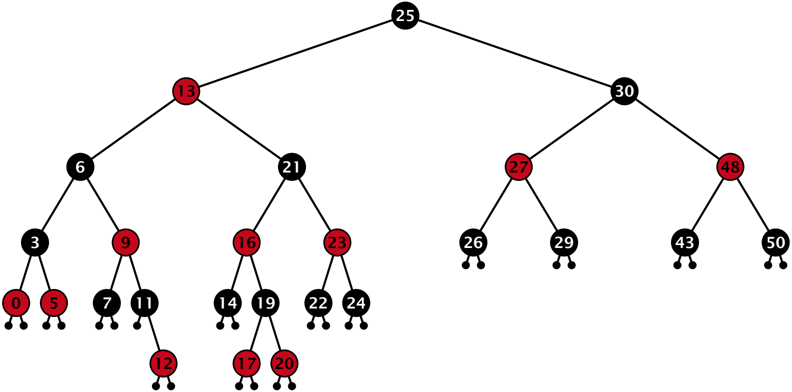
Red Black Trees: Example



7.2 Red Black Trees

Lemma 2

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

Definition 3

The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 4

A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

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Proof of Lemma 4.

Induction on the height of v.

base case ($\mathrm{height}(v) = 0$)

- If height(v) (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.
- The black height of v is 0.
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Proof (cont.)

induction step

- Supose v is a node with $\text{height}(v) > 0$.
- v has two children with strictly smaller height.
- These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
- Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

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Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

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The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

Hence, $h \leq 2\log(n+1) = \mathcal{O}(\log n)$. $\square$

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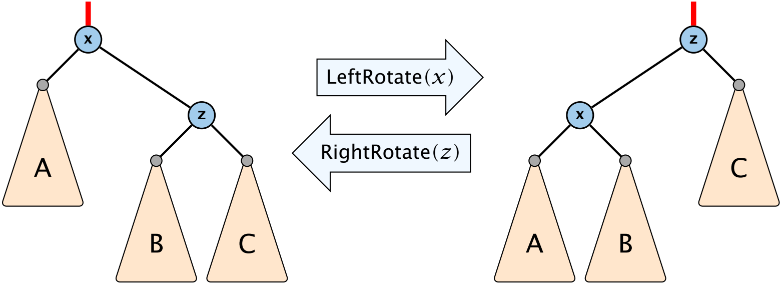
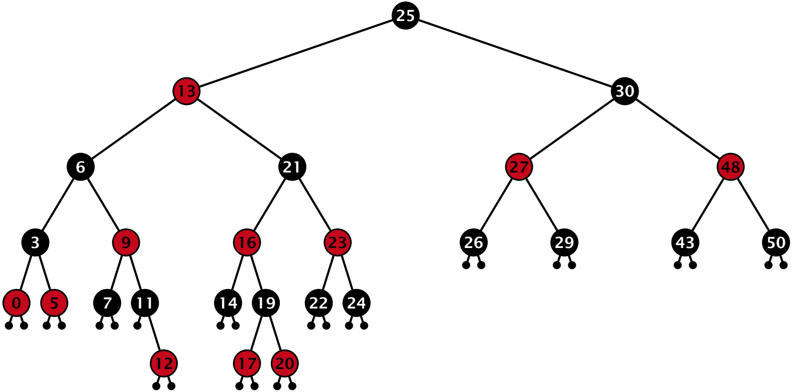
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

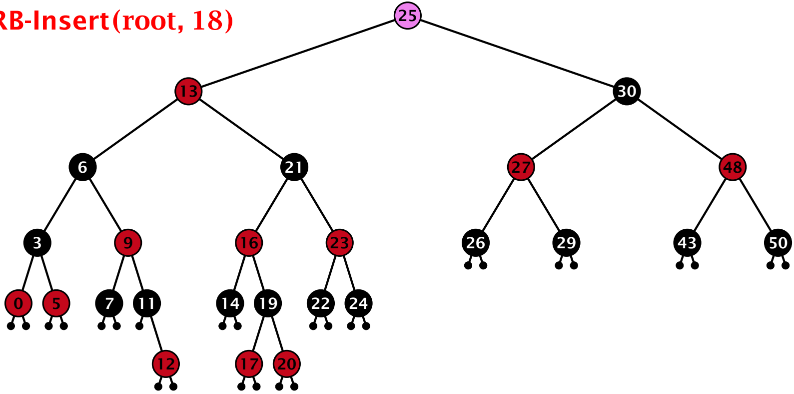


Insert:

- first make a normal insert into a binary search tree
- then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)



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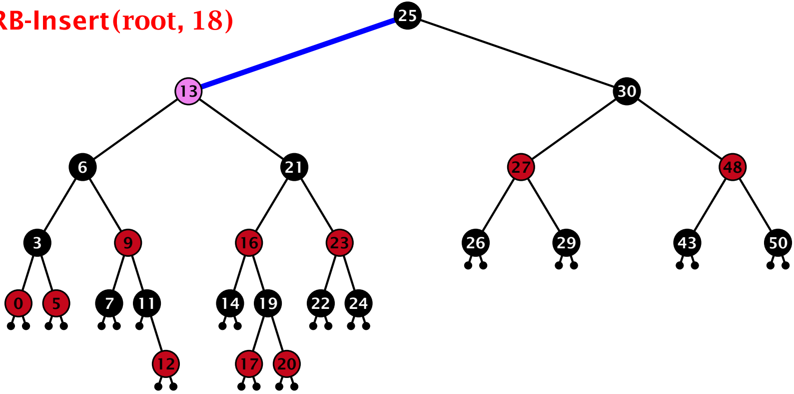

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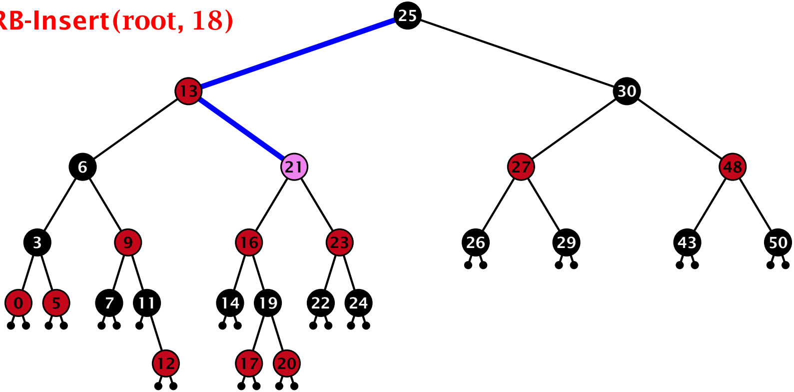

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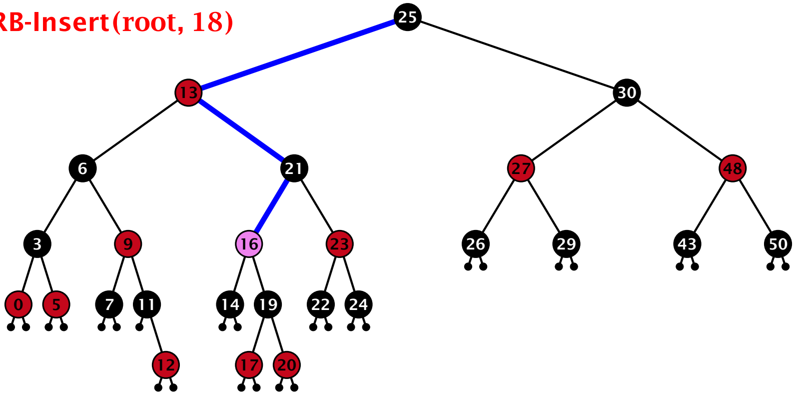

Insert:

- first make a normal insert into a binary search tree
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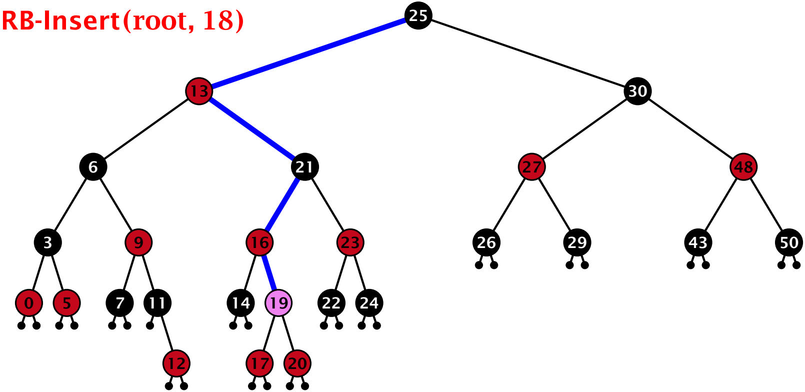

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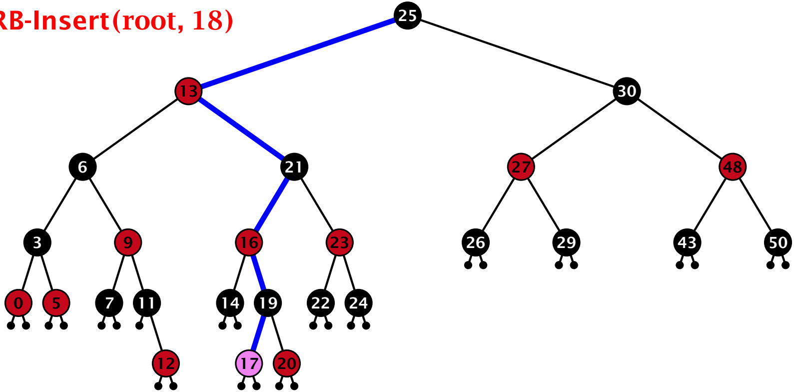

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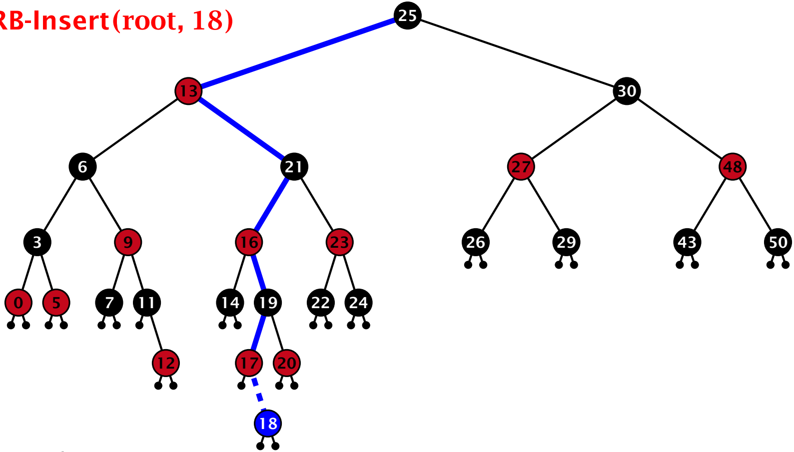

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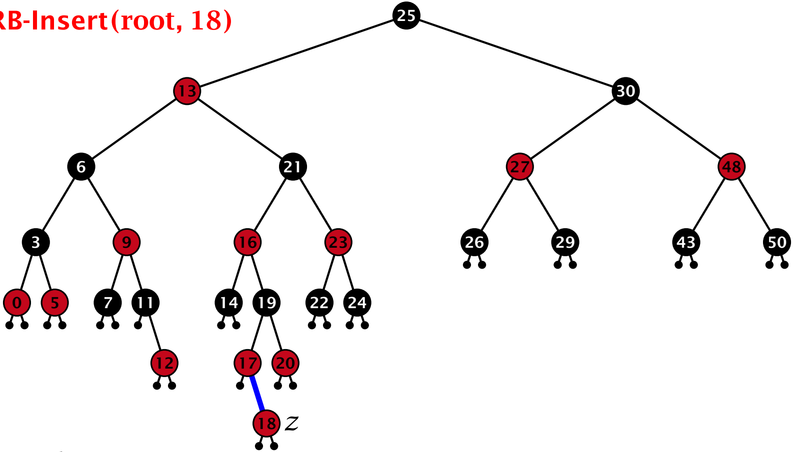

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Red Black Trees: Insert

Invariant of the fix-up algorithm:

- z is a red node
- the black-height property is fulfilled at every node
- the only violation of red-black properties occurs at z and $\text{parent}[z]$
  - either both of them are red (most important case)
  - or the parent does not exist (violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
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Red Black Trees: Insert

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while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then        z in left subtree of grandparent
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        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
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        else                                        Case 2: uncle black
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            col[p[z]] ← black; col[gp[z]] ← red;    2b: z left child
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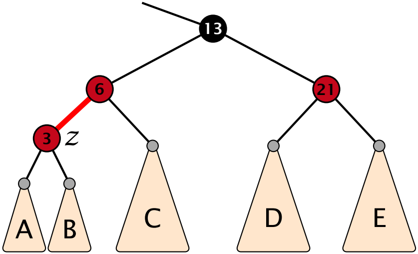
Case 1: Red Uncle



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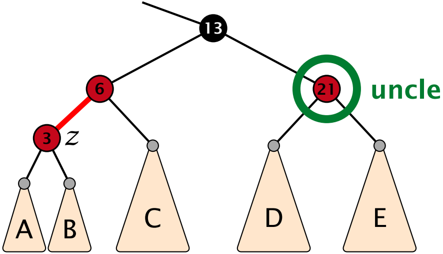

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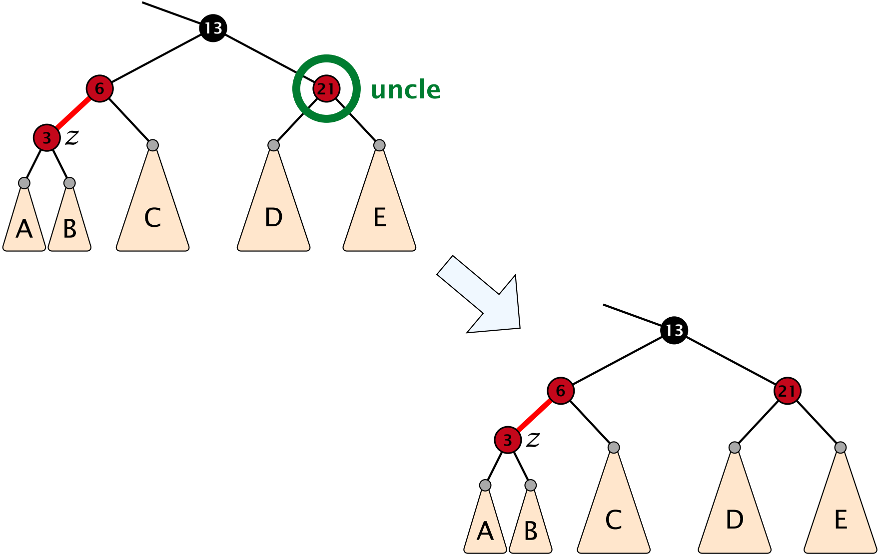

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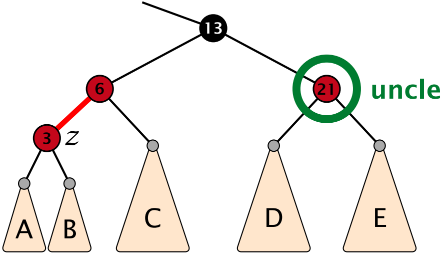
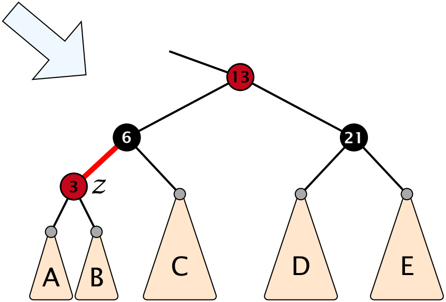
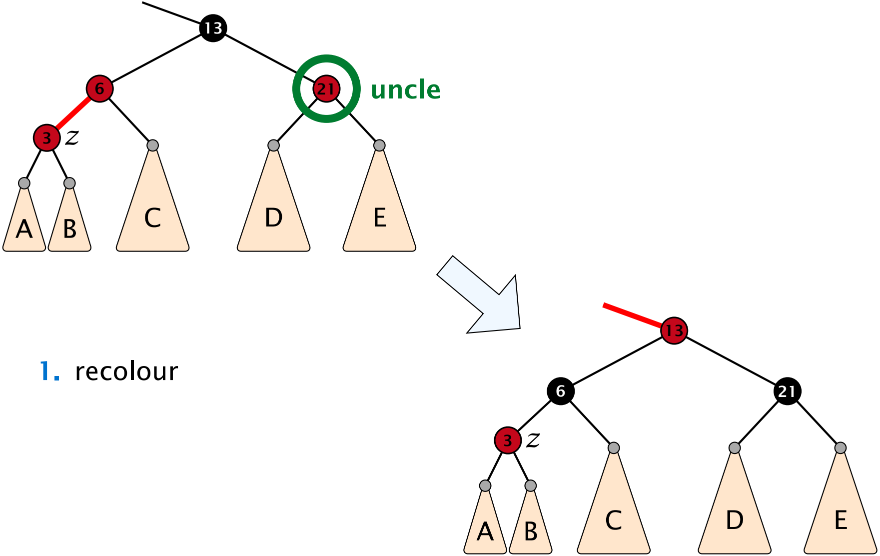




1. recolour

Case 1: Red Uncle



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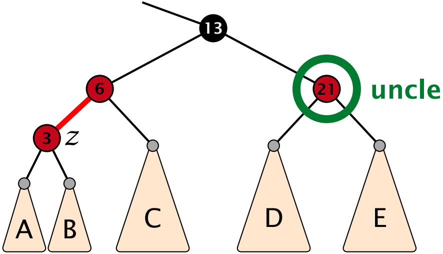
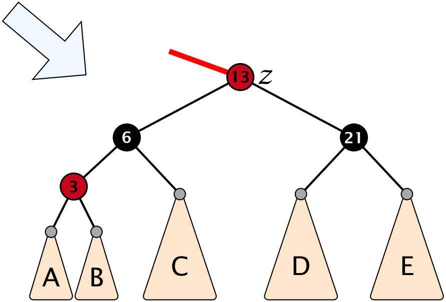
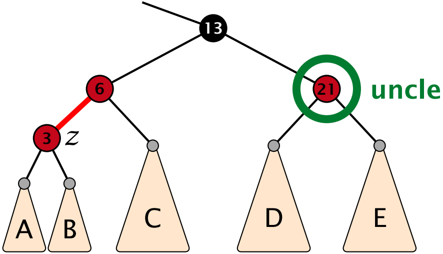




1. recolour
2. move z to grand-parent

Case 1: Red Uncle



1. recolour
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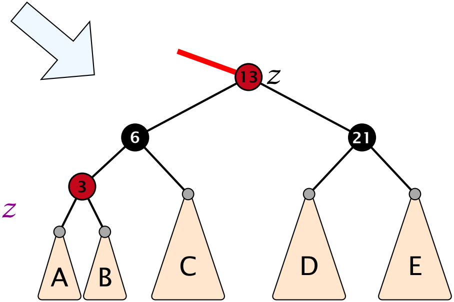
3. invariant is fulfilled for new z



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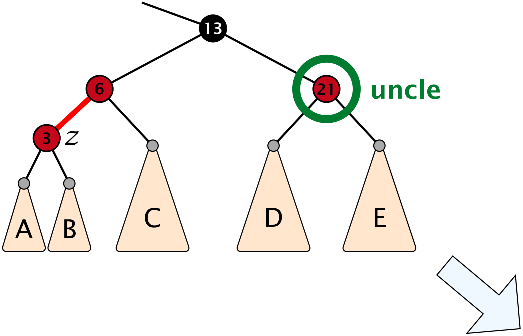
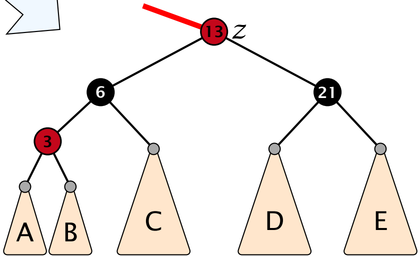
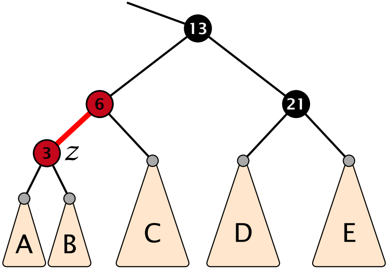




1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



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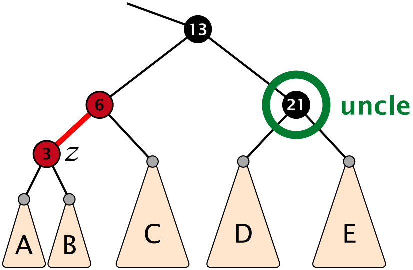

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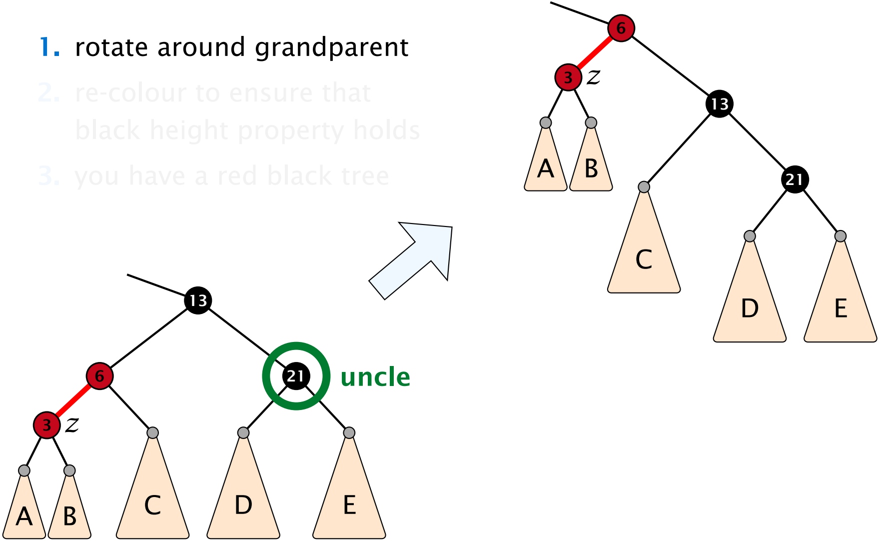

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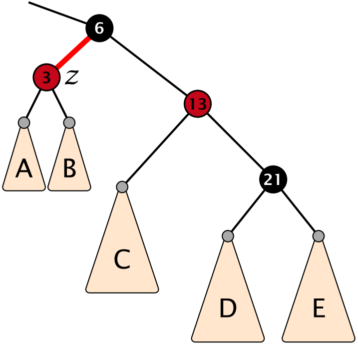

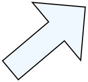

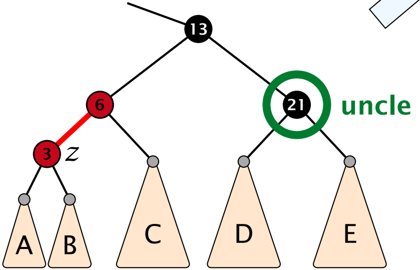

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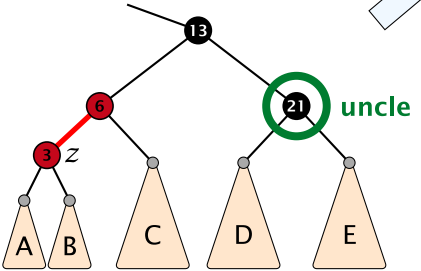
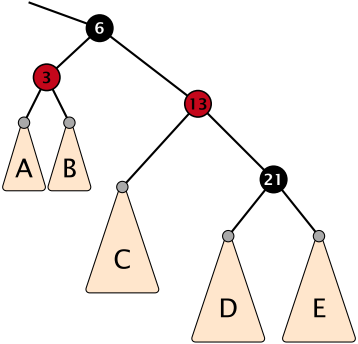
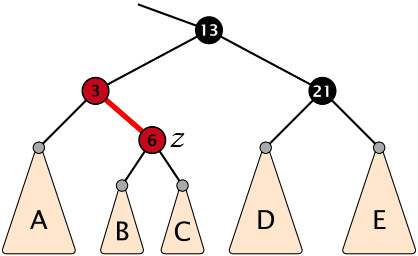




Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



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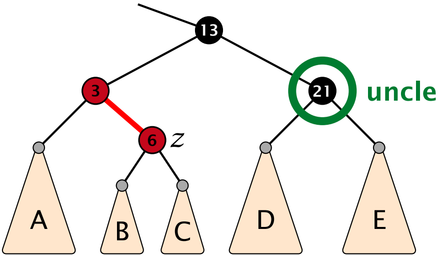

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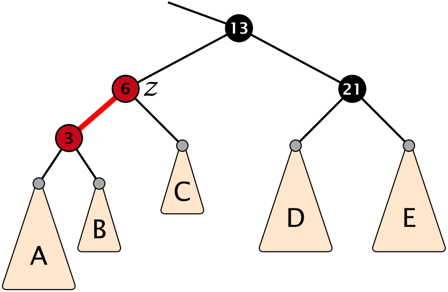

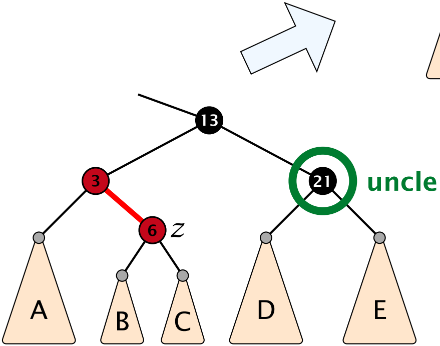

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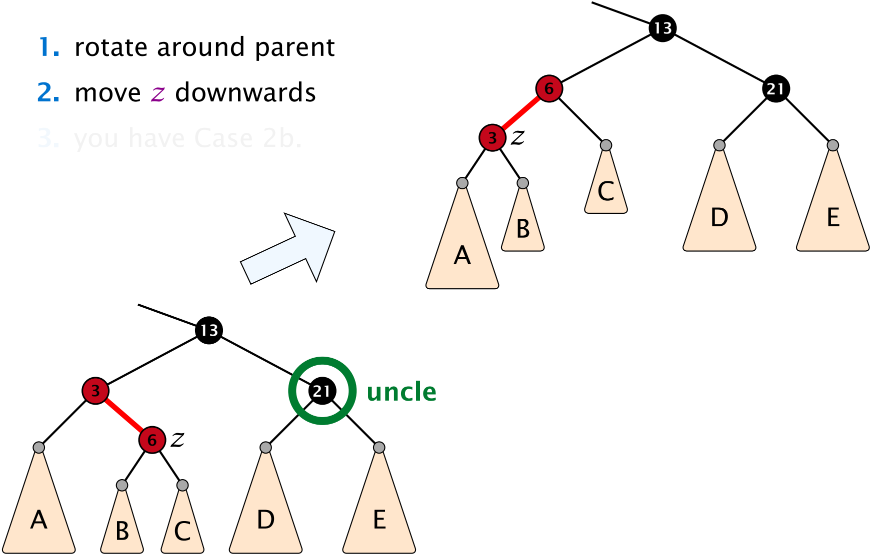

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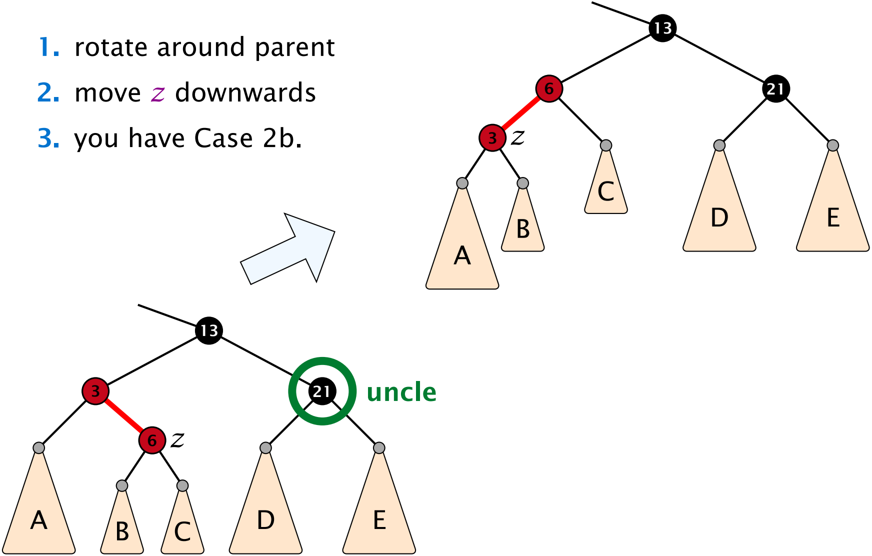

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a → Case 2b → red-black tree
- Case 2b → red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

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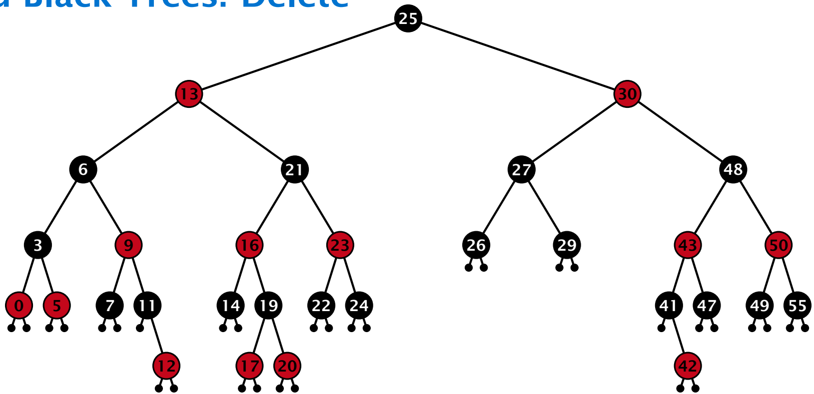

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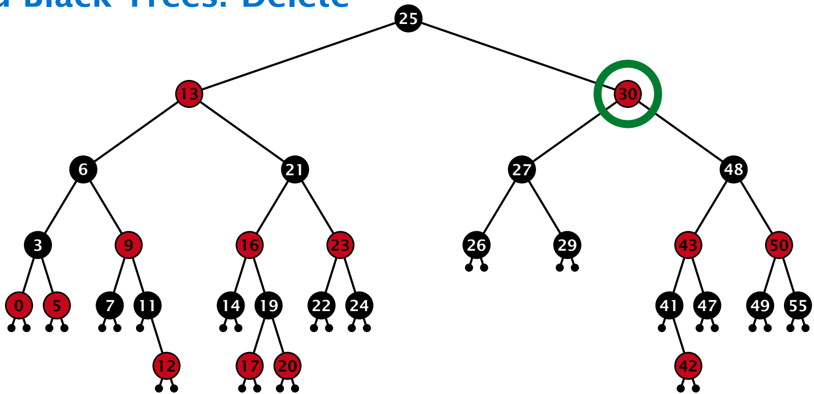

Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

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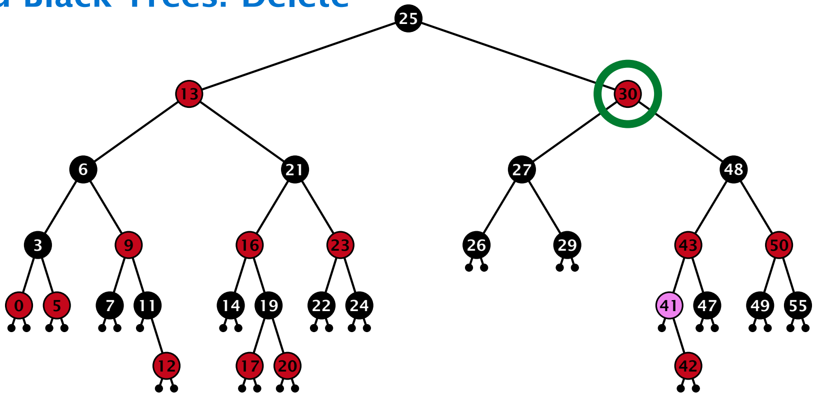

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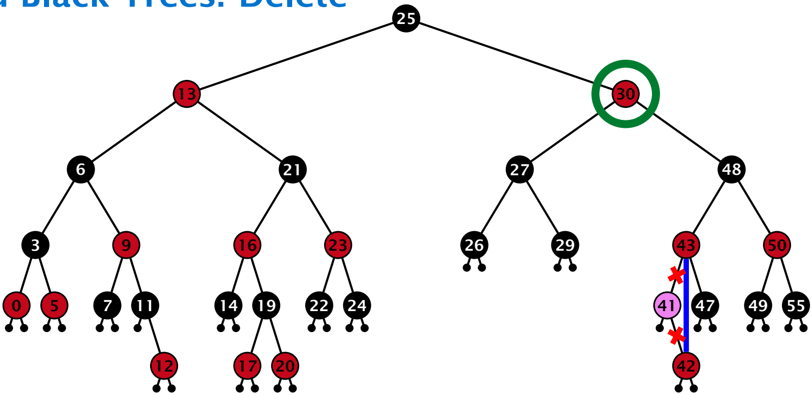

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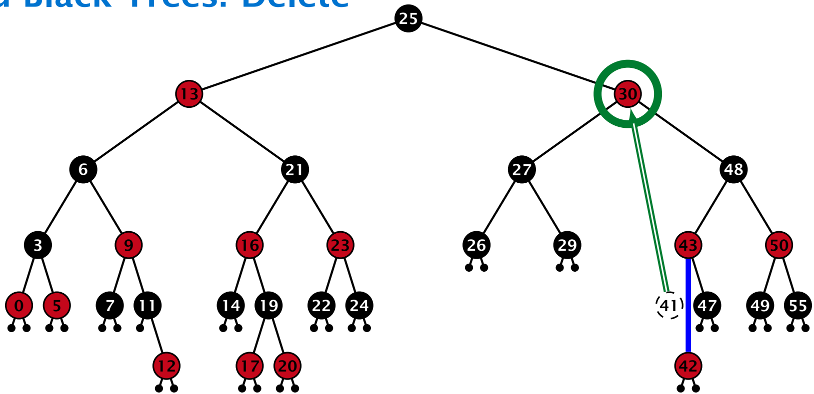

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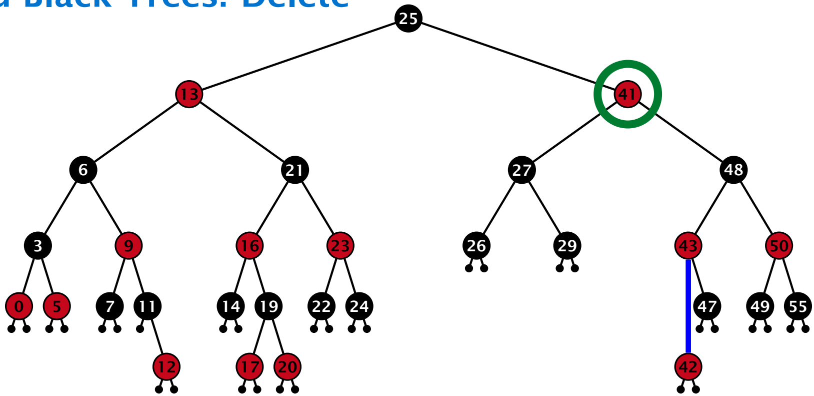

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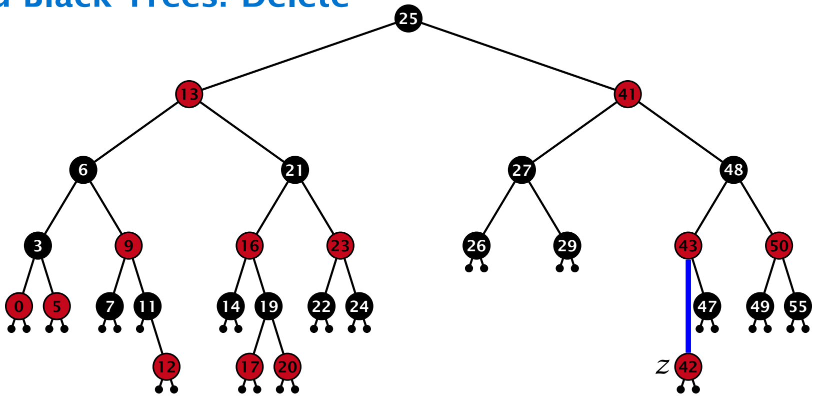
Red Black Trees: Delete



Delete:

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- if z is red, we can simply color it black and everything is fine
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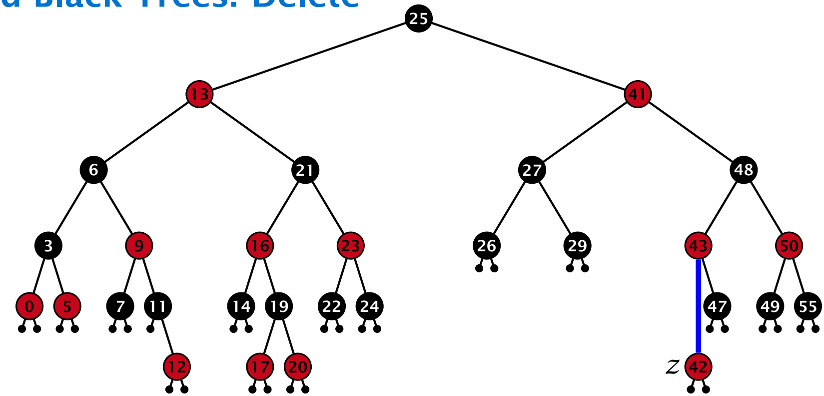

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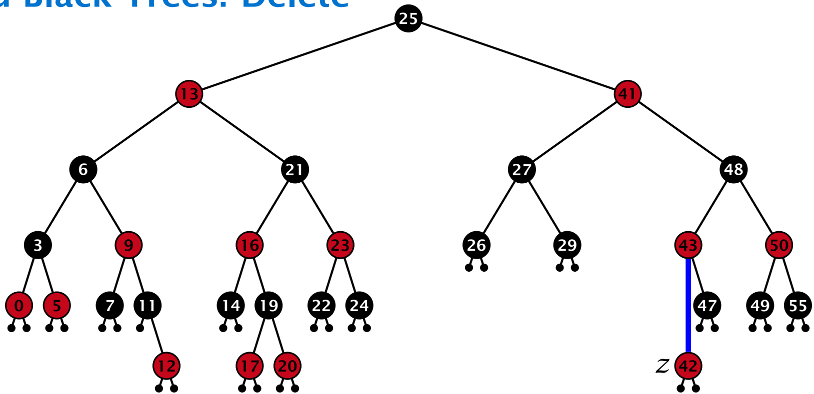

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Invariant of the fix-up algorithm

- the node z is black
- if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

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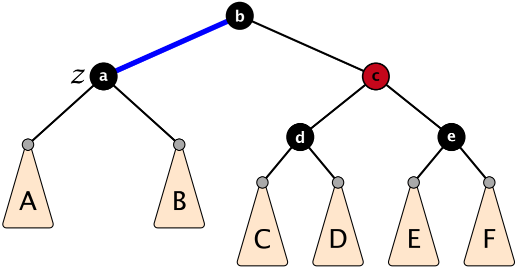

1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
4. Case 2 (special),
   or Case 3, or Case 4

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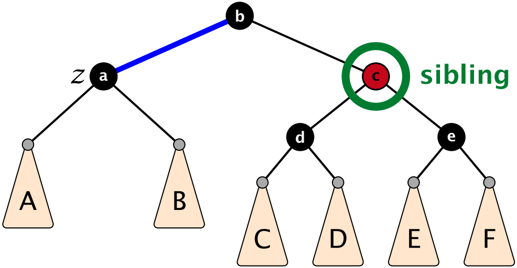

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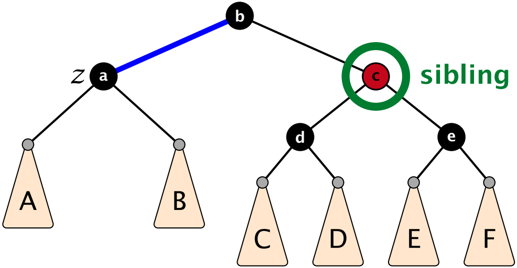

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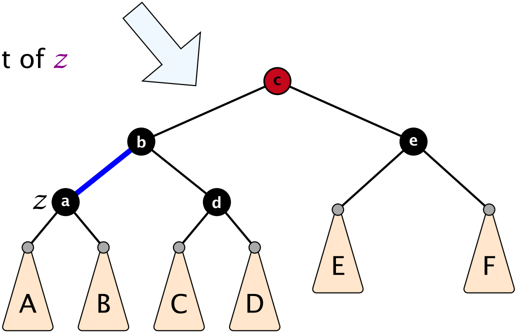

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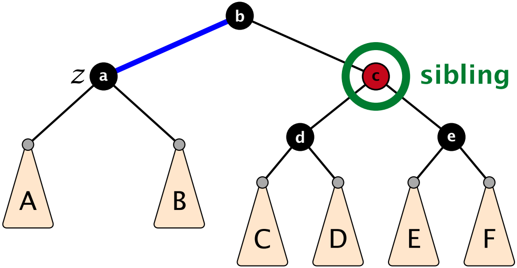

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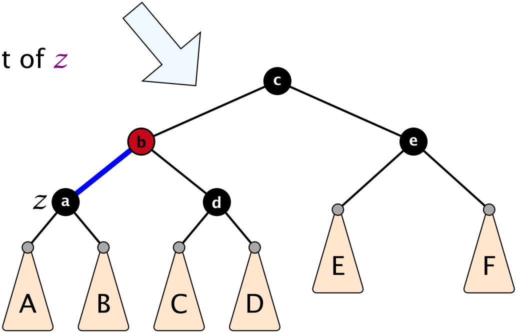

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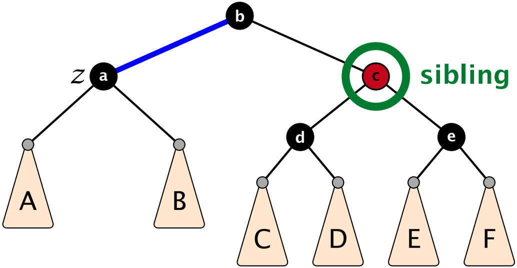

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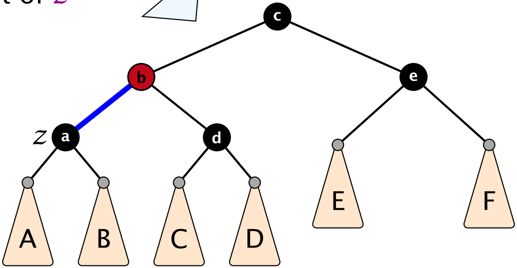

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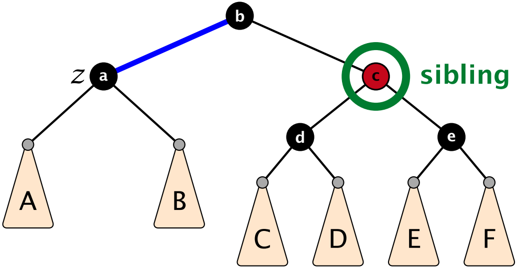

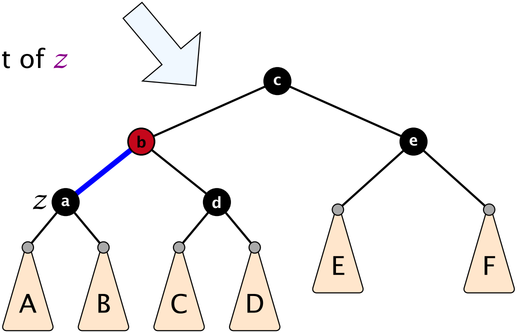

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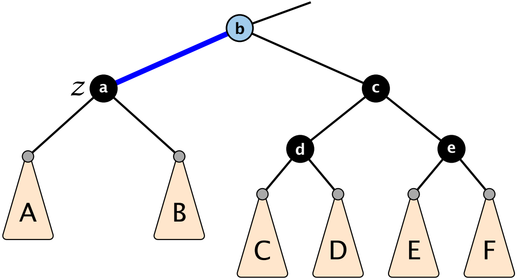

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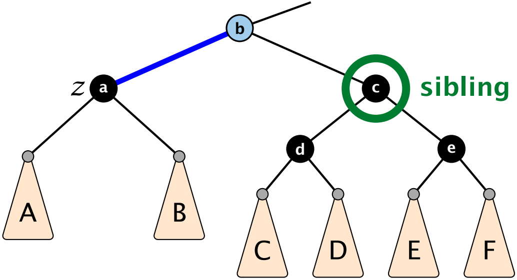

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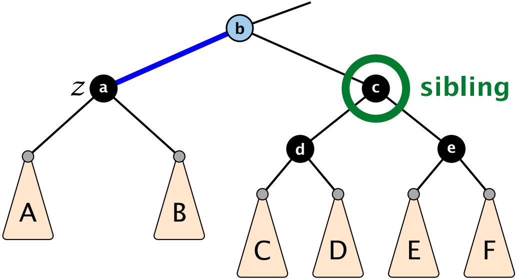

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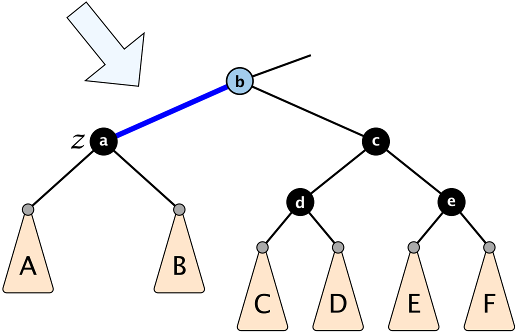

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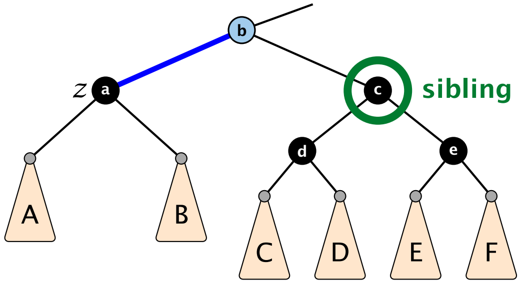

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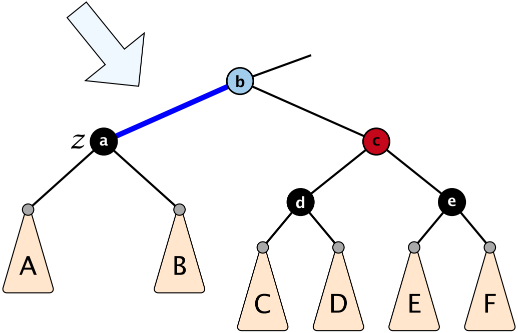

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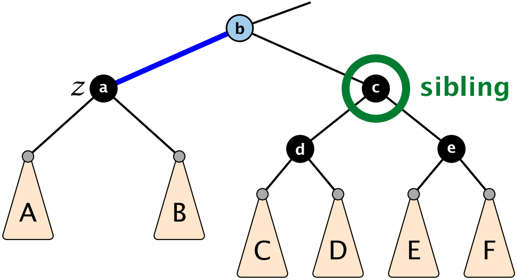

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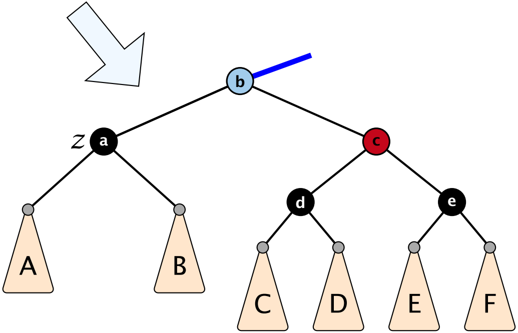

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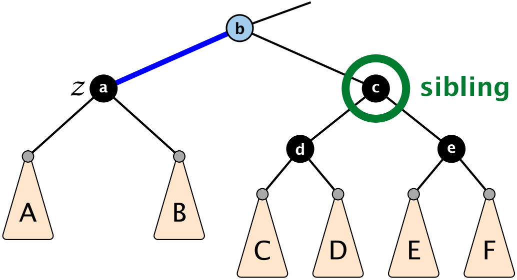

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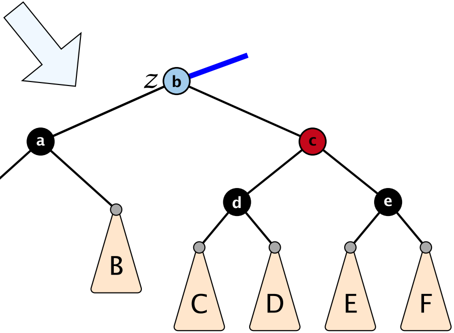

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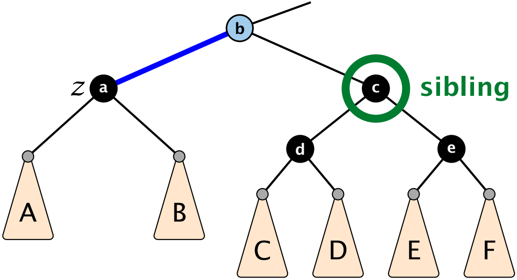

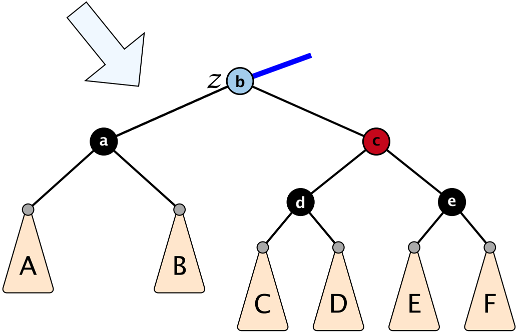

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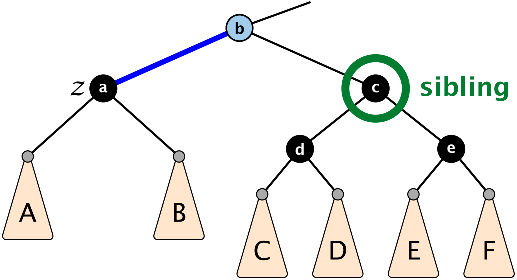

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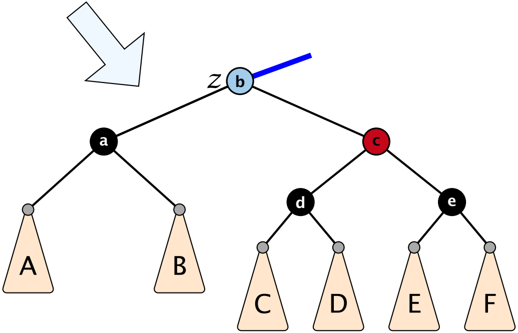

Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
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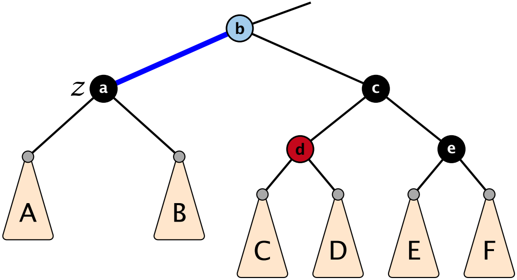

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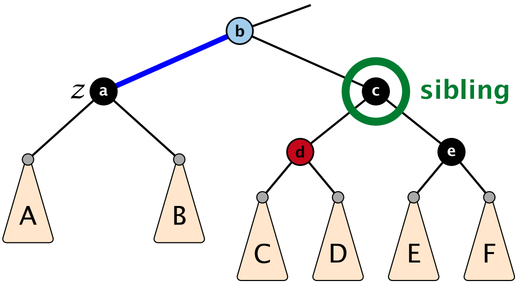

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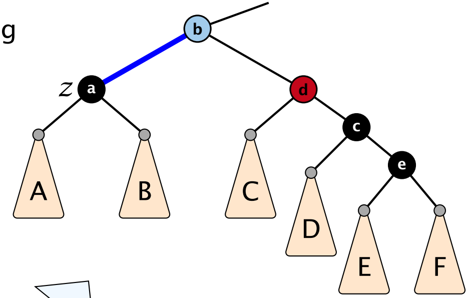

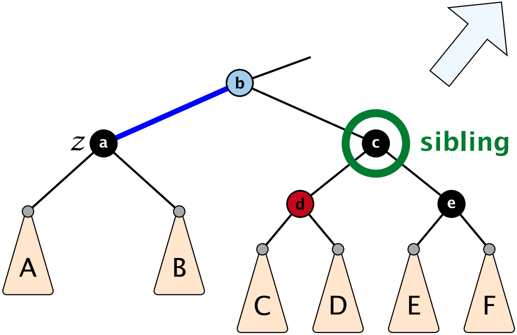

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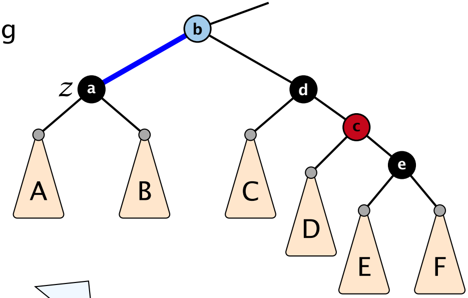

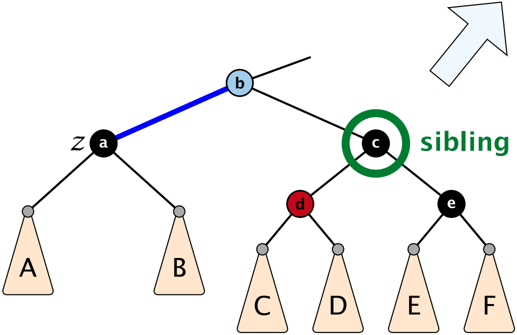

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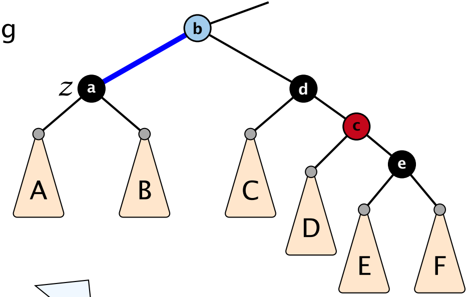

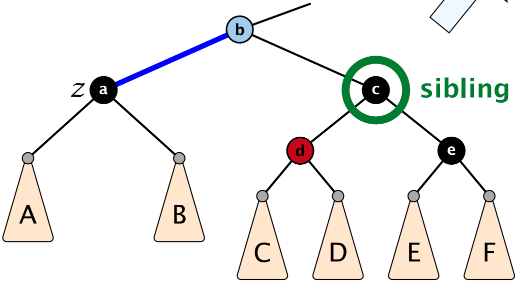

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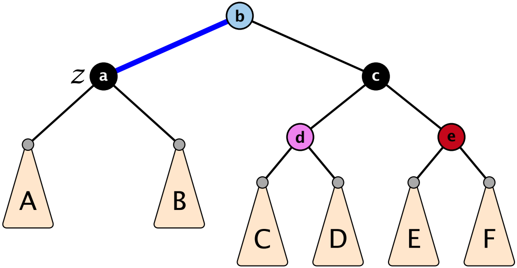

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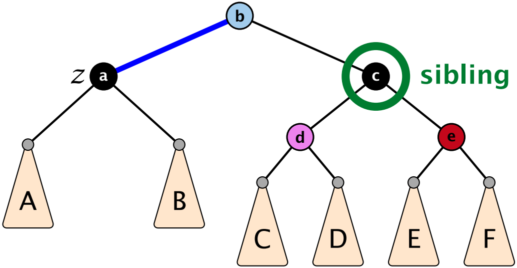

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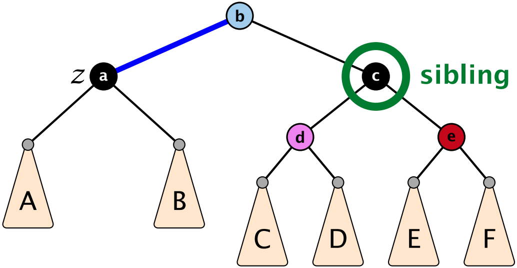

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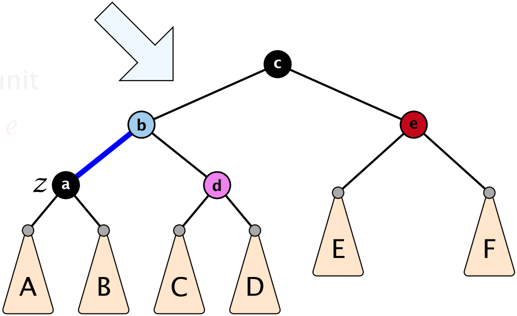

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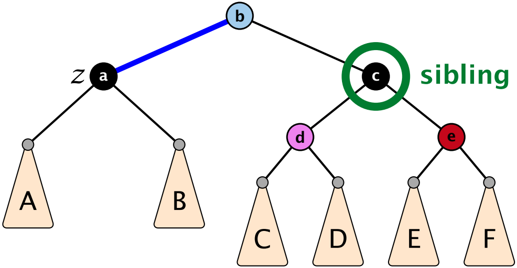

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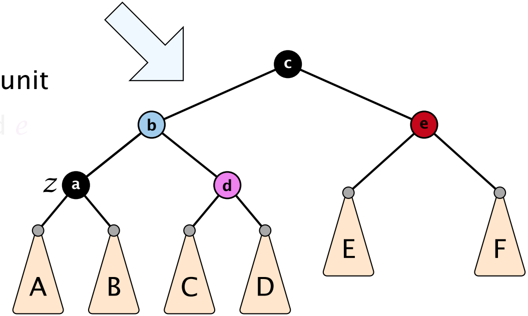

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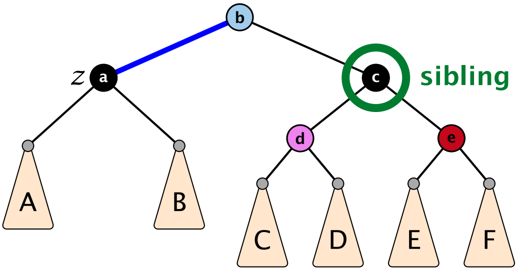

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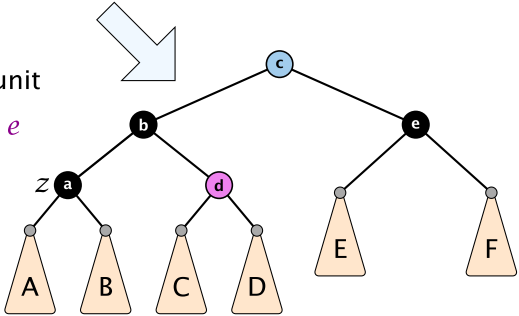

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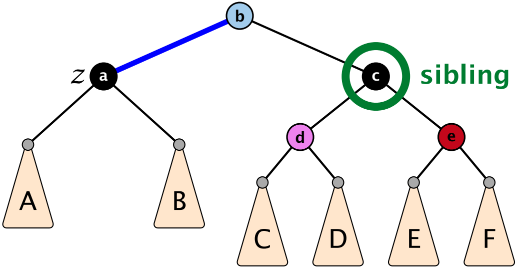

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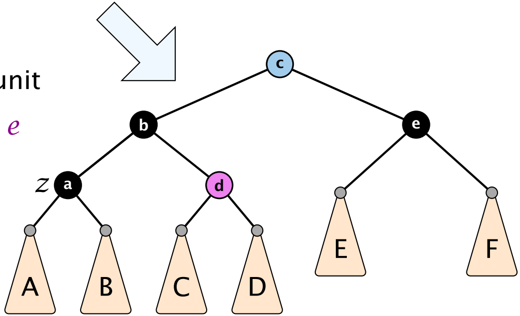

Running time:

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Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

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Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

Splay Trees:

- after access, an element is moved to the root; splay(x)
- repeated accesses are faster
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Splay Trees

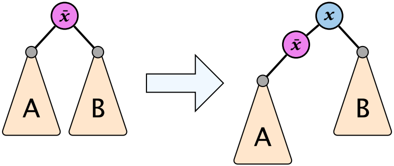
find(x)

- search for x according to a search tree
- let $\bar{x}$ be last element on search-path
- splay$(\bar{x})$

Splay Trees

insert(x)

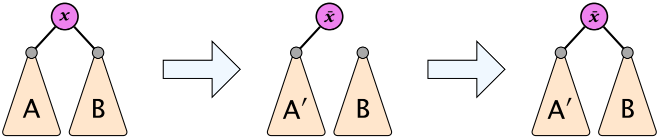
- search for x; $\bar{x}$ is last visited element during search (successer or predecessor of x)
- splay($\bar{x}$) moves $\bar{x}$ to the root
- insert x as new root



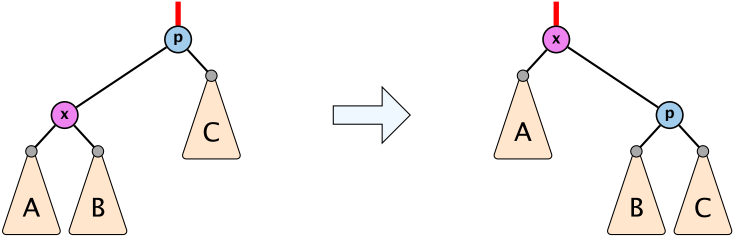
Splay Trees

delete(x)

- search for x; splay(x); remove x
- search largest element $\bar{x}$ in A
- splay($\bar{x}$) (on subtree A)
- connect root of B as right child of $\bar{x}$



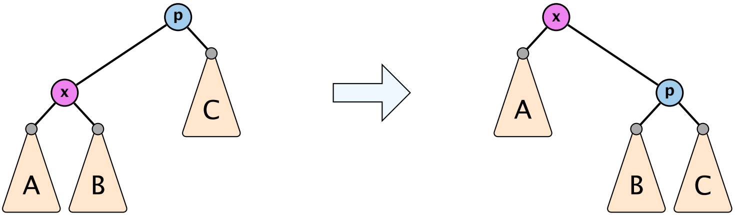
Move to Root



How to bring element to root?

- one (bad) option: moveToRoot(x)
- iteratively do rotation around parent of x until x is root
- if x is left child do right rotation otw. left rotation

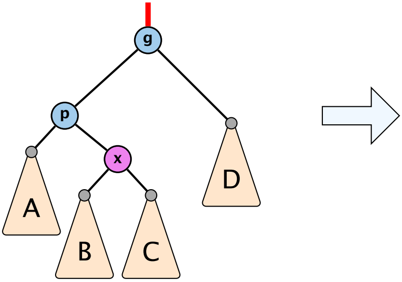
Splay: Zig Case



better option splay(x):

- zig case: if x is child of root do left rotation or right rotation around parent

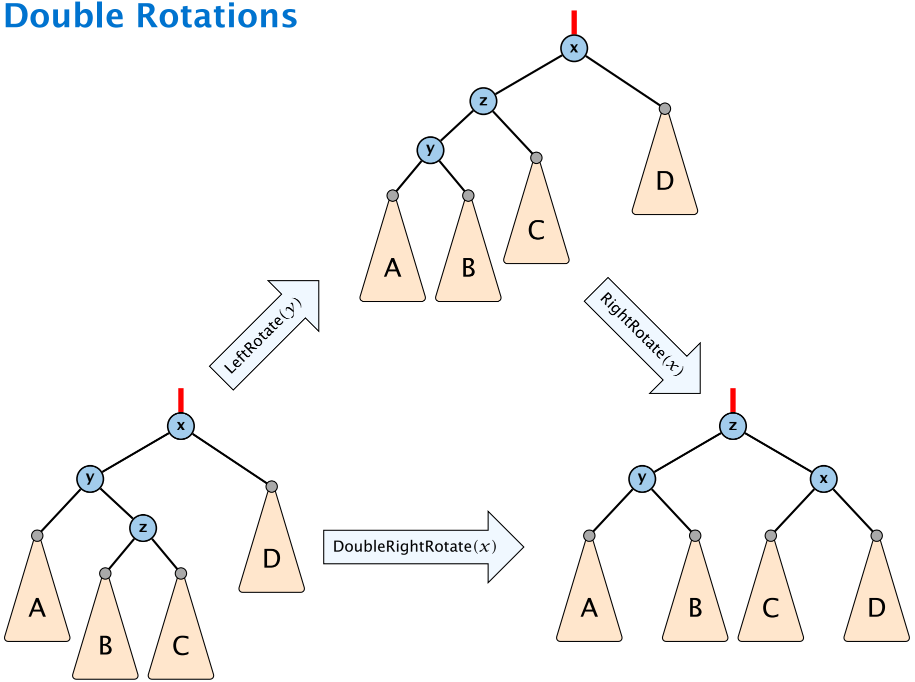
Splay: Zigzag Case



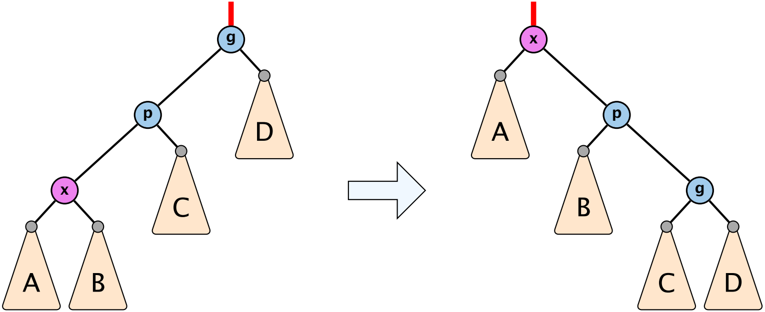
better option splay(x):

- zigzag case: if x is right child and parent of x is left child (or x left child parent of x right child)
- do double right rotation around grand-parent (resp. double left rotation)

Double Rotations



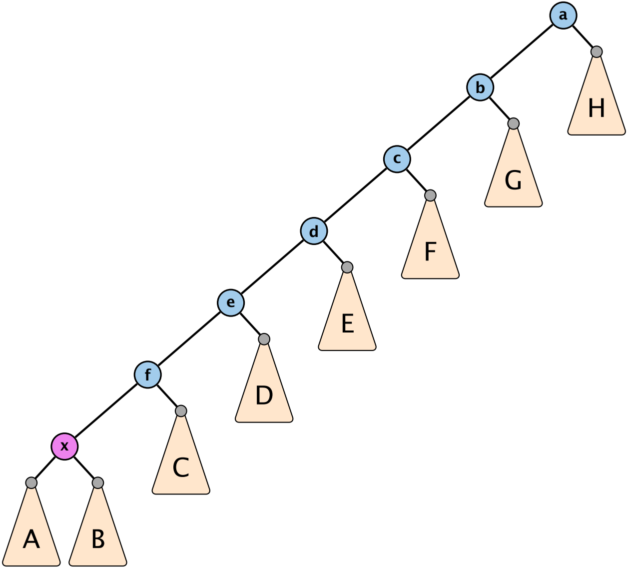
Splay: Zigzig Case



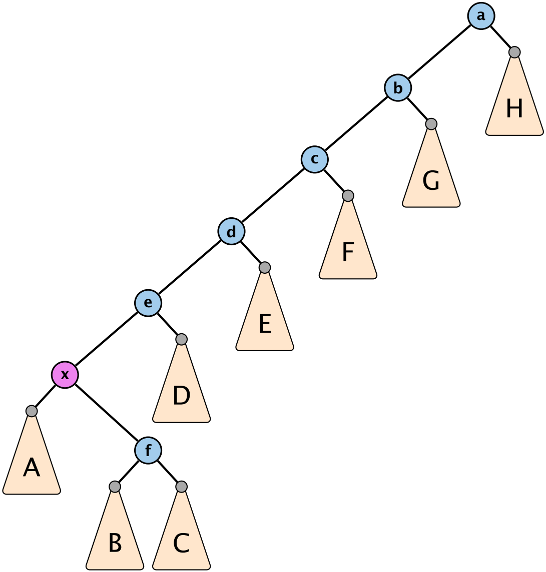
better option splay(x):

- zigzig case: if x is left child and parent of x is left child (or x right child, parent of x right child)
- do right roation around grand-parent followed by right rotation around parent (resp. left rotations)

Splay vs. Move to Root



Splay vs. Move to Root



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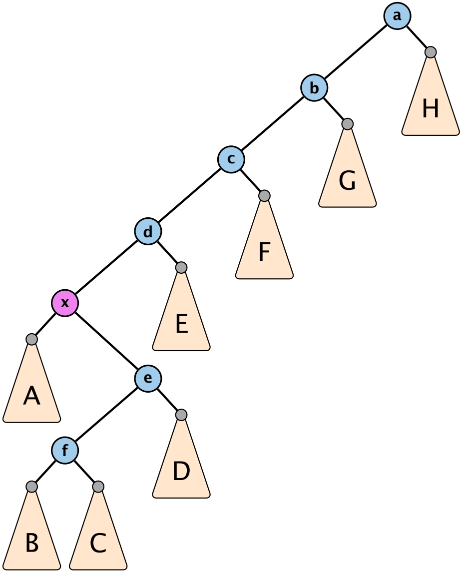

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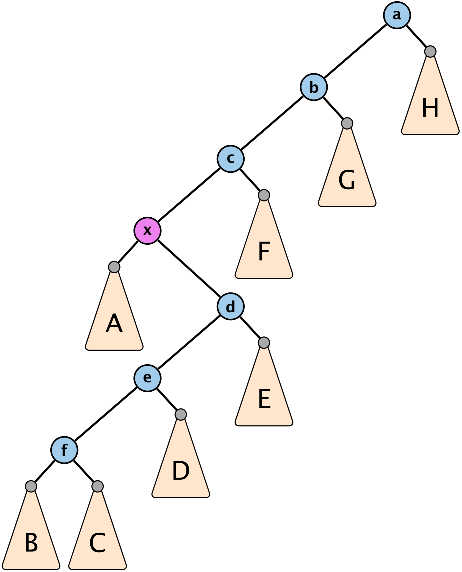

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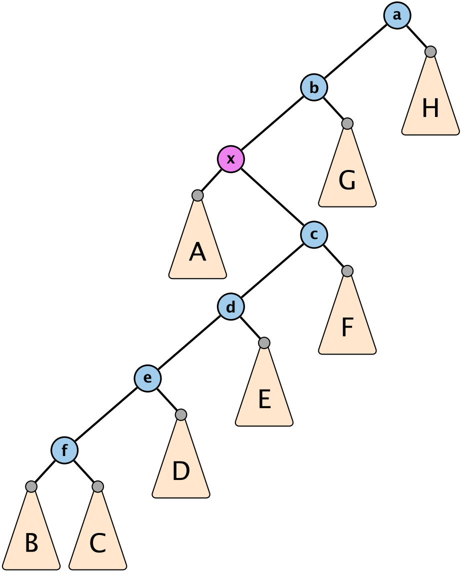

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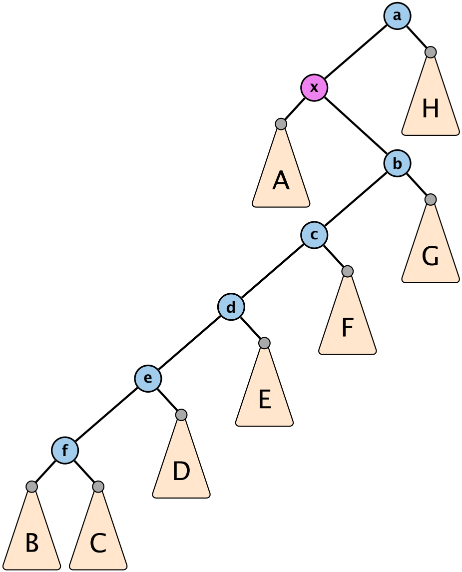

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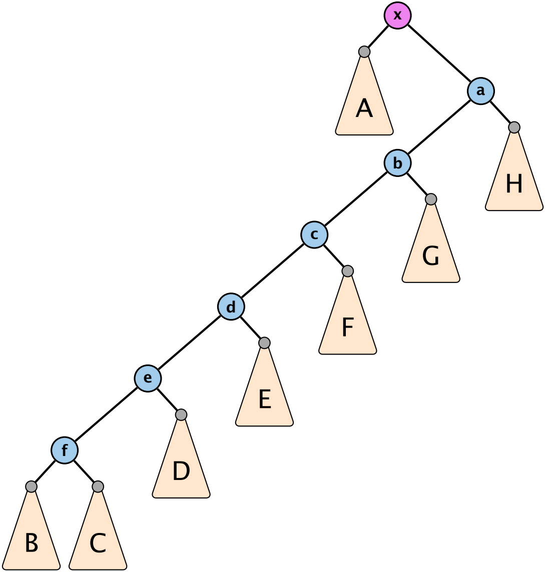

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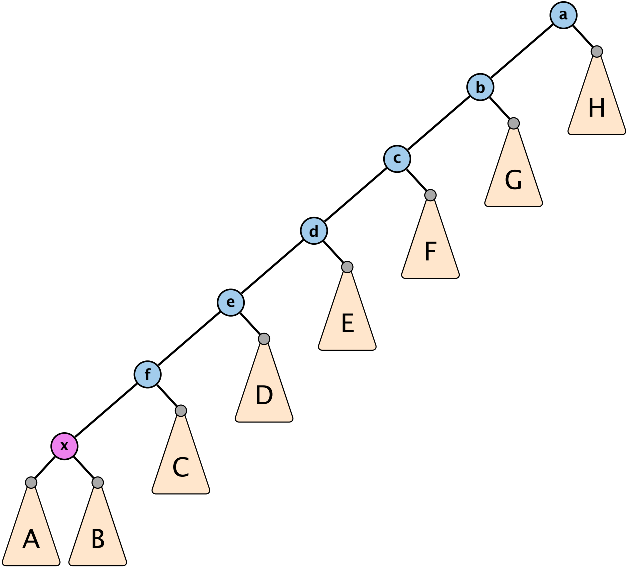

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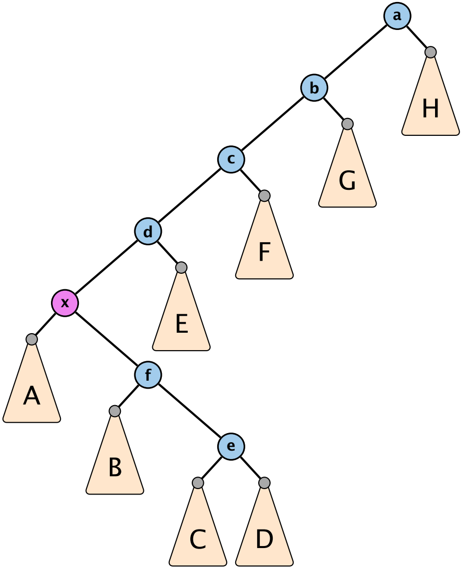

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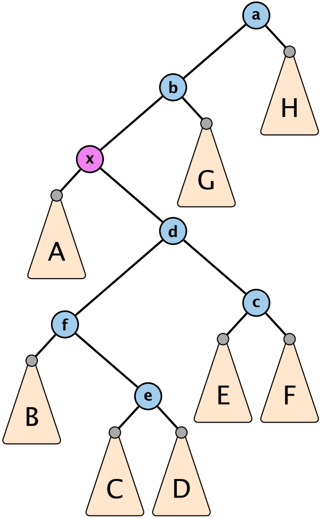

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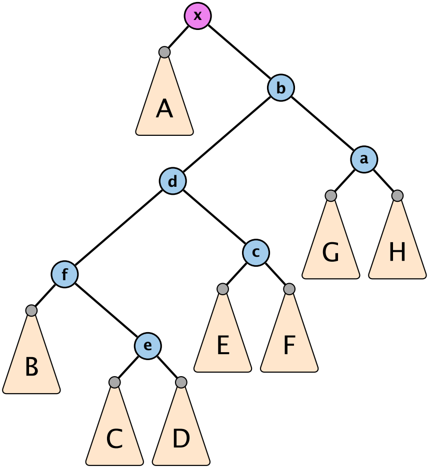

Static Optimality

Suppose we have a sequence of m find-operations. find(x) appears h_x times in this sequence.

The cost of a static search tree T is:

$$\text{cost}(T) = m + \sum_x h_x \,\text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is $\mathcal{O}(\text{cost}(T_{\text{min}}))$, where T_{min} is an optimal static search tree.

Dynamic Optimality

Let S be a sequence with m find-operations.

Let A be a data-structure based on a search tree:

- the cost for accessing element x is $1 + \mathrm{depth}(x)$;
- after accessing x the tree may be re-arranged through rotations;

Conjecture:
A splay tree that only contains elements from S has cost $\mathcal{O}(\mathrm{cost}(A, S))$, for processing S.

Lemma 5

Splay Trees have an amortized running time of $\mathcal{O}(\log n)$ for all operations.

Amortized Analysis

Definition 6

A data structure with operations $\mathrm{op}_1(), \dots, \mathrm{op}_k()$ has amortized running times $t_1, \dots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

Potential Method

Introduce a potential for the data structure.

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Then

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^{k} \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- $S.\,\mathrm{push}()$
- $S.\,\mathrm{pop}()$
- $S.\,\mathrm{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

Actual cost:

- $S.\,\mathrm{push}()$: cost 1.
- $S.\,\mathrm{pop}()$: cost 1.
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Amortized cost:

- $S.\mathbf{push}()$: cost

$$\hat{C}_{\text{push}} = C_{\text{push}} + \Delta\Phi = 1 + 1 \leq 2 \ .$$

- $S.\mathbf{pop}()$: cost

$$\hat{C}_{\text{pop}} = C_{\text{pop}} + \Delta\Phi = 1 - 1 \leq 0 \ .$$

- $S.\mathbf{multipop}(k)$: cost

$$\hat{C}_{\text{mp}} = C_{\text{mp}} + \Delta\Phi = \min\{\text{size}, k\} - \min\{\text{size}, k\} \leq 0 \ .$$

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Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n-bit binary counter may require to examine n-bits, and maybe change them.

Actual cost:

- Changing bit from 0 to 1: cost 1.
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- Increment: cost is $k + 1$, where k is the number of consecutive ones in the least significant bit-positions (e.g, 001101 has $k = 1$).

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Choose potential function $\Phi(x) = k$, where k denotes the number of ones in the binary representation of x.

Amortized cost:

$0 \to 1$:

$$\hat{C}_{0\to 1} = C_{0\to 1} + \Delta\Phi = 1 + 1 \le 2 .$$

$1 \to 0$:

$$\hat{C}_{1\to 0} = C_{1\to 0} + \Delta\Phi = 1 - 1 \le 0 .$$

Let k denotes the number of consecutive ones in the least significant bit-positions. An increment involves k $1 \to 0$ operations, and one $0 \to 1$ operation.

Hence, the amortized cost is

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  Hence, the amortized cost is $k\hat{C}_{1\to0} + \hat{C}_{0\to1} \le 2$.

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Splay Trees

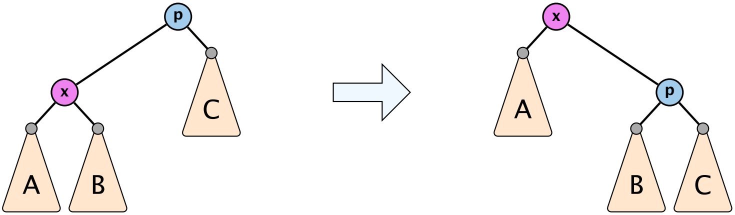
potential function for splay trees:

- size $\mathrm{s}(x) = |T_x|$
- rank $\mathrm{r}(x) = \log_2(s(x))$
- $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

Splay: Zig Case



$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$
$$= r'(p) - r(x)$$
$$\le r'(x) - r(x)$$

$$\text{cost}_{\text{zig}} \le 1 + 3(r'(x) - r(x))$$

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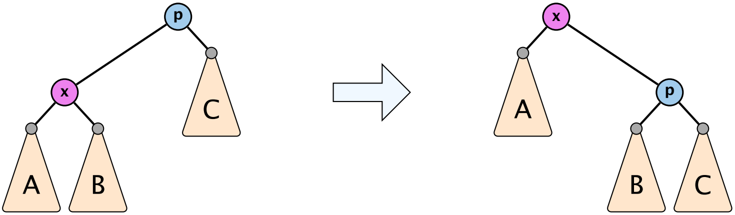

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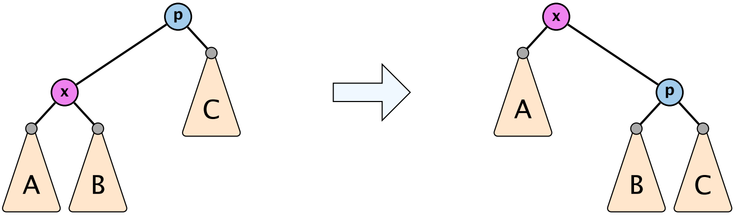

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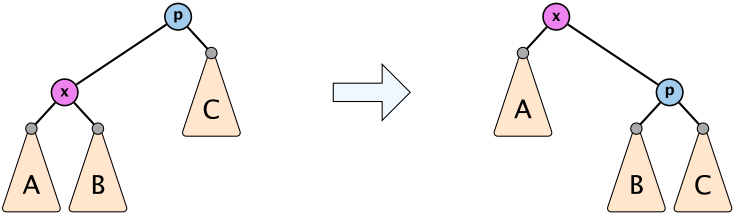

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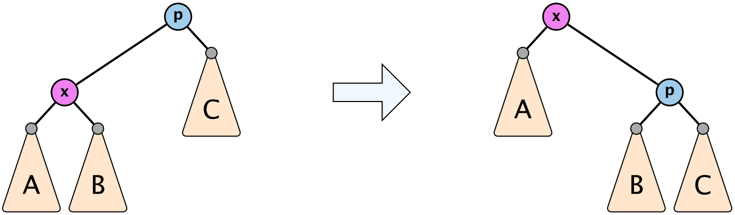

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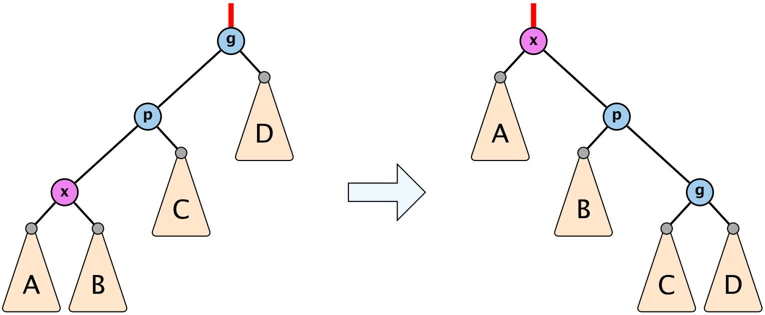

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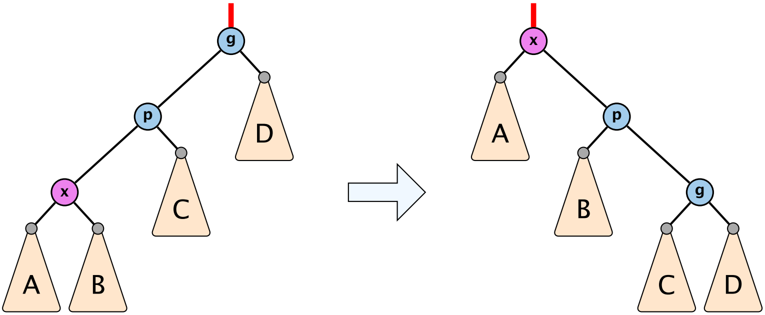

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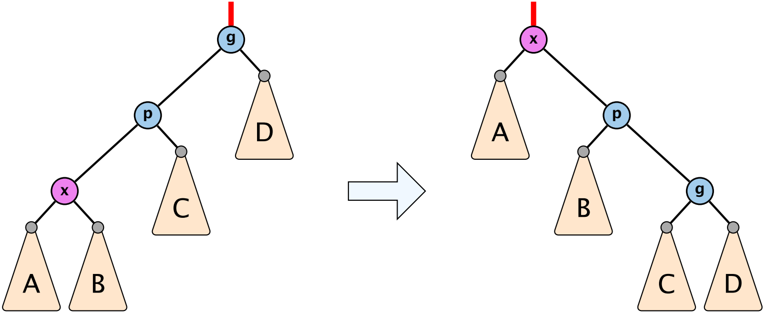

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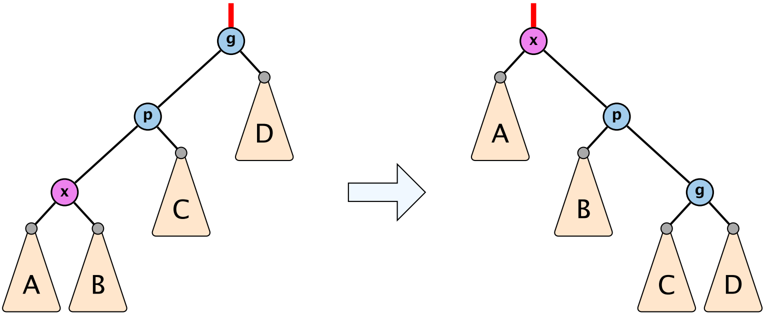

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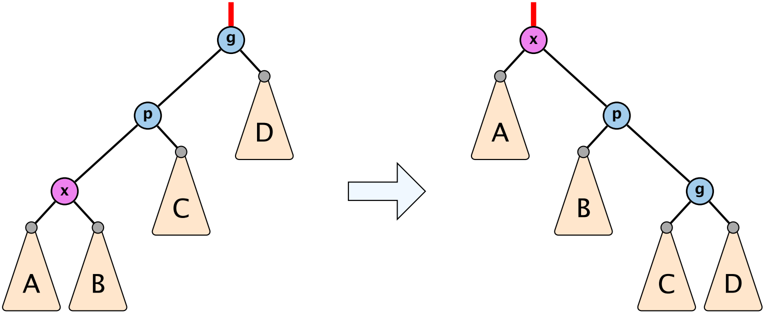

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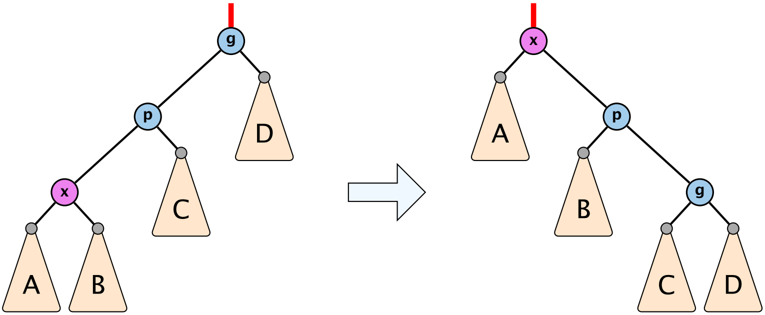

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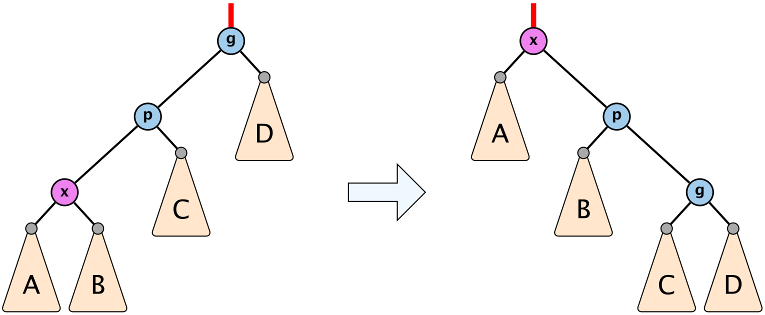

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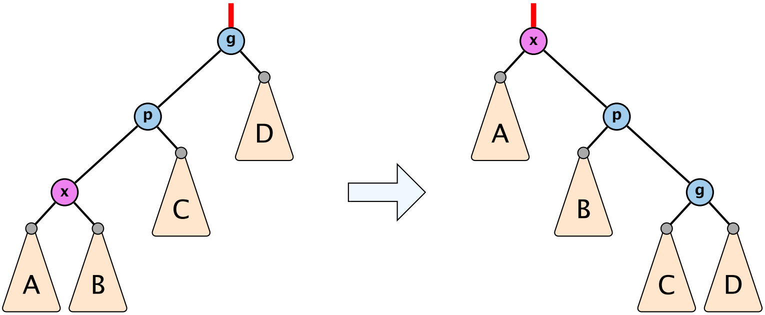

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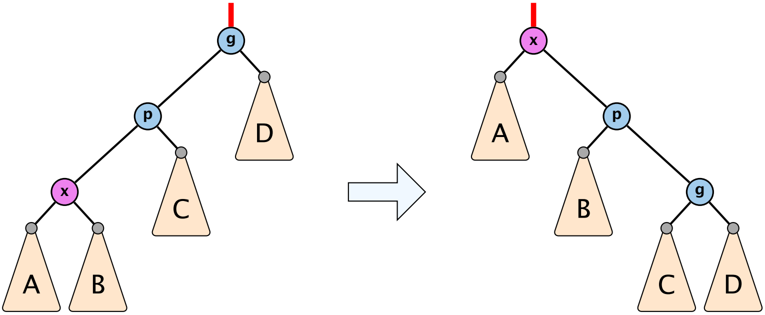

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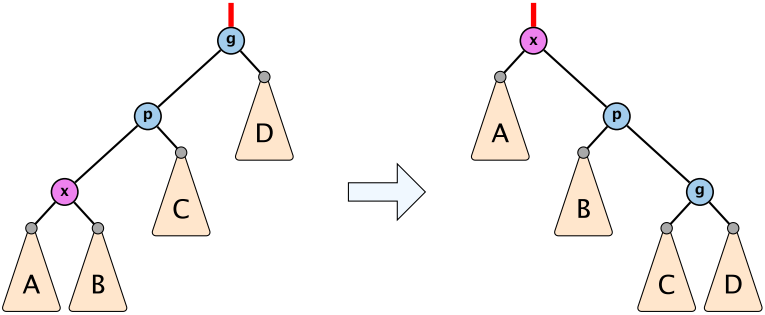

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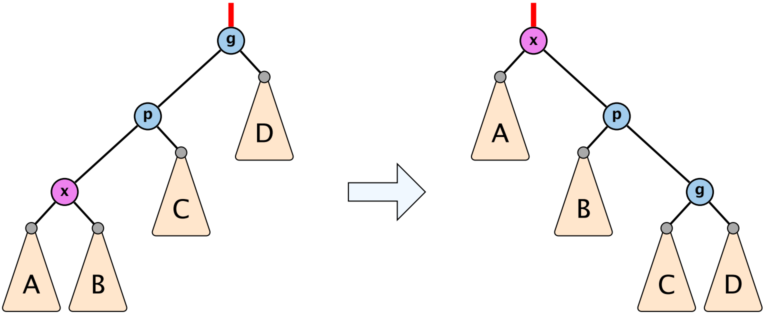

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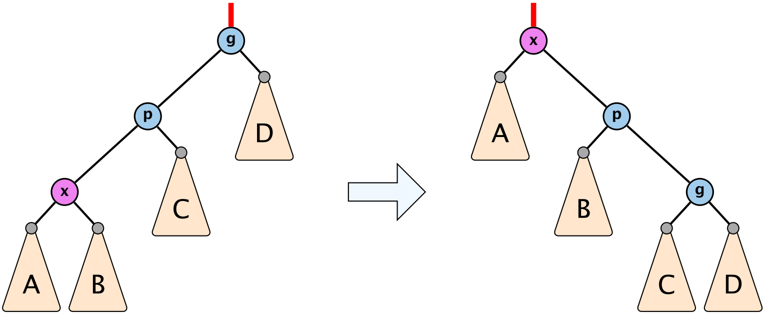

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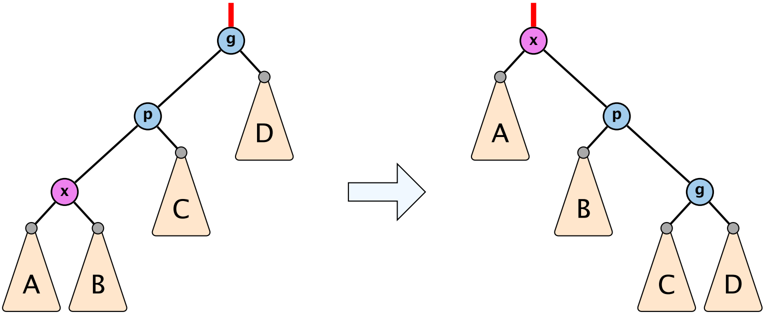

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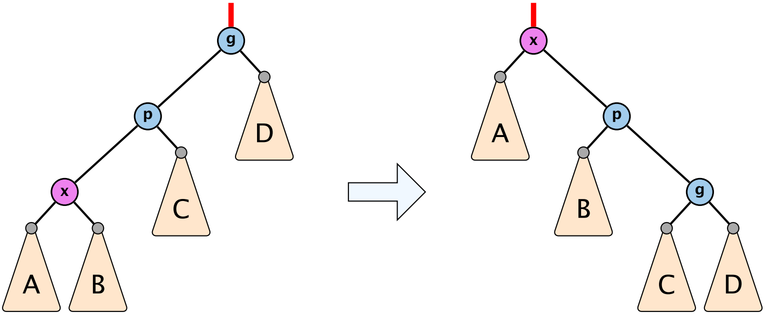

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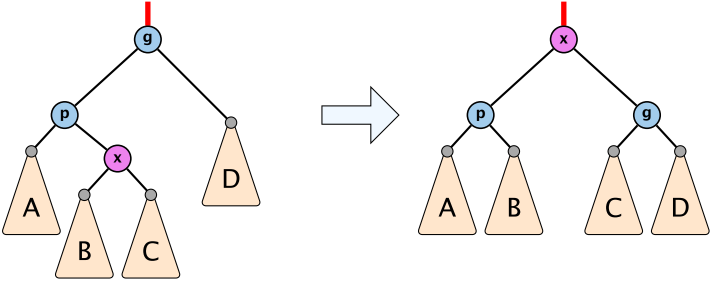

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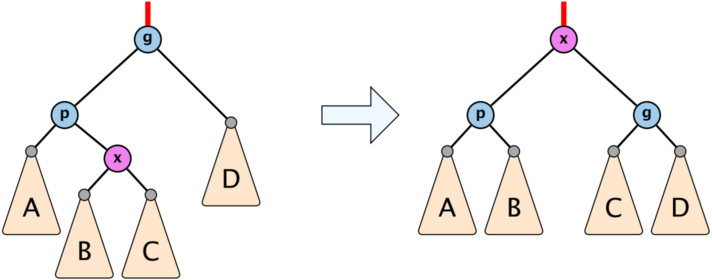

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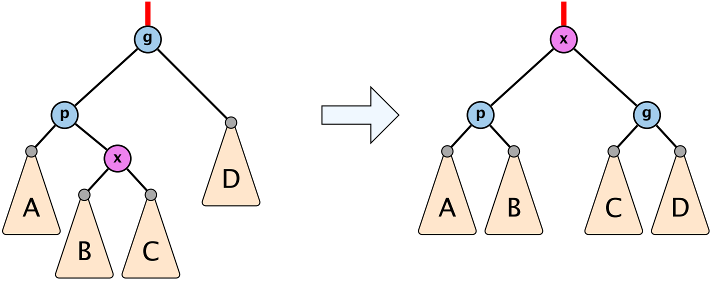

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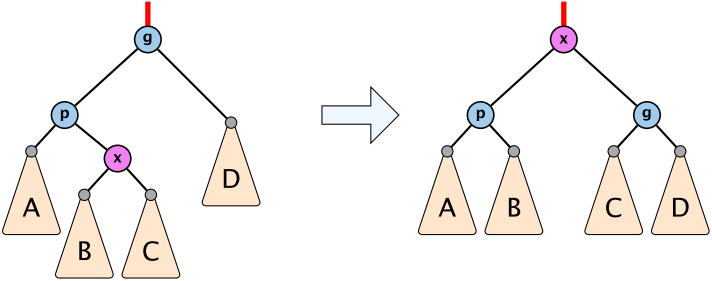

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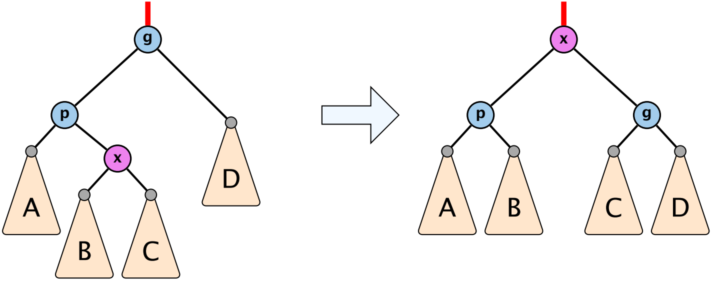

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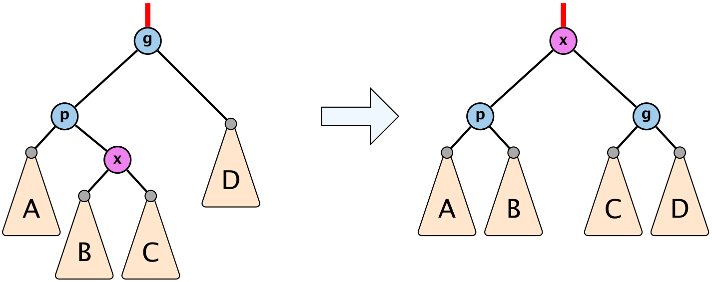

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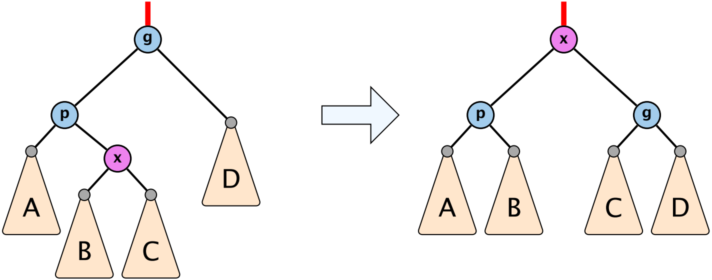

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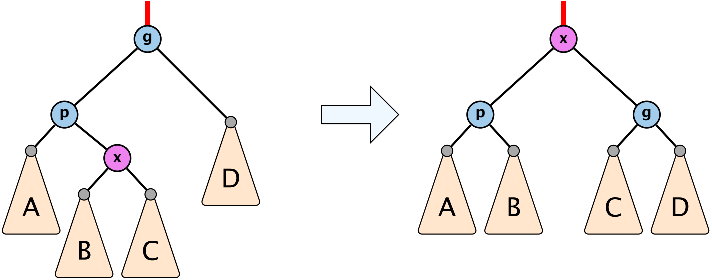

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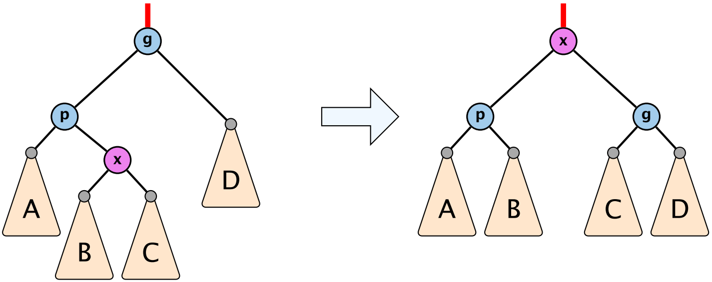

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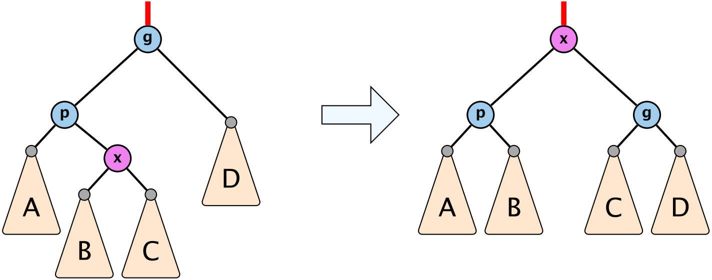

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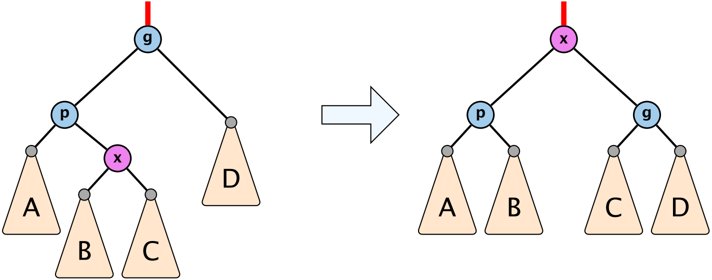

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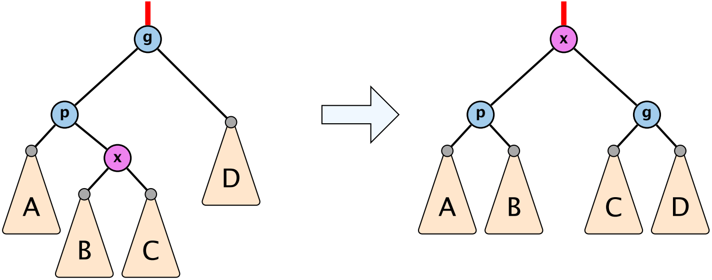

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Amortized cost of the whole splay operation:

$$
\begin{aligned}
&\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\
&= 2 + r(\text{root}) - r_0(x) \\
&\leq \mathcal{O}(\log n)
\end{aligned}
$$

7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- **Insert(x)**: insert element x.
- **Search(k)**: search for element with key k.
- **Delete(x)**: delete element referenced by pointer x.
- **find-by-rank(ℓ)**: return the ℓ-th element; return “error” if the data-structure contains less than ℓ elements.

Augment an existing data-structure instead of developing a new one.

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How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
2. We store in each node v the size of the sub-tree rooted at v.
3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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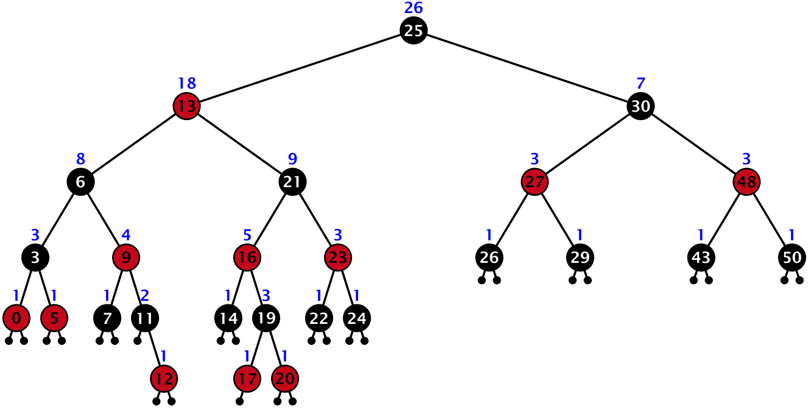
Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

4. How does find-by-rank work?
   Find-by-rank(k) $:=$ Select(root, k) with

Algorithm 11 Select(x, i)

```
if x = null then return error
if left[x] ≠ null then r ← left[x].size + 1 else r ← 1
if i = r then return x
if i < r then
    return Select(left[x], i)
else
    return Select(right[x], i − r)
```

Select(x, i)

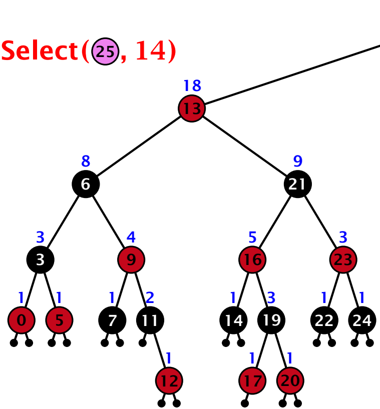


Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

Select(x, i)

Select(25, 14)



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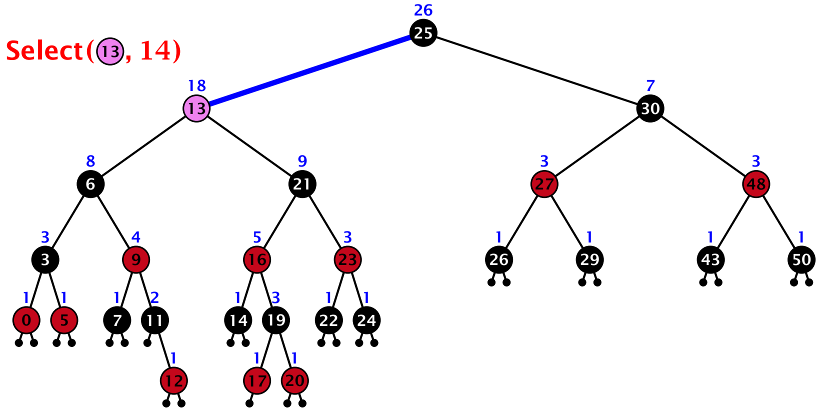

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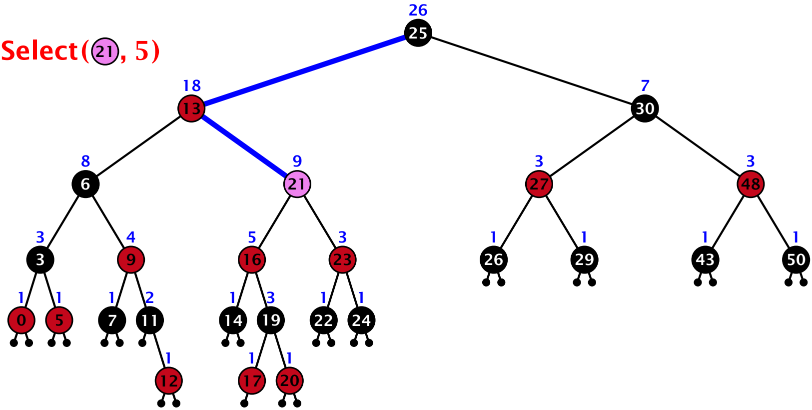

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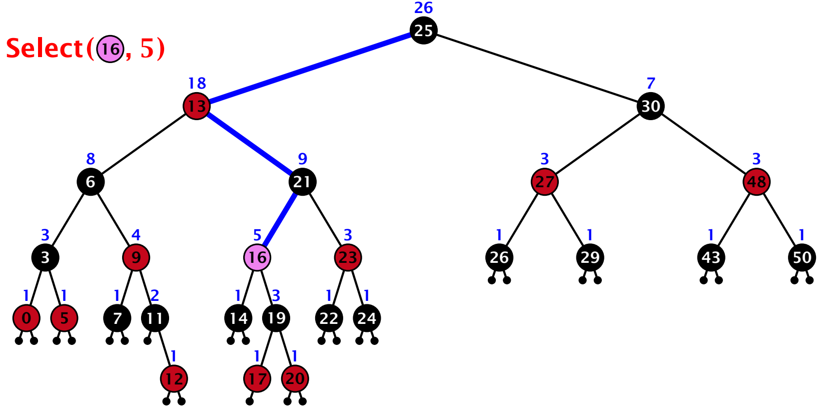

Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
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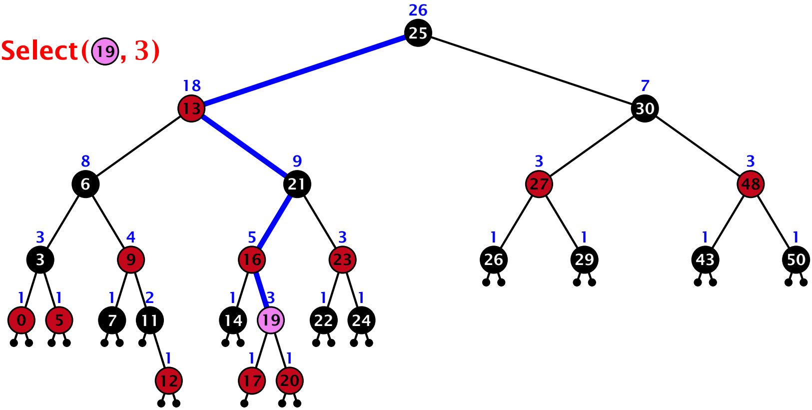

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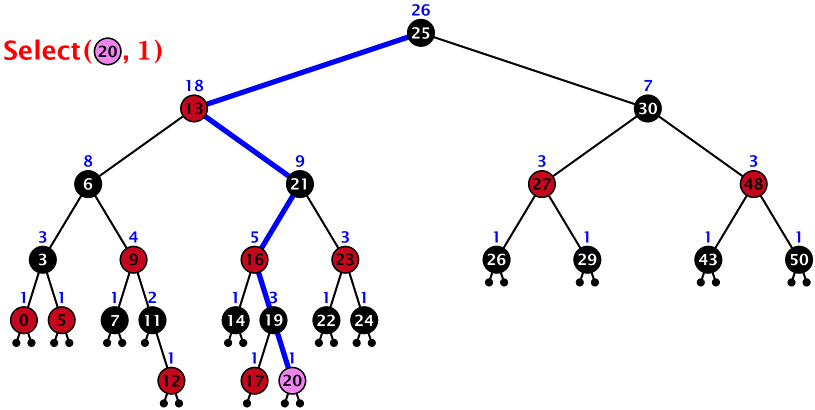

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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

Search(k): Nothing to do.

Insert(x): When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

Delete(x): Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

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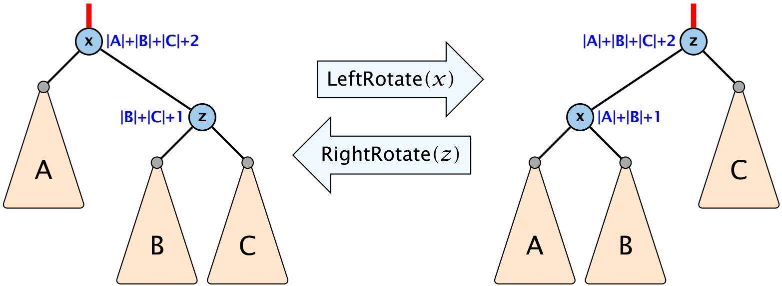
Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

7.5 (a, b)-trees

Definition 7

For $b \geq 2a - 1$ an (a, b)-tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex v has at least a and at most b children
3. the root has degree at least 2 if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
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7.5 (a, b)-trees

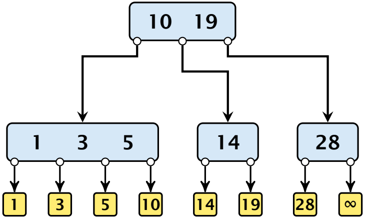
Each internal node v with $d(v)$ children stores $d-1$ keys $k_1, \dots, k_{d-1}$. The i-th subtree of v fulfills

$$k_{i-1} < \text{ key in } i\text{-th sub-tree } \leq k_i \ ,$$

where we use $k_0 = -\infty$ and $k_d = \infty$.

7.5 (a,b)-trees

Example 8



7.5 (a, b)-trees

Variants

- The dummy leaf element may not exist; it only makes implementation more convenient.
- Variants in which $b = 2a$ are commonly referred to as B-trees.
- A B-tree usually refers to the variant in which keys and data are stored at internal nodes.
- A B^+ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
- A B^* tree requires that a node is at least $2/3$-full as opposed to $1/2$-full (the requirement of a B-tree).

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Lemma 9

Let T be an (a, b)-tree for $n > 0$ elements (i.e., $n + 1$ leaf nodes) and height h (number of edges from root to a leaf vertex). Then

1. $2a^{h-1} \leq n + 1 \leq b^h$
2. $\log_b(n + 1) \leq h \leq 1 + \log_a(\frac{n+1}{2})$

Proof.

- If $n > 0$ the root has degree at least 2 and all other nodes have degree at least a. This gives that the number of leaf nodes is at least $2a^{h-1}$.
- Analogously, the degree of any node is at most b and hence, the number of leaf nodes at most b^h.

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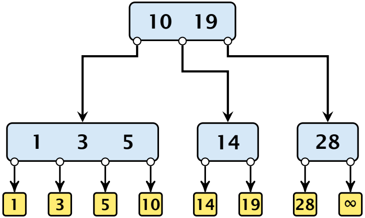
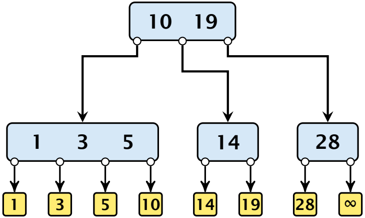


Search

Search(8)



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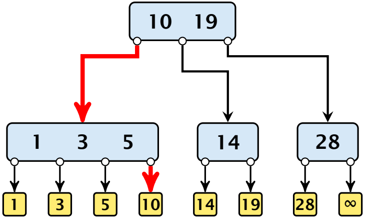

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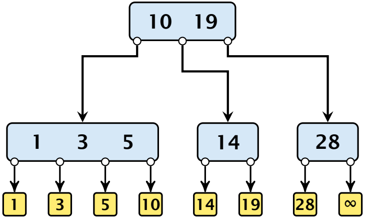
Search(19)



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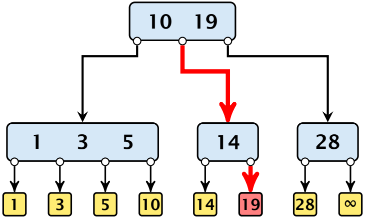

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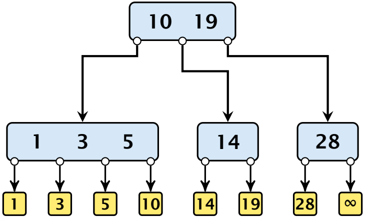

The search is straightforward. It is only important that you need to go all the way to the leaf.

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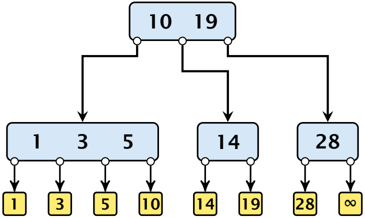

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Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

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Insert element x:

- Follow the path as if searching for $\text{key}[x]$.
- If this search ends in leaf ℓ, insert x before this leaf.
- For this add $\text{key}[x]$ to the key-list of the last internal node v on the path.
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Rebalance(v):

- Let k_i, $i = 1, \dots, b$ denote the keys stored in v.
- Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.
- Create two nodes v_1, and v_2. v_1 gets all keys $k_1, \dots, k_{j-1}$ and v_2 gets keys $k_{j+1}, \dots, k_b$.
- Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.
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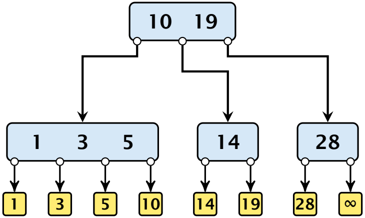

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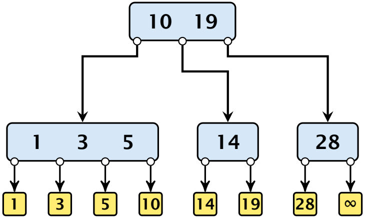
Insert(8)



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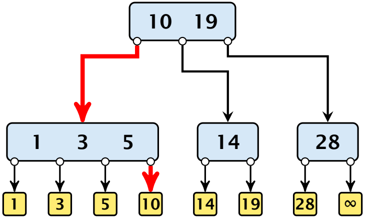

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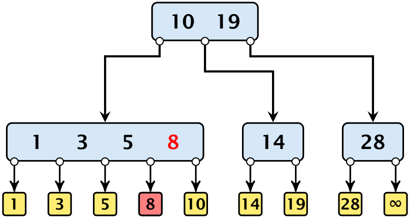

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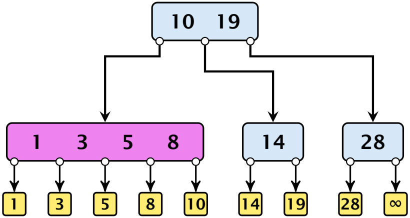

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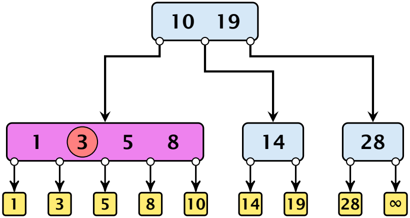

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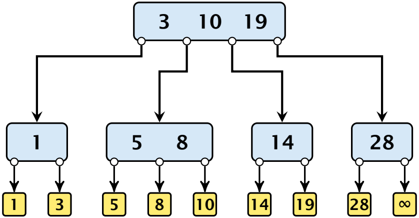

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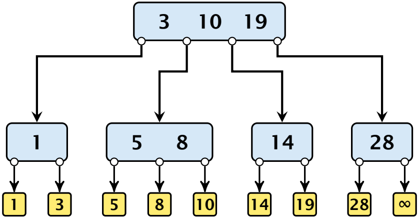
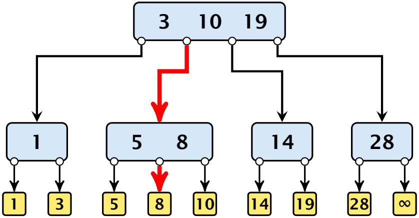


Insert

Insert(6)



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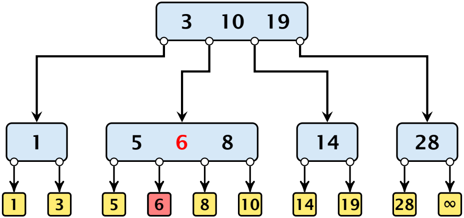

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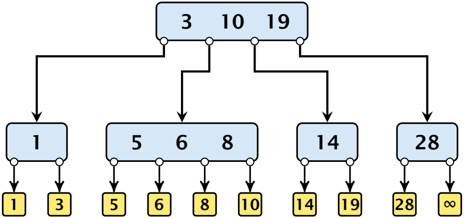

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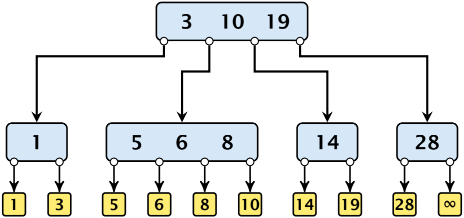

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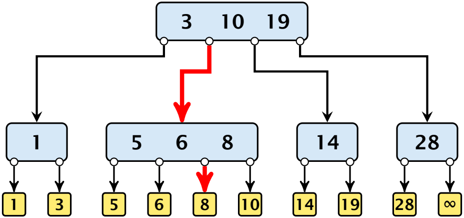

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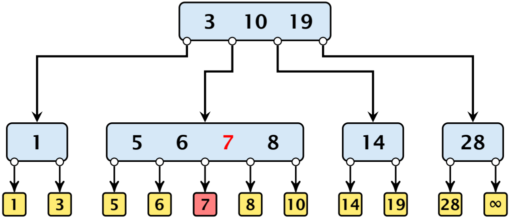

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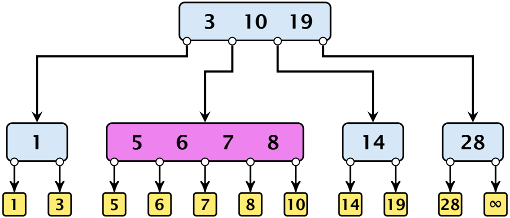

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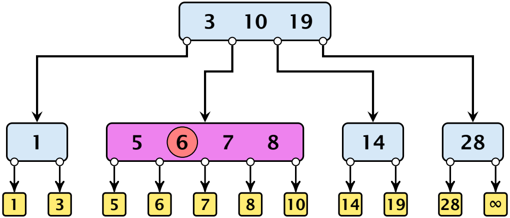

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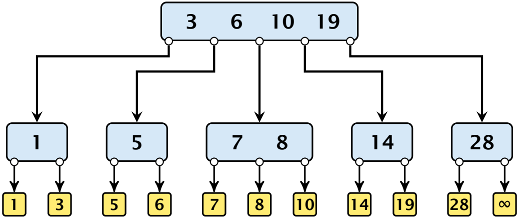

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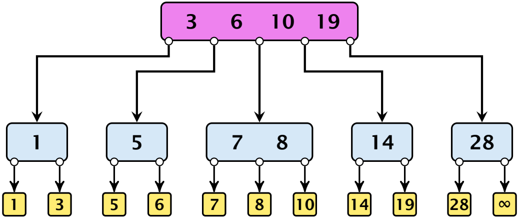

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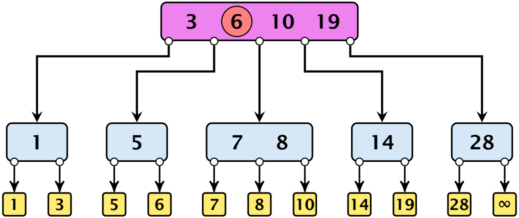

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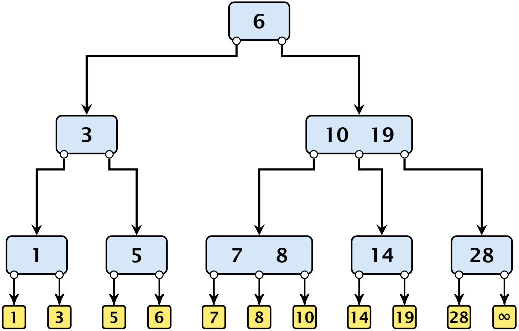

Delete

Delete element x (pointer to leaf vertex):

- Let v denote the parent of x. If $\text{key}[x]$ is contained in v, remove the key from v, and delete the leaf vertex.
- Otherwise delete the key of the predecessor of x from v; delete the leaf vertex; and replace the occurrence of $\text{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
- If now the number of keys in v is below $a - 1$ perform $\text{Rebalance}'(v)$.

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- If now the number of keys in v is below $a - 1$ perform Rebalance′(v).

Delete

Rebalance'(v):

- If there is a neighbour of v that has at least a keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
- If not: merge v with one of its neighbours.
- The merged node contains at most $(a-2)+(a-1)+1$ keys, and has therefore at most $2a-1 \leq b$ successors.
- Then rebalance the parent.
- During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

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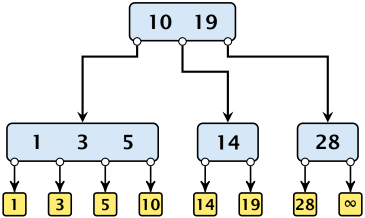

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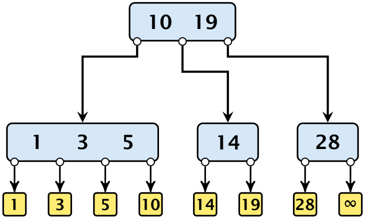

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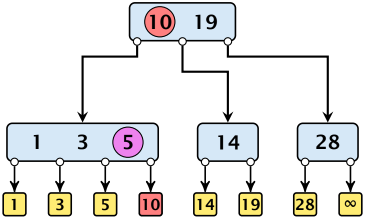

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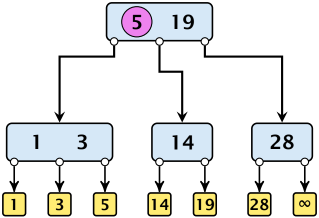

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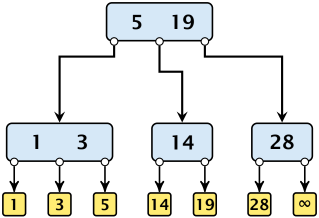

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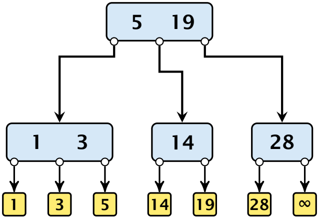

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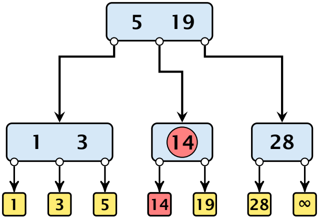

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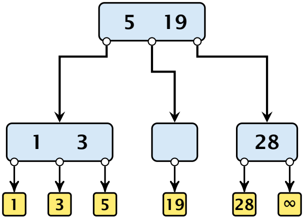

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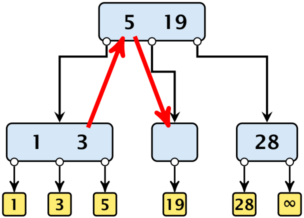

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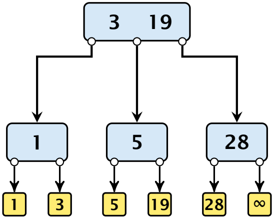

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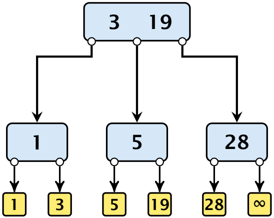

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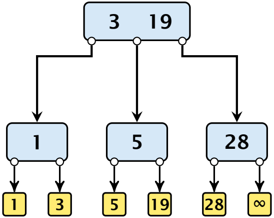

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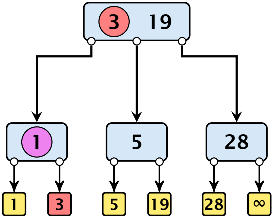

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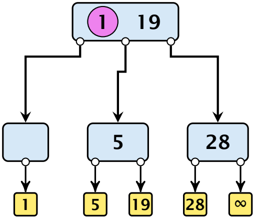

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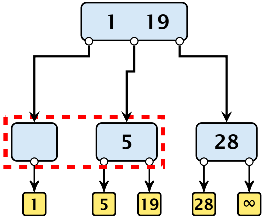

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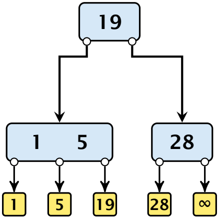

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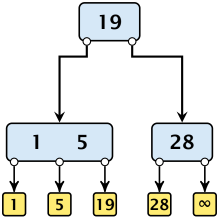

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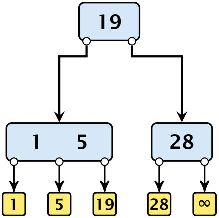

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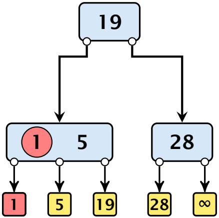

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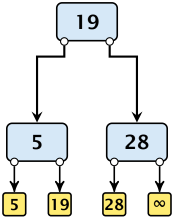

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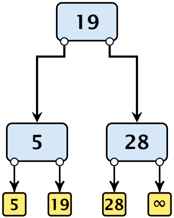

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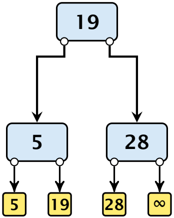

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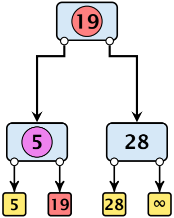

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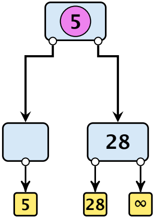

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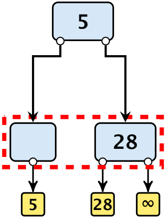

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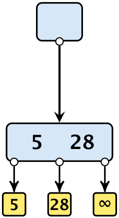

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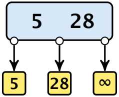
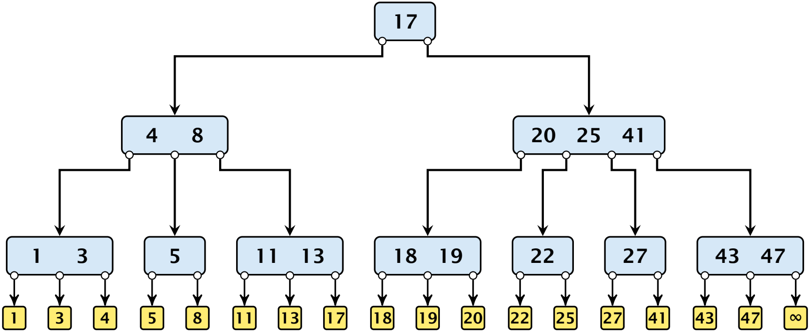


$(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:



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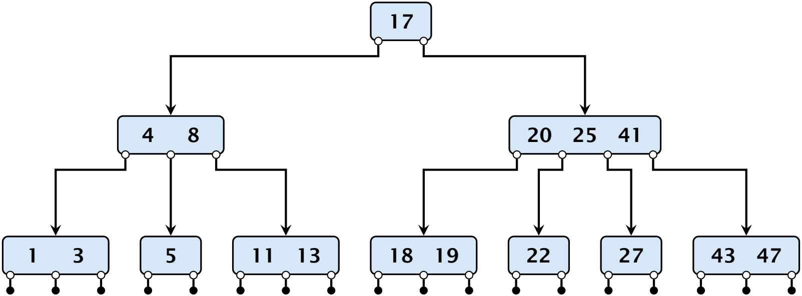

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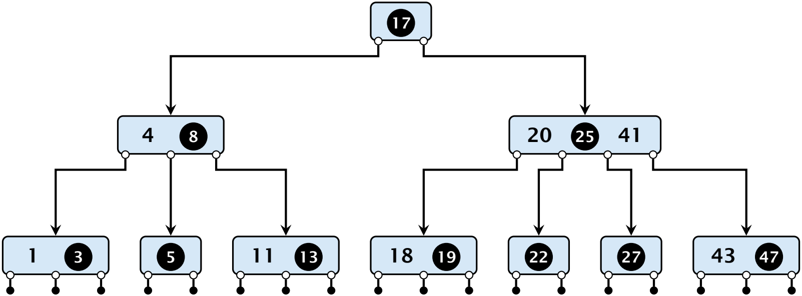

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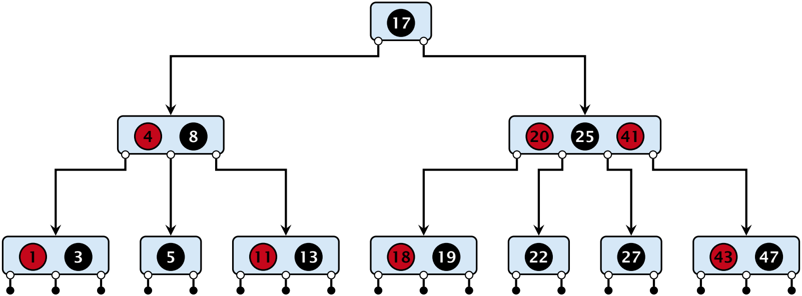

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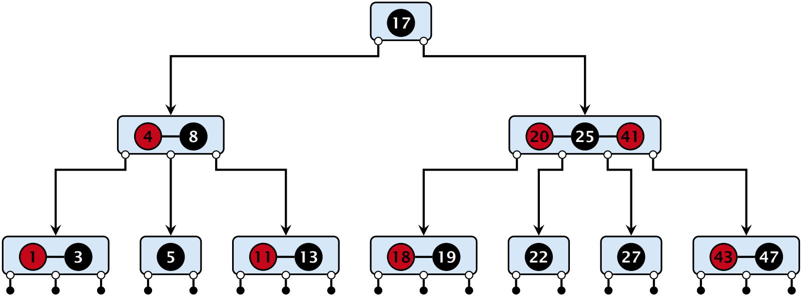

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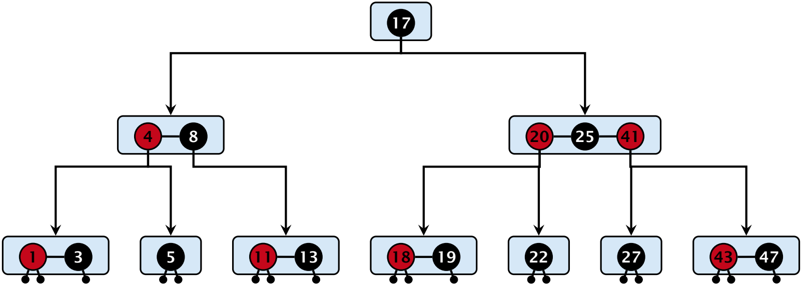

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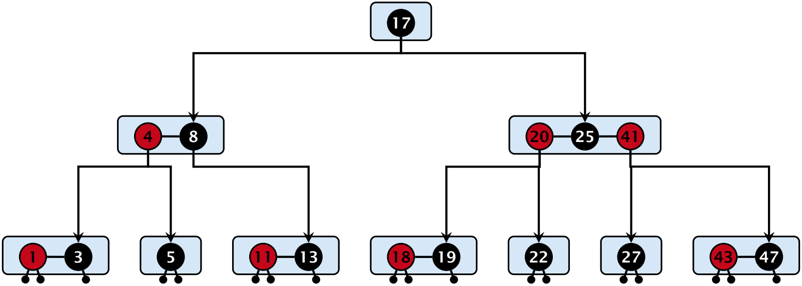

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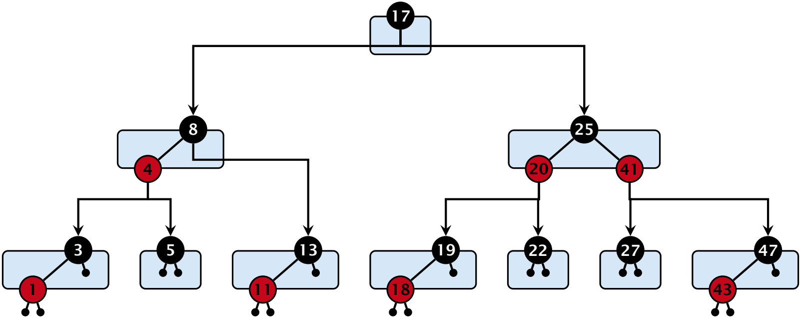

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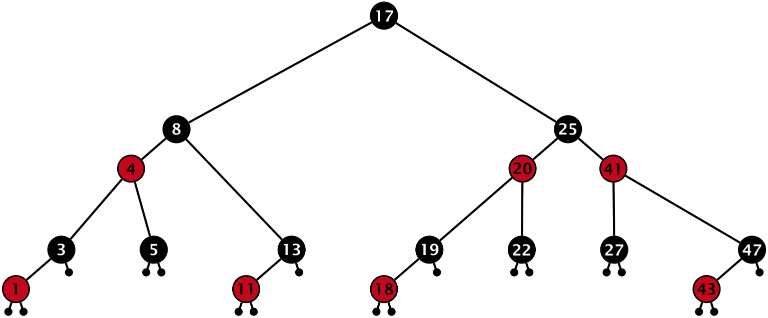

$(2,4)$-trees and red black trees

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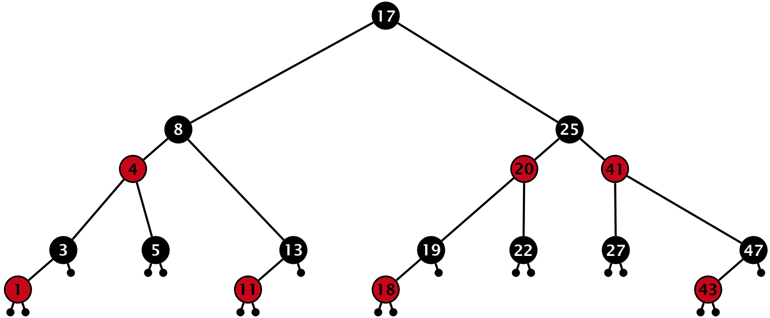

Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same $(2,4)$-tree.

7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

- time for search $\Theta(n)$
- time for insert $\Theta(n)$ (dominated by searching the item)
- time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$

7.6 Skip Lists

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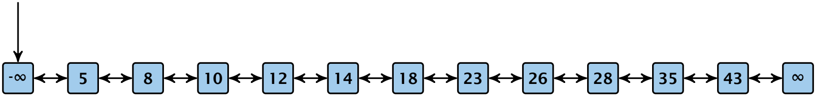

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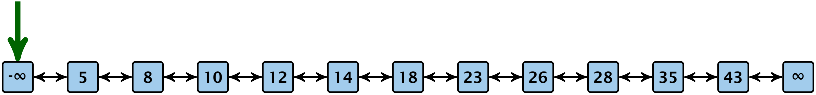

7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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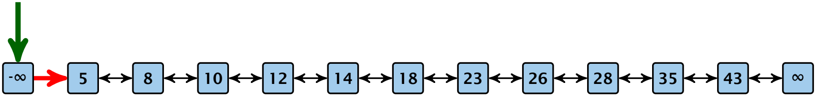

7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

- time for search $\Theta(n)$
- time for insert $\Theta(n)$ (dominated by searching the item)
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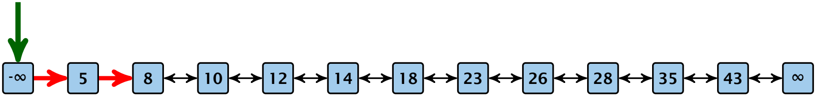

7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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- time for insert $\Theta(n)$ (dominated by searching the item)
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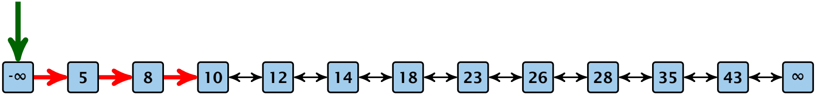

7.6 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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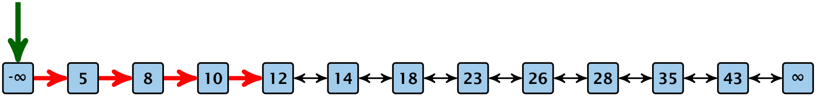
- time for delete $\Theta(1)$ if we are given a handle to the object, otw. $\Theta(n)$



7.6 Skip Lists

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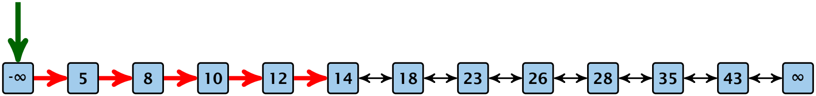

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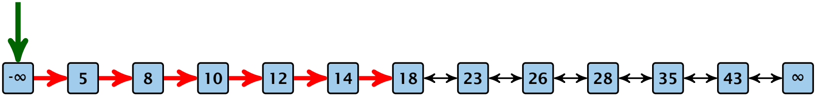

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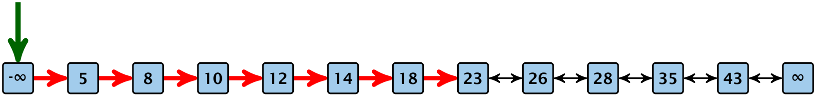

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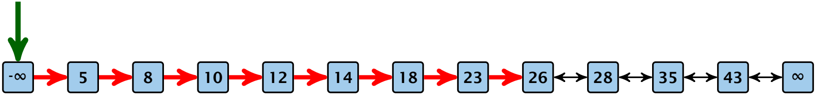

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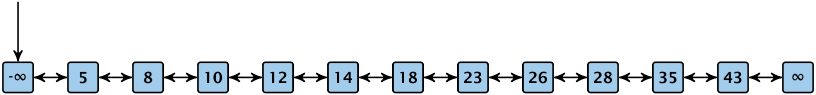

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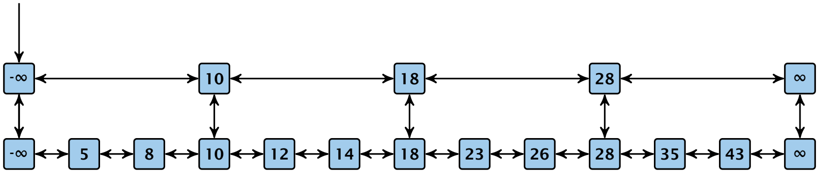

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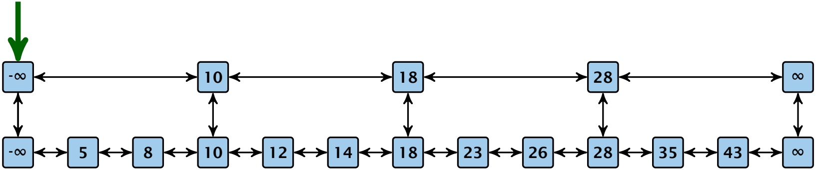

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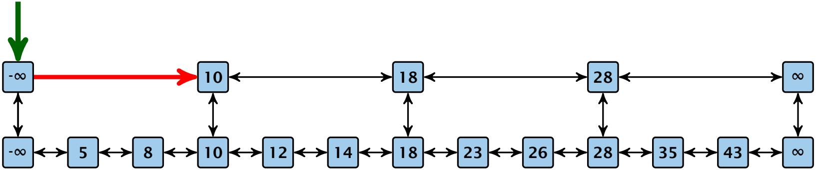

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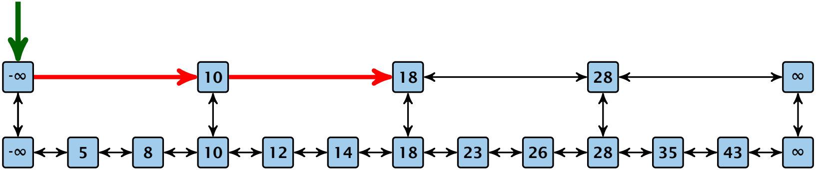

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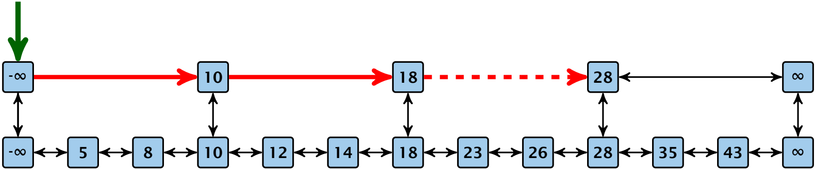

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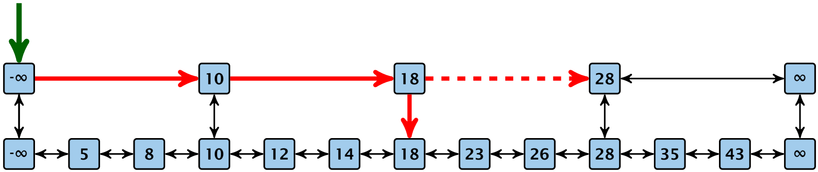

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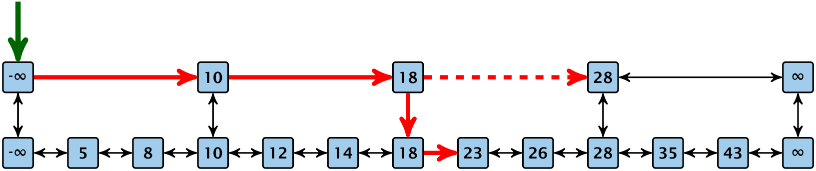

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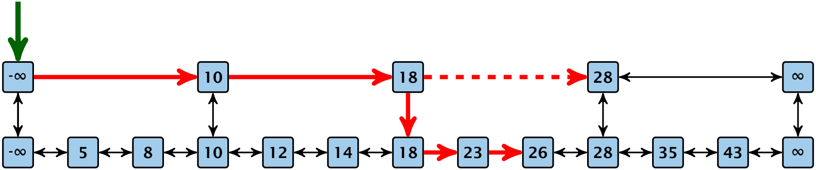

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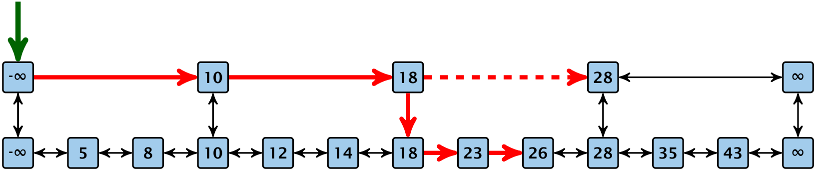

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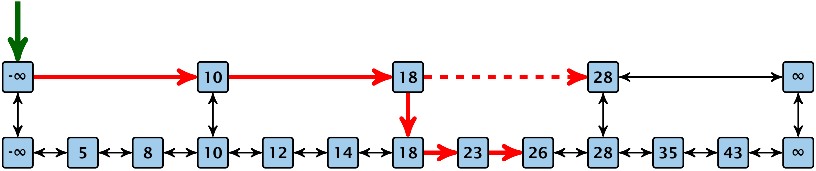

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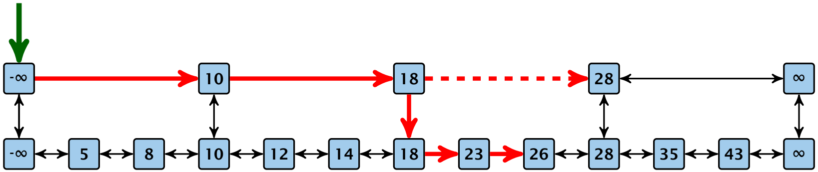

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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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- If we want that in L_i we always skip over roughly the same number of elements in L_{i-1} an insert or delete may require a lot of re-organisation.

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- A search operation gives you the insert position for element x in every list.
- Flip a coin until it shows head, and record the number $t \in \{1, 2, \dots\}$ of trials needed.
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- You get all predecessors via backward pointers.
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The time for both operations is dominated by the search time.

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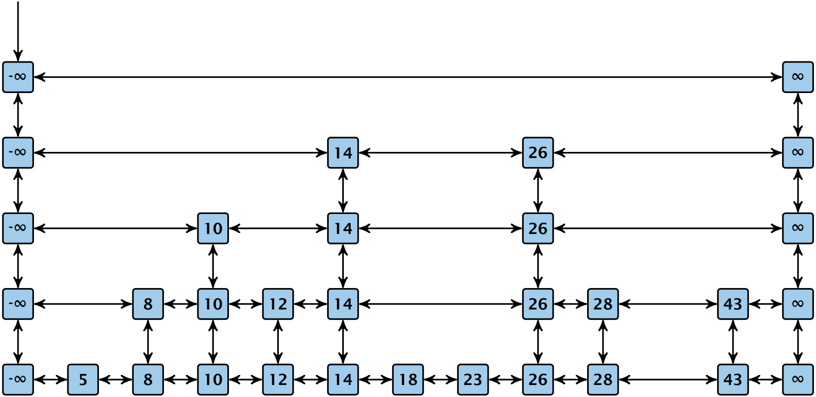

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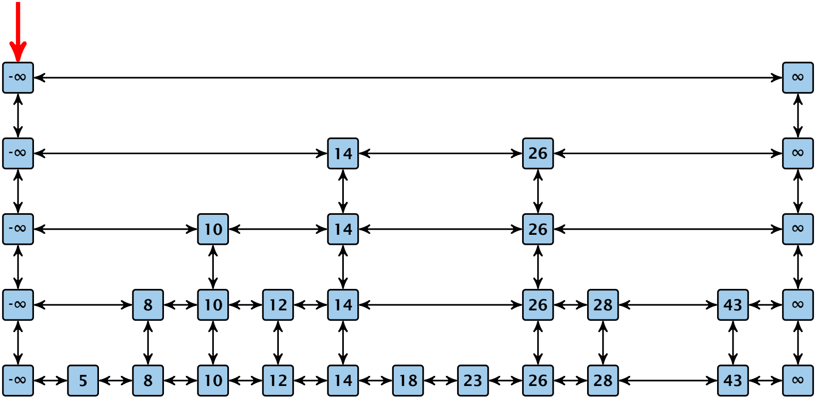

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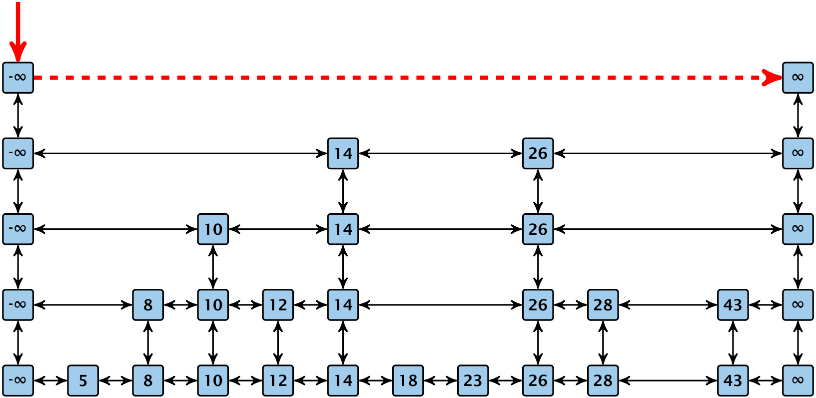

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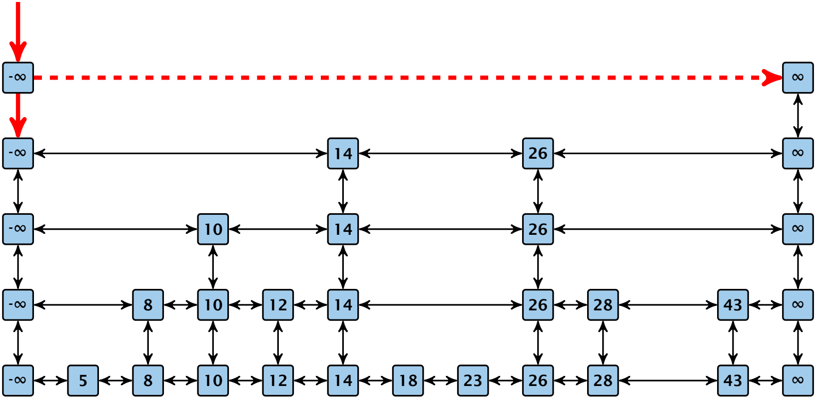

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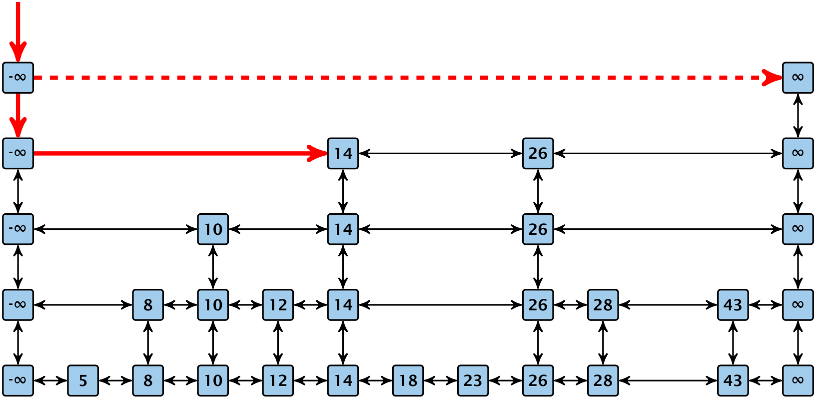

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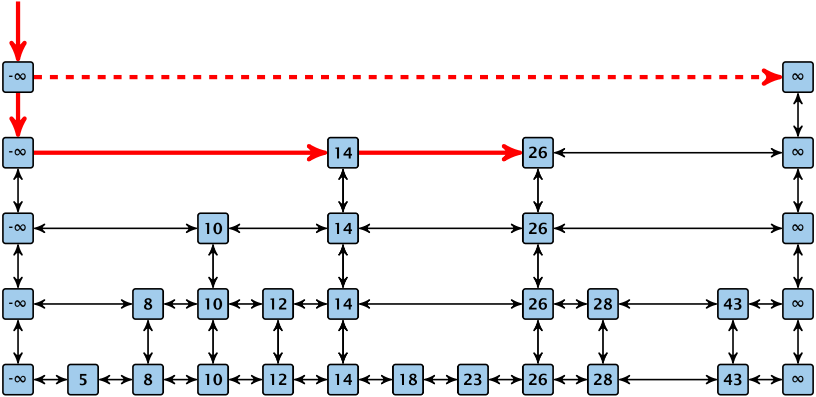

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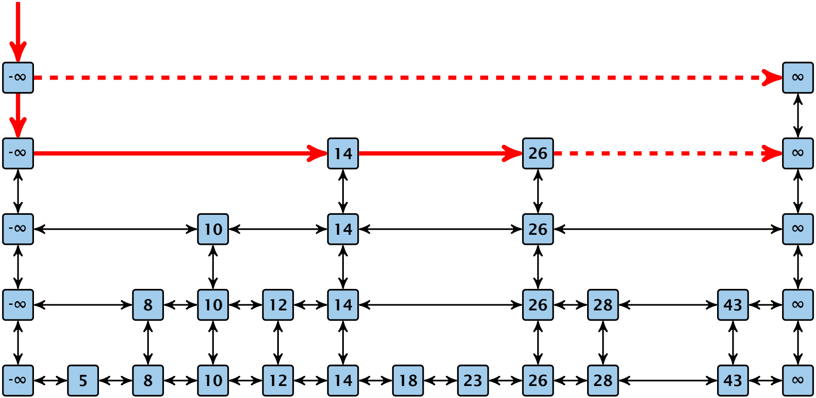

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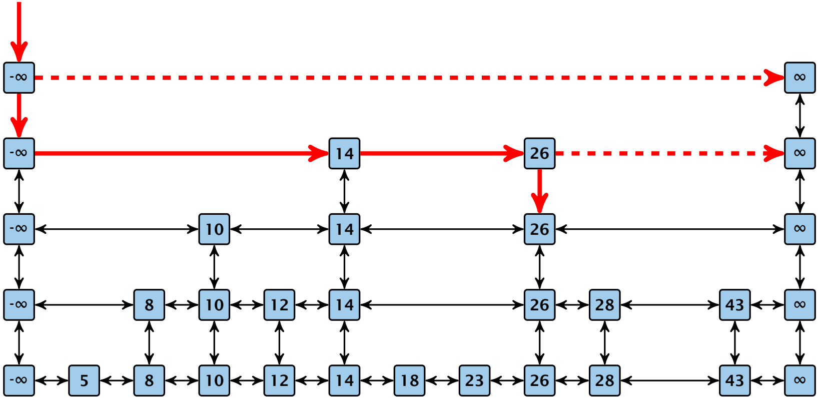

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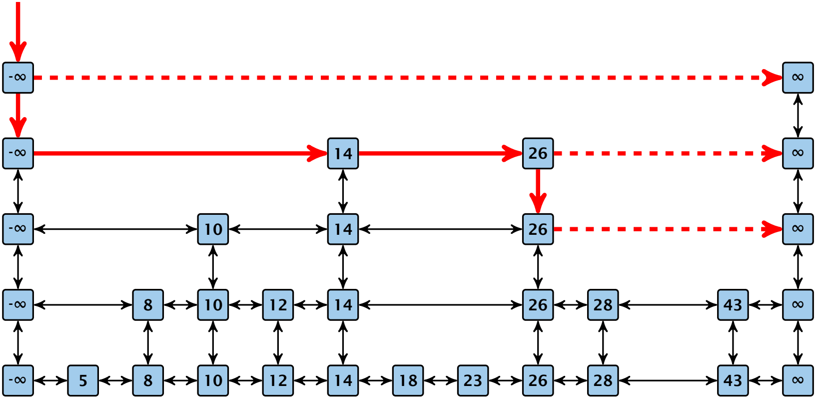

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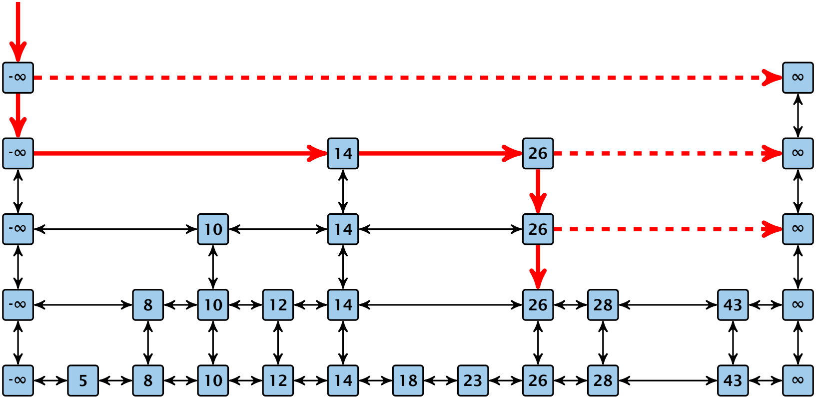

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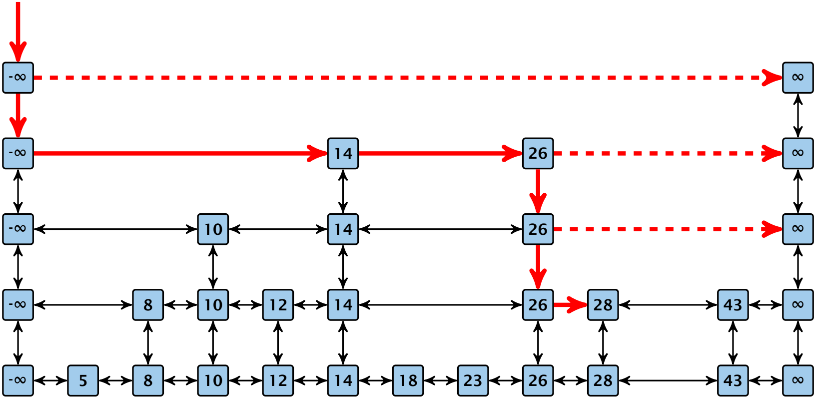

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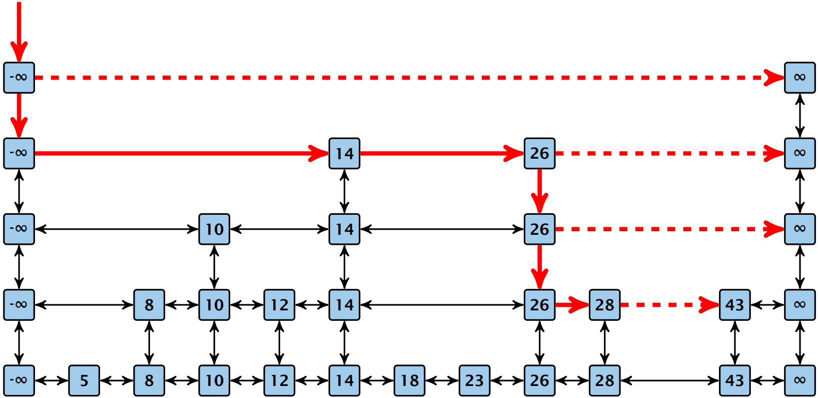

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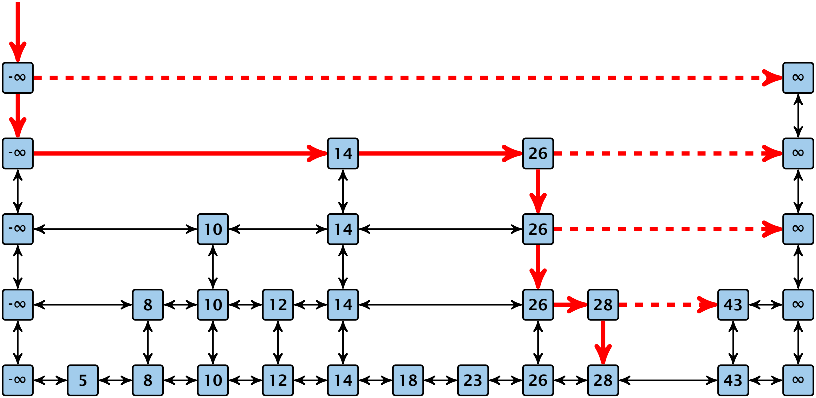

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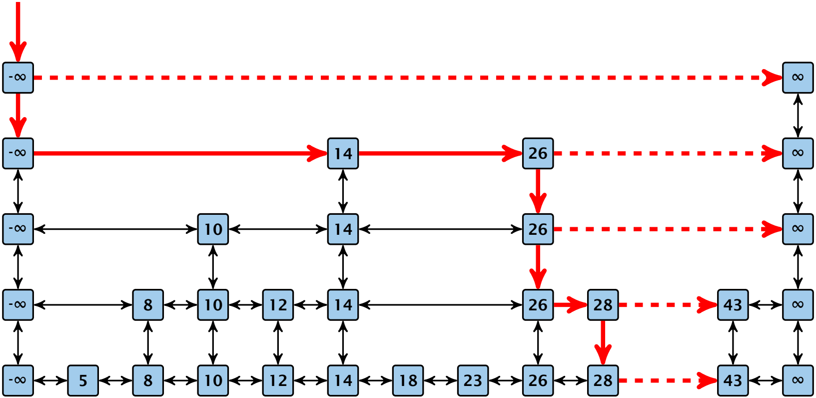

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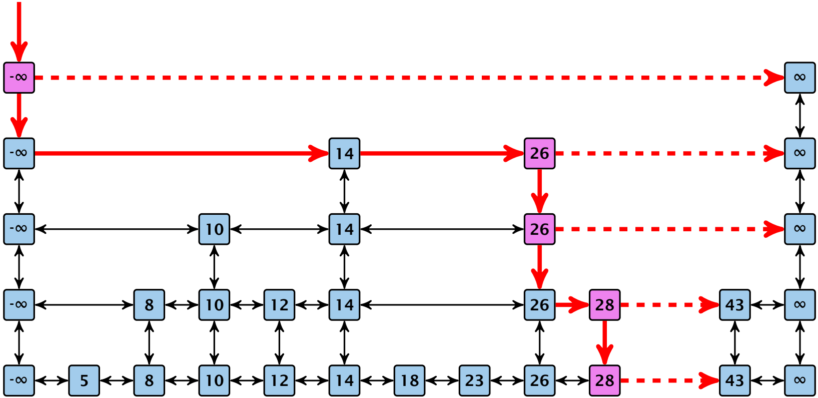

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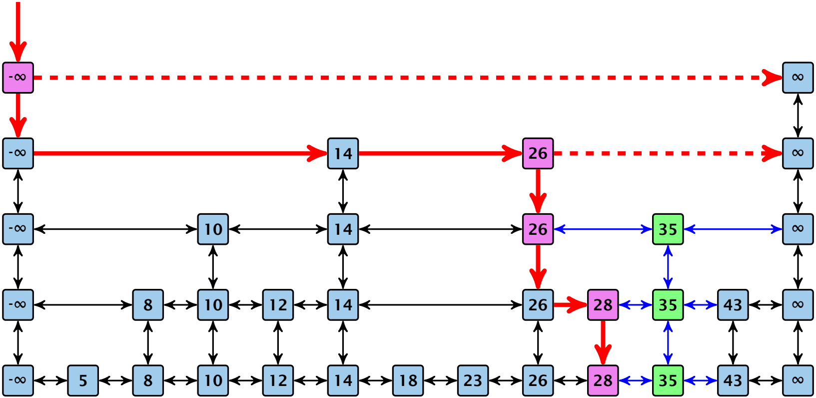

High Probability

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We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

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Suppose there are polynomially many events $E_1, E_2, \dots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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$$\begin{aligned}\Pr[E_1 \wedge \cdots \wedge E_\ell] &= 1 - \Pr[\bar{E}_1 \vee \cdots \vee \bar{E}_\ell] \\ &\geq 1 - n^c \cdot n^{-\alpha}\end{aligned}$$

High Probability

Suppose there are polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

7.6 Skip Lists

Lemma 11

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

7.6 Skip Lists

Backward analysis:

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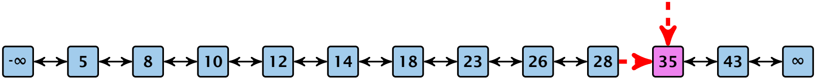

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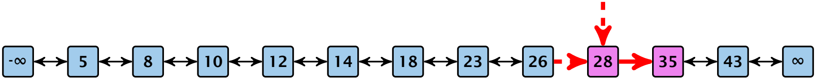

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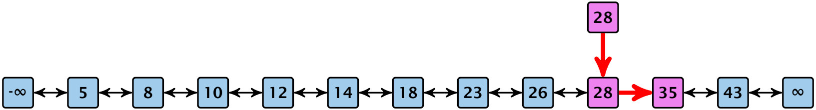

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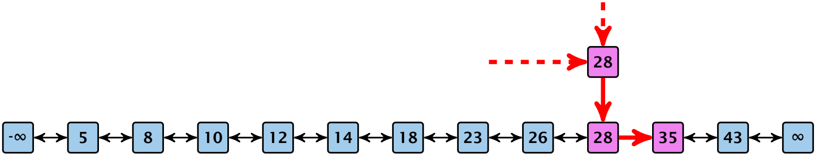

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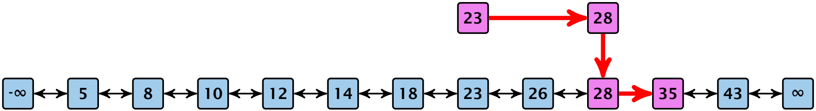

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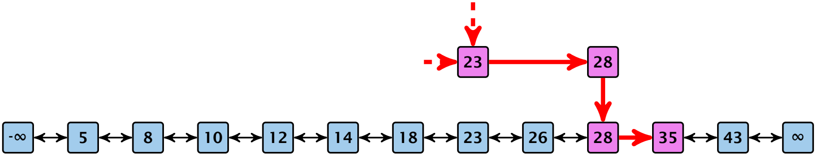

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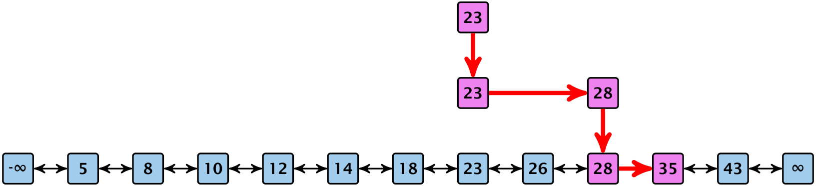

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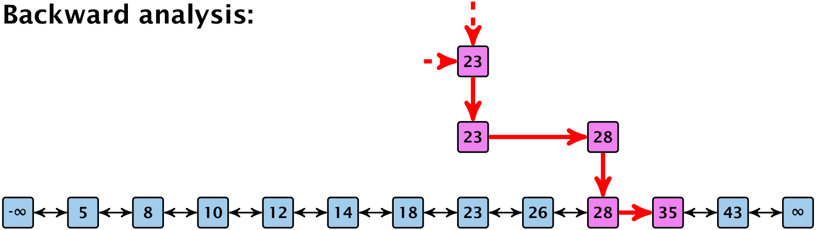

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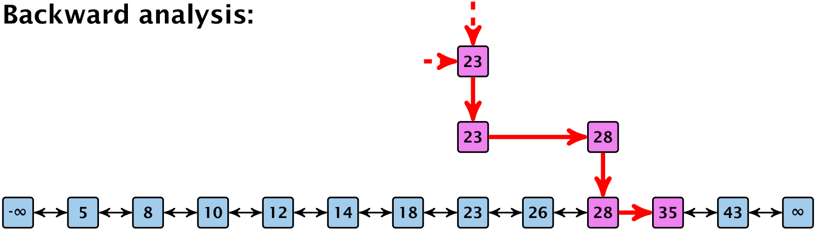

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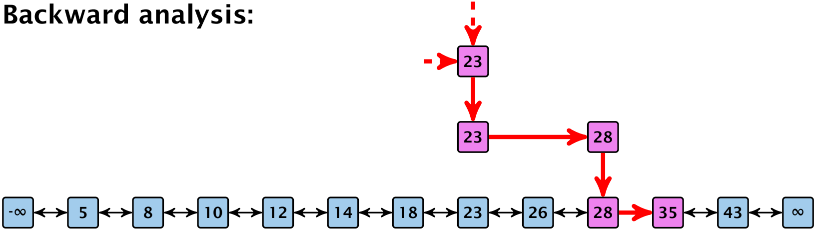

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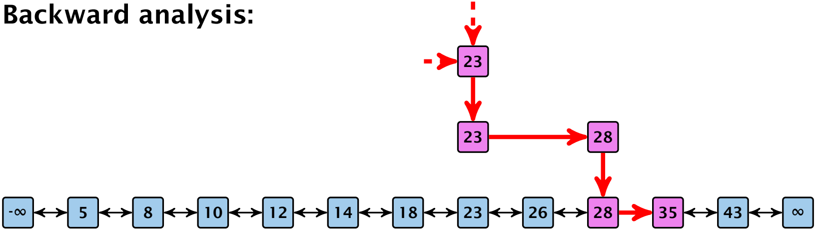

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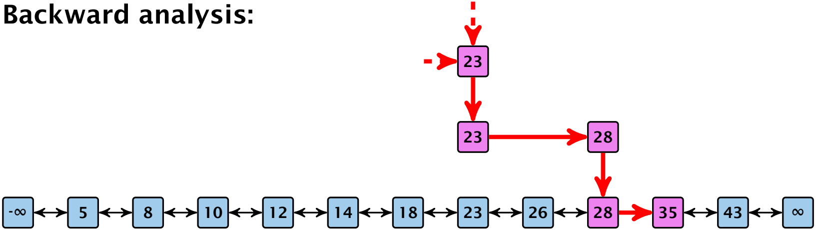

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We show that w.h.p:

- A "long" search path must also go very high.
- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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This means, the search requires at most z steps, w. h. p.

7.7 Hashing

Dictionary:

- $S.\textbf{insert}(x)$: Insert an element x.
- $S.\textbf{delete}(x)$: Delete the element pointed to by x.
- $S.\textbf{search}(k)$: Return a pointer to an element e with $\text{key}[e] = k$ in S if it exists; otherwise return null.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

Hashing tries to directly compute the memory location from the given key. The goal is to have constant search time.

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Definitions:

- Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- Array $T[0, \ldots, n-1]$ hash-table.
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- Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
- Array $T[0,\ldots,n-1]$ hash-table.
- Hash function $h : U \to [0,\ldots,n-1]$.

The hash-function h should fulfill:

- Fast to evaluate.
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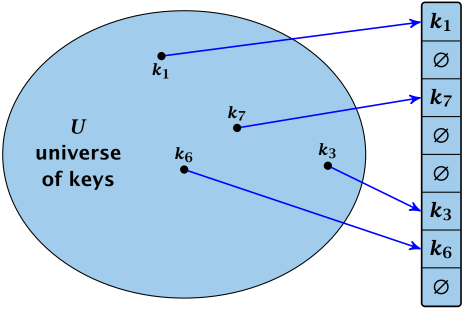
Direct Addressing

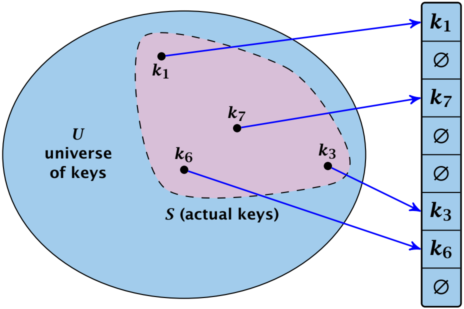
Ideally the hash function maps all keys to different memory locations.



This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.



Such a hash function h is called a perfect hash function for set S.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions

Usually the universe U is much larger than the table-size n.

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

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Collisions

Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 12

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \to [0, \ldots, n-1]$.

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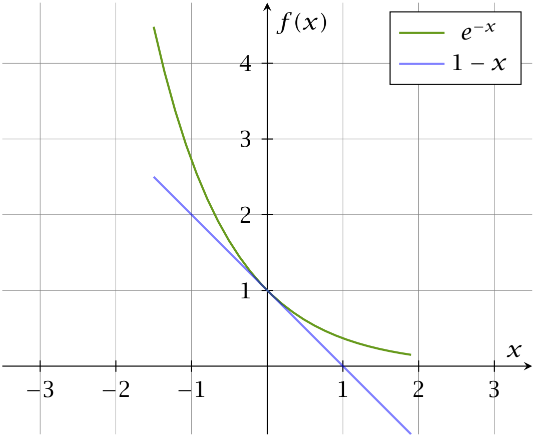
Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

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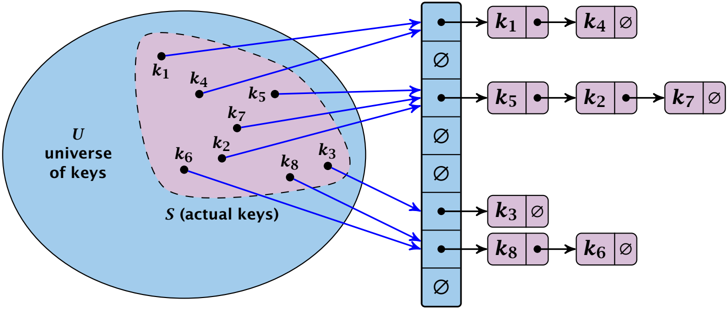
Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- Access: compute $h(x)$ and search list for $\mathrm{key}[x]$.
- Insert: insert at the front of the list.



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Let A denote a strategy for resolving collisions. We use the following notation:

- A^+ denotes the average time for a **successful** search when using A;
- A^- denotes the average time for an **unsuccessful** search when using A;
- We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called fill factor of the hash-table.

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$$A^{-} = 1 + \alpha \ .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

$$\mathrm{E}\Bigg[\frac{1}{m}\sum_{i=1}^{m}\Bigg(1 + \sum_{j=i+1}^{m} X_{ij}\Bigg)\Bigg]$$

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Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- pointers increase memory requirements
- pointers may lead to bad cache efficiency

Advantages:

- no à priori limit on the number of elements
- deletion can be implemented efficiently
- by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k, 0)$,. . . ,$h(k, n - 1)$ must form a permutation of $0, \ldots, n - 1$.

$\textbf{Search}(k)$: Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1)$, $h(k, 2)$, ...

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Choices for $h(k,j)$:

- Linear probing:
  $h(k,i) = h(k) + i \mod n$
  (sometimes: $h(k,i) = h(k) + ci \mod n$).
- Quadratic probing:
  $h(k,i) = h(k) + c_1 i + c_2 i^2 \mod n$.
- Double hashing:
  $h(k,i) = h_1(k) + i h_2(k) \mod n$.

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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- Quadratic probing:
  $h(k, i) = h(k) + c_1 i + c_2 i^2 \mod n$.
- Double hashing:
  $h(k, i) = h_1(k) + i h_2(k) \mod n$.

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

Open Addressing

Choices for $h(k,j)$:

- Linear probing:
  $h(k,i) = h(k) + i \mod n$
  (sometimes: $h(k,i) = h(k) + ci \mod n$).
- Quadratic probing:
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Linear Probing

- Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.
- Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 13

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

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$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

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- Any probe into the hash-table usually creates a cache-miss.

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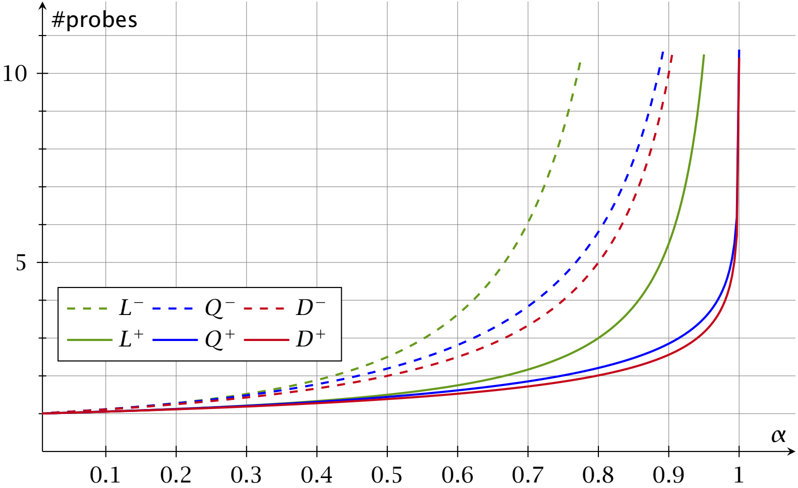
Open Addressing

Some values:

α	*Linear Probing*		*Quadratic Probing*		*Double Hashing*	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

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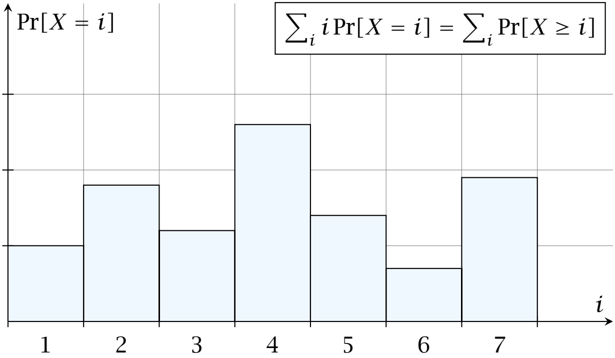
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Analysis of Idealized Open Address Hashing



Analysis of Idealized Open Address Hashing

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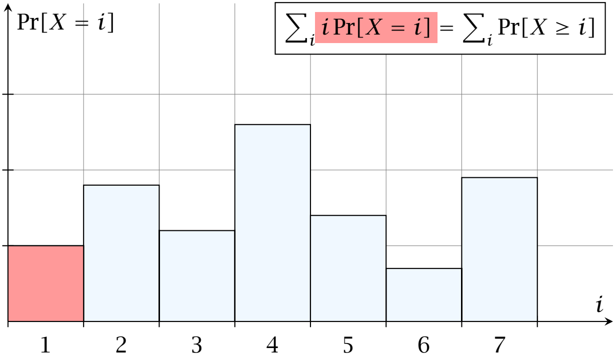

Analysis of Idealized Open Address Hashing

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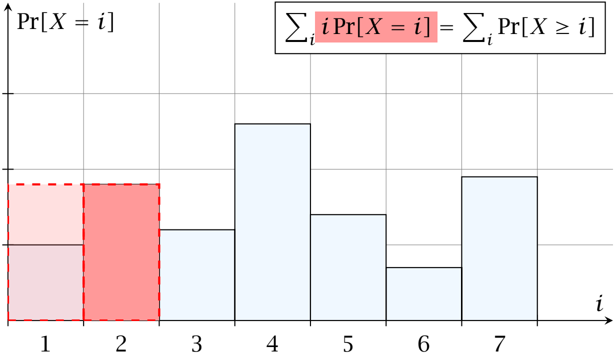

Analysis of Idealized Open Address Hashing

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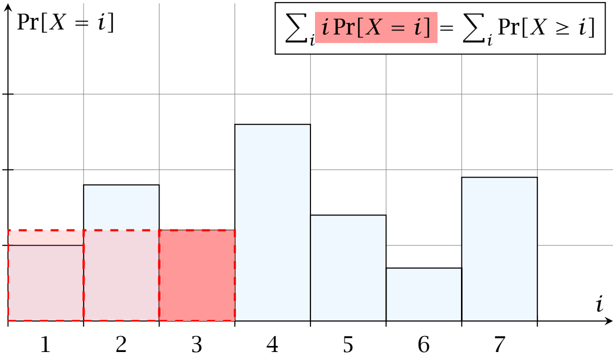

Analysis of Idealized Open Address Hashing

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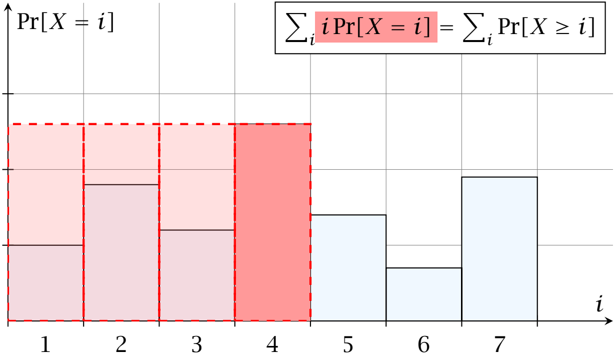

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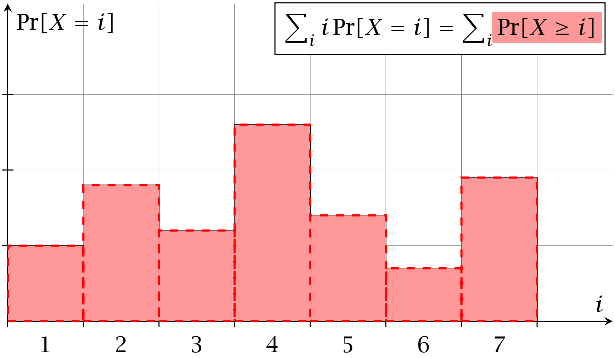

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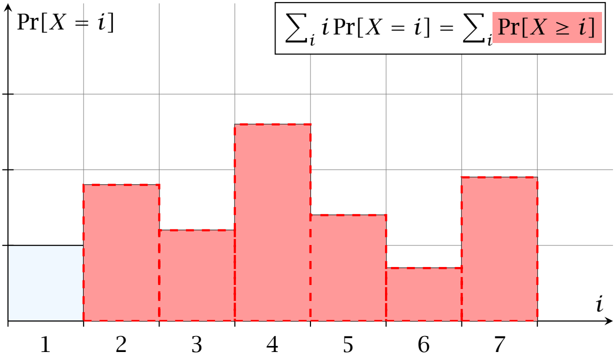

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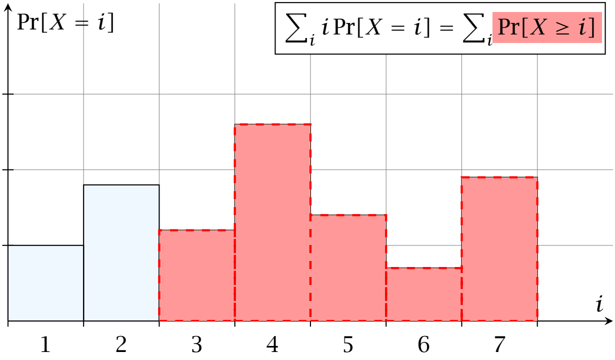

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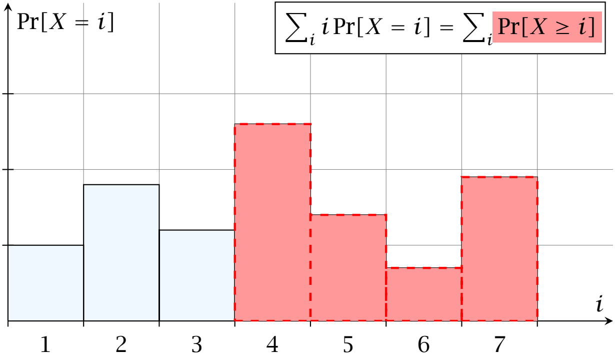

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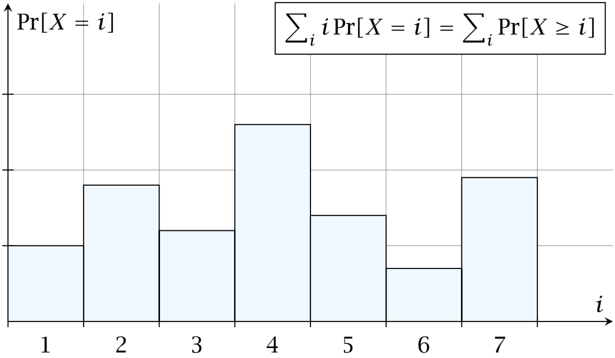

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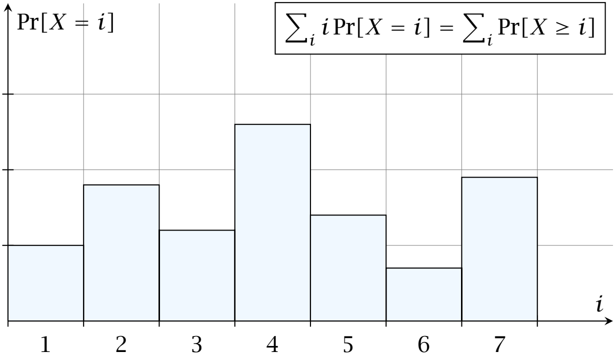

The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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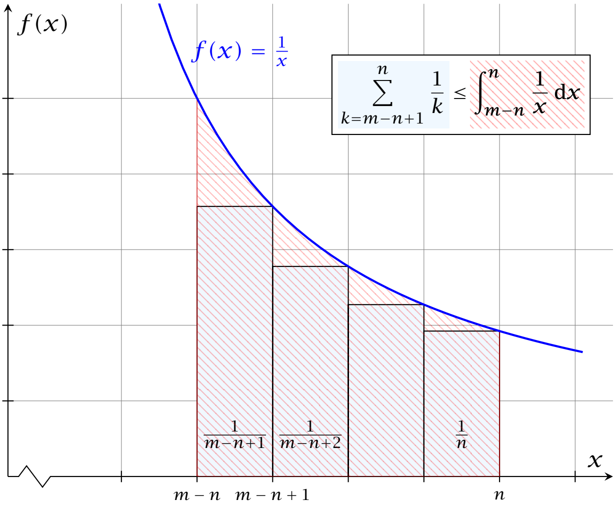

Deletions in Hashtables

How do we delete in a hash-table?

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- Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- One can delete an element by replacing it with a deleted-marker.
  - During an insertion if a deleted-marker is encountered an element can be inserted there.
  - During a search a deleted-marker must not be used to terminate the probe sequence.
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$p \leftarrow \text{succ}(p)$
while $T[p] \neq \text{null}$ **do**
    $y \leftarrow T[p]$
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p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

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Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \dots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

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However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \dots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

Universal Hashing

Definition 16

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

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Universal Hashing

Definition 17

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \dots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

- For any key $u \in U$, and $t \in \{0, \dots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} \ .$$

This requirement clearly implies a universal hash-function.

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Definition 18
A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\dots,n-1\}$ is called k-independent if for any choice of $\ell \le k$ distinct keys $u_1,\dots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\dots,t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \le \frac{1}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

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A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\dots,n-1\}$ is called (μ,k)-independent if for any choice of $\ell \le k$ distinct keys $u_1,\dots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\dots,t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \le \frac{\mu}{n^\ell} ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 20

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \ldots, n-1\}$.

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Proof.

Let $x, y \in U$ be two distinct keys. We have to show that the probability of a collision is only $1/n$.

- $ax + b \not\equiv ay + b \pmod{p}$

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  If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

  Multiplying with $a \not\equiv 0 \pmod{p}$ gives

  $$a(x - y) \not\equiv 0 \pmod{p}$$

  where we use that $\mathbb{Z}_p$ is a field (Körper) and, hence, has no zero divisors (nullteilerfrei).

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- The hash-function does not generate collisions before the $\pmod n$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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\begin{aligned}
a &\equiv (t_x - t_y)(x - y)^{-1} \pmod{p} \\
b &\equiv t_y - ay \pmod{p}
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Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs (a, b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the $\bmod n$-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

What happens when we do the $\bmod n$ operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \ldots, p-1$ the values $t_x, t_x + n, t_x + 2n, \ldots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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As $t_y \neq t_x$ there are

$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

possibilities for choosing t_y such that the final hash-value creates a collision.

This happens with probability at most $\frac{1}{n}$.

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right]$$

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$$\frac{\left\lfloor\frac{p}{n}\right\rfloor^2}{p(p-1)} \le \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2}\left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array}\right] \le \frac{\left\lceil\frac{p}{n}\right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor\frac{p}{n}\rfloor$ and $\lceil\frac{p}{n}\rceil$.

Universal Hashing

Definition 21

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$. Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \ .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

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The polynomial is defined by $d+1$ distinct points.

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Fix $\ell \leq d+1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$
Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.
We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

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possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Fix $\ell \leq d+1$; let $x_1,\dots,x_\ell \in \{0,\dots,q-1\}$ be keys, and let $t_1,\dots,t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1,\dots,\ell\}\}$
Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0,\dots,\lceil \tfrac{q}{n} \rceil - 1\}\}}_{=:B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

We first fix the values for inputs $x_1,\dots,x_\ell$.
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Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Therefore we have

$$|B_1| \cdot \ldots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}$$

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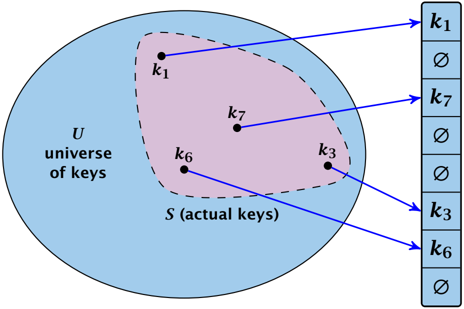
This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \ge (d+1)n$, and $\ell \le d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$\mathrm{E}[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \ .$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

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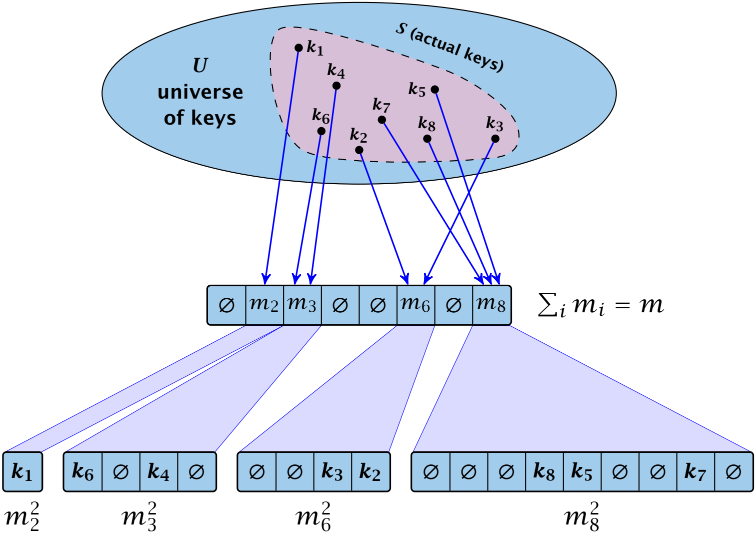

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$$
= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .
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Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

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Goal:
Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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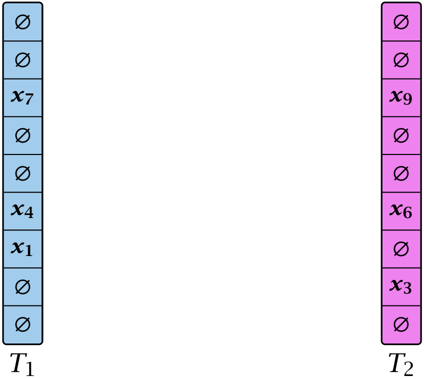

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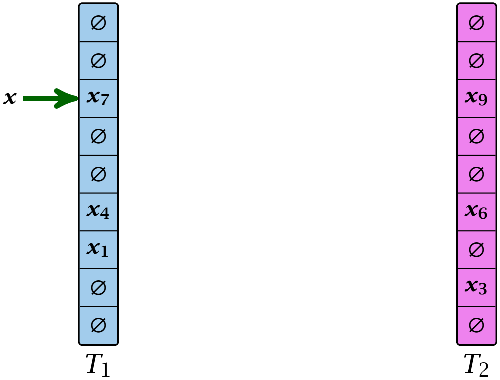

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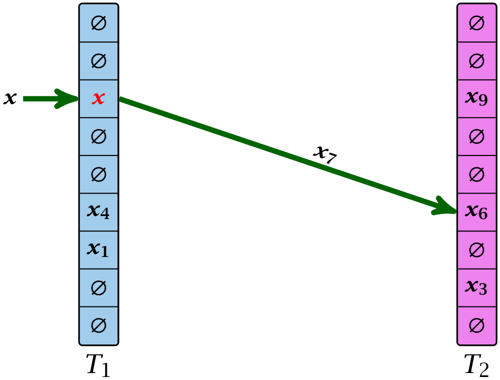

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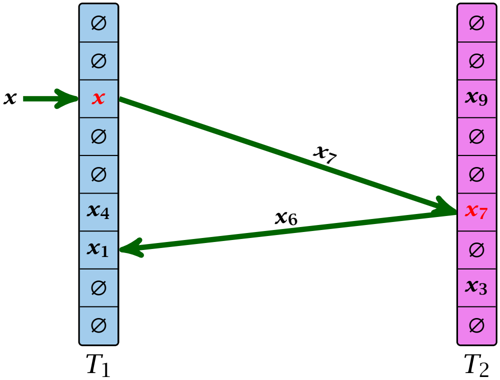

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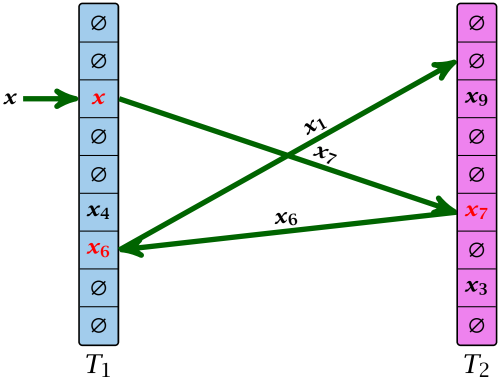

Cuckoo Hashing

Algorithm 13 Cuckoo-Insert(x)

if $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$ **then return**
steps $\leftarrow 1$
while steps $\leq$ maxsteps **do**
    exchange x and $T_1[h_1(x)]$
    if x = null **then return**
    exchange x and $T_2[h_2(x)]$
    if x = null **then return**
    steps $\leftarrow$ steps $+1$
rehash() // change hash-functions; rehash everything
Cuckoo-Insert(x)

Cuckoo Hashing

- We call one iteration through the while-loop a step of the algorithm.
- We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.
- We say a phase is successful if it is not terminated by the maxstep-condition, but the while loop is left because $x = \text{null}$.

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What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after $\mathrm{maxsteps}$ steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

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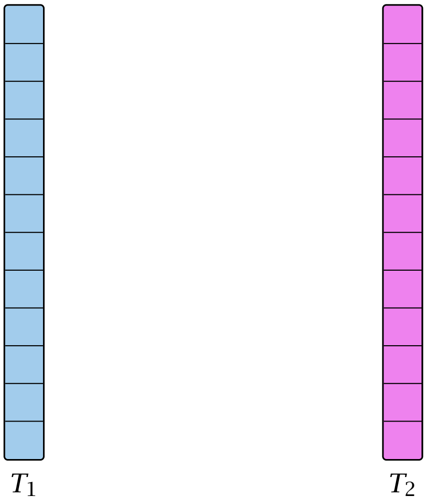

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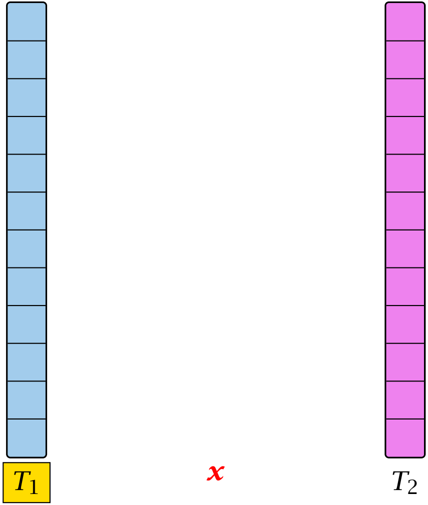

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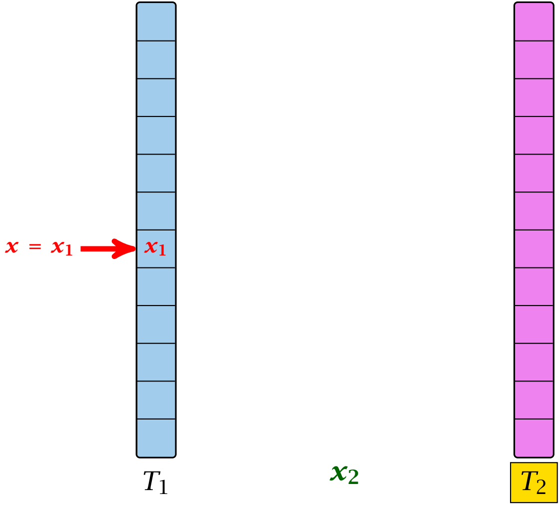

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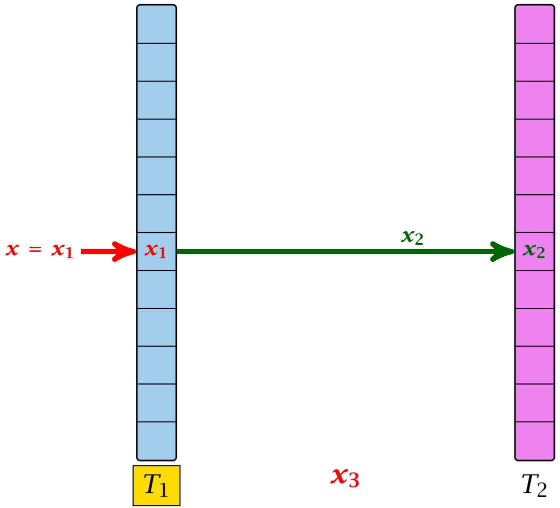

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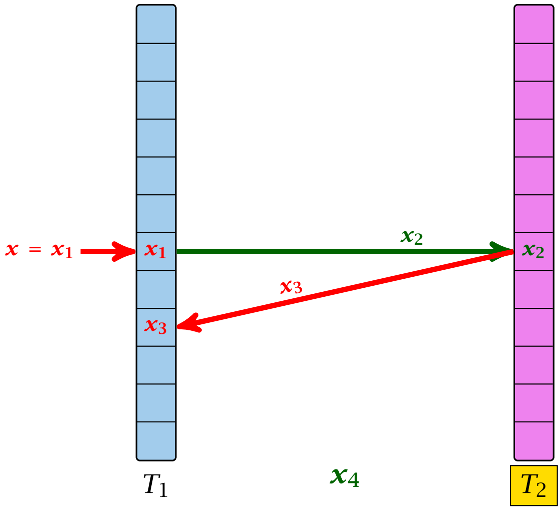

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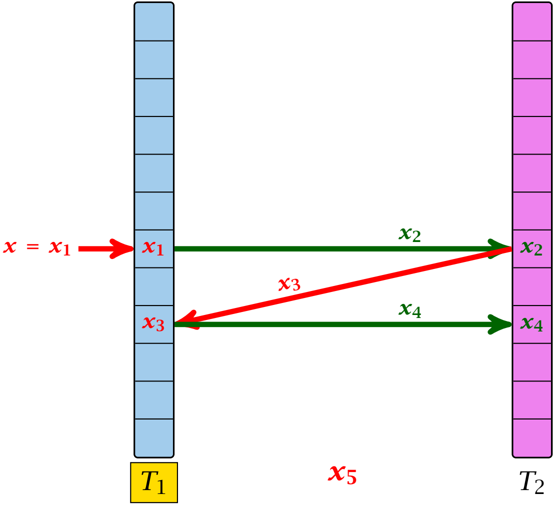

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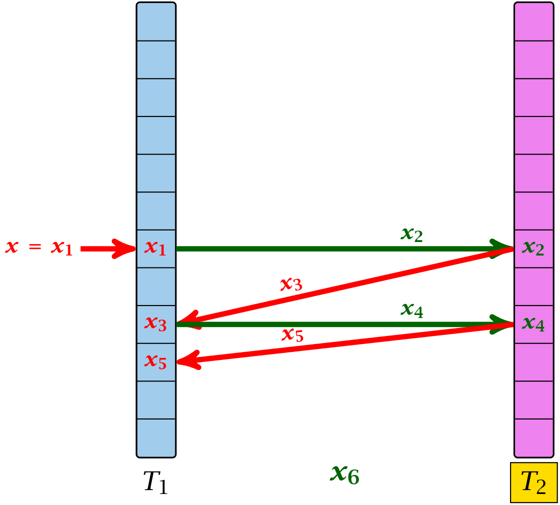

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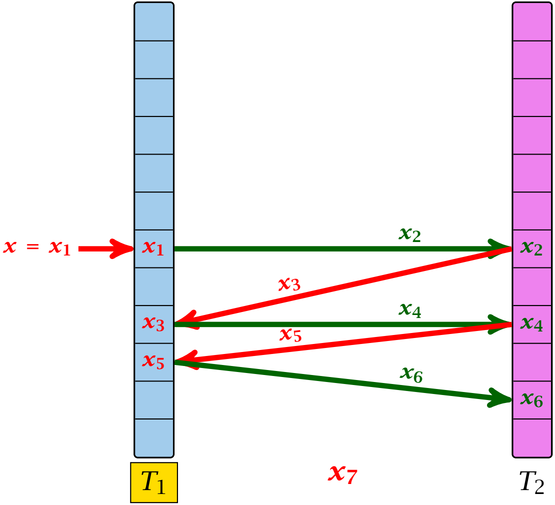

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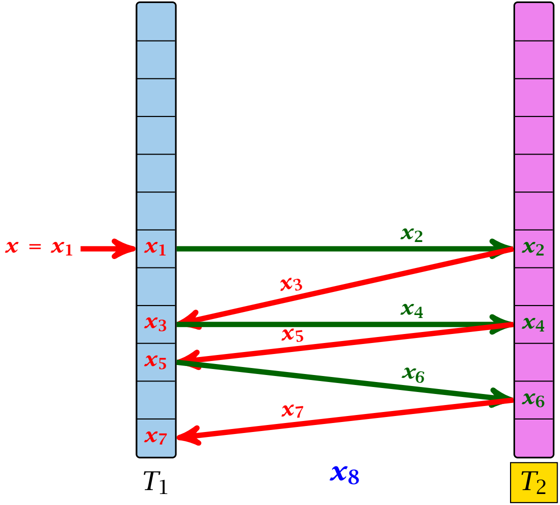

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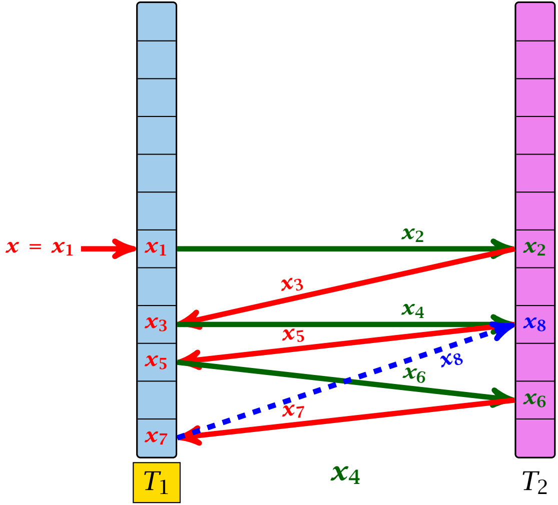

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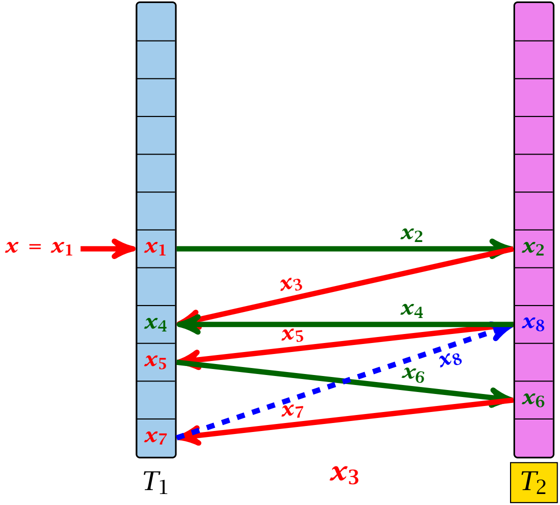

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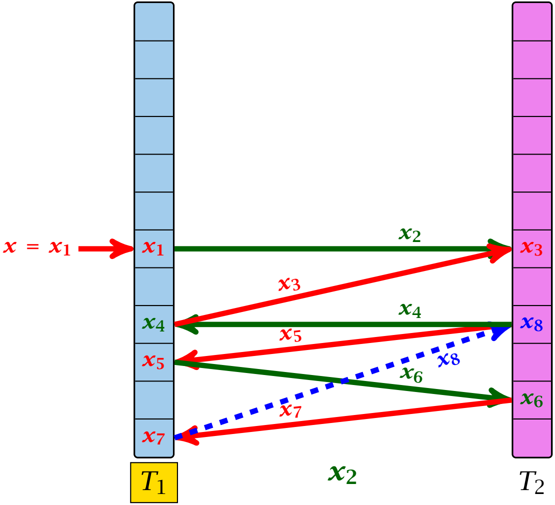

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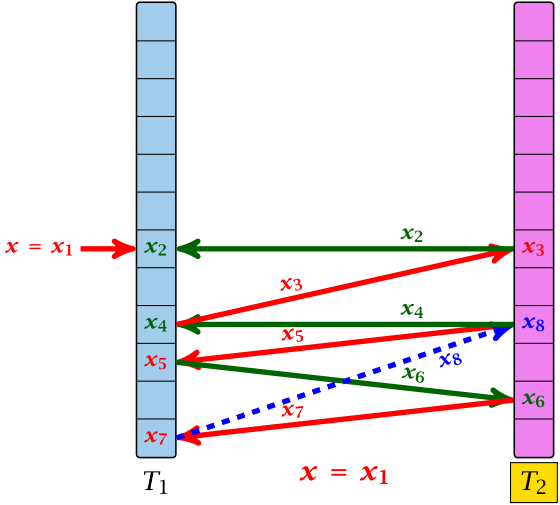

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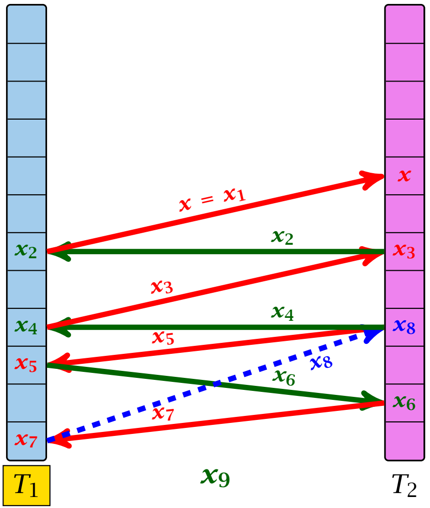

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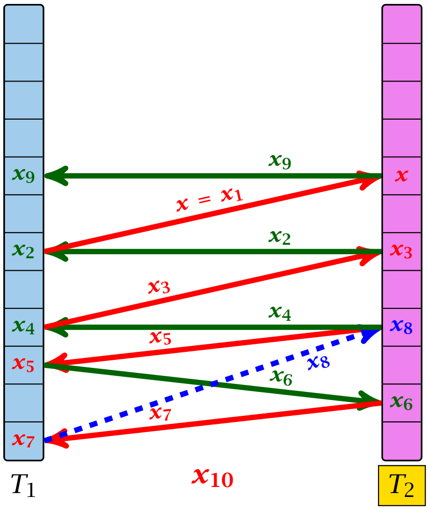

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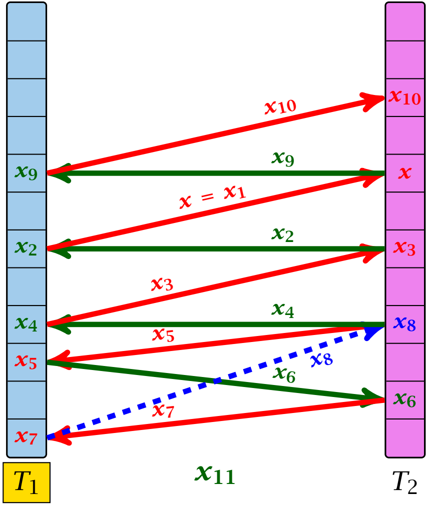

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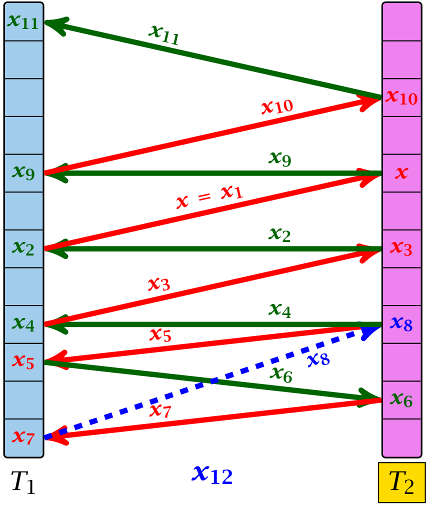

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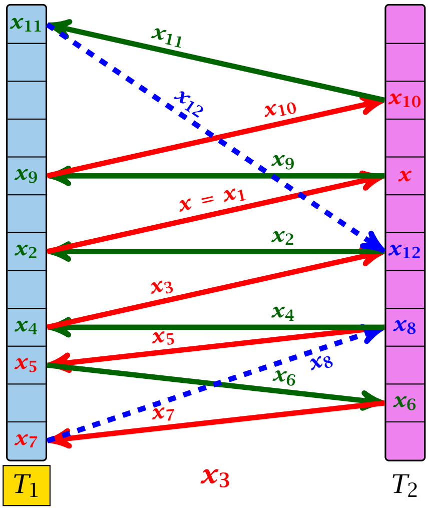

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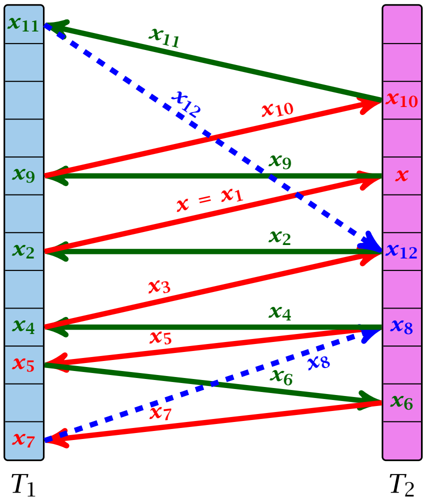

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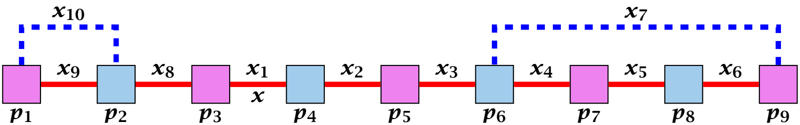

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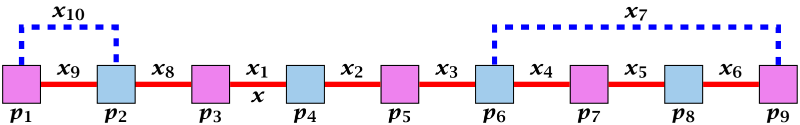

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- $s-1$ different cells (alternating btw. cells from T_1 and T_2).
- s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.
- The leftmost cell is “linked forward” to some cell on the right.
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- One link represents key x; this is where the counting starts.

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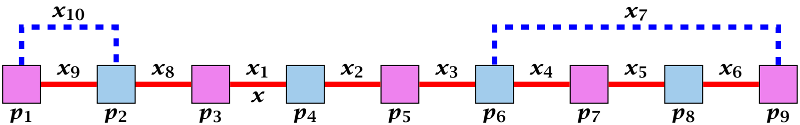

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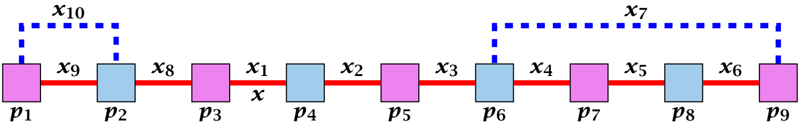

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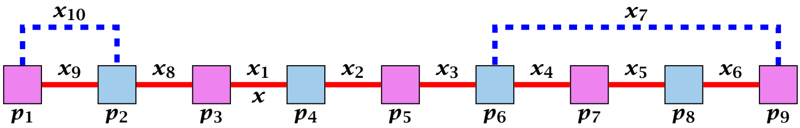

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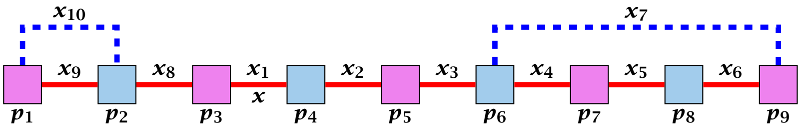

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A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

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What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

These events are independent.

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- There are at most s^2 possibilities where to attach the forward and backward links.
- There are at most s possibilities to choose where to place key x.
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The probability that there exists an active cycle-structure is therefore at most

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Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) \quad .$$

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Now, we analyze the probability that a phase is not successful without running into a closed cycle.

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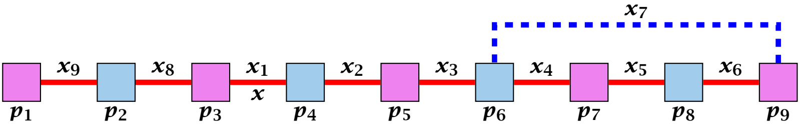

Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 22

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

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Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$

As $r \le i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \le i + j - 1 \ .$$

Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. □

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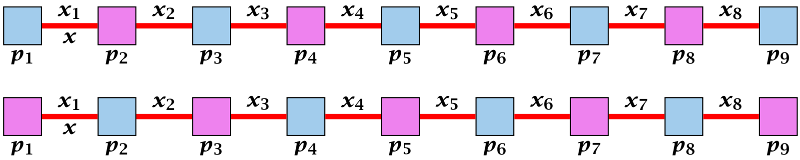

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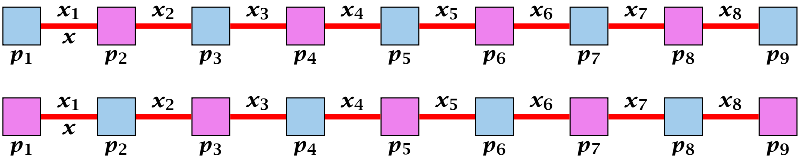

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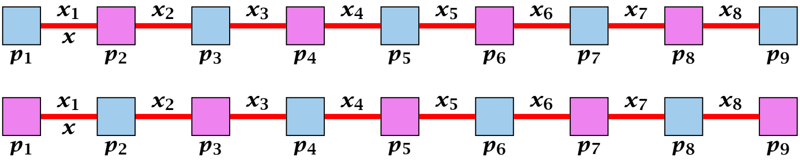

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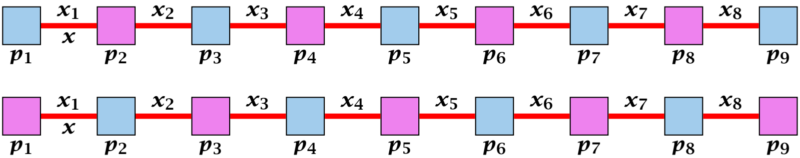

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Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t+2)/3$.

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by choosing $\ell \geq \log\big(\frac{1}{2\mu^2 m^2}\big)/\log\big(\frac{1}{1+\epsilon}\big) = \log\big(2\mu^2 m^2\big)/\log\big(1+\epsilon\big)$

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This gives $\mathrm{maxsteps} = \Theta(\log m)$.

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

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$$\begin{aligned}\Pr[\text{successful}] &= \Pr[\text{no cycle}] - \Pr[\text{unsuccessful} \mid \text{no cycle}] \\ &\ge c \cdot \Pr[\text{no cycle}]\end{aligned}$$

for a suitable constant $c > 0$.

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

- A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).
- The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching $\mathrm{maxsteps}$ without running into a cycle).
- A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.
- The expected number of unsuccessful rehashes is $\sum_{i\geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.
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The expected cost for all rehashes is

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What kind of hash-functions do we need?

Since $\mathrm{maxsteps}$ is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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How do we make sure that $n \geq (1+\epsilon)m$?

- Let $\alpha := 1/(1+\epsilon)$.
- Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).
- Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).
- Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.
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Lemma 23

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

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