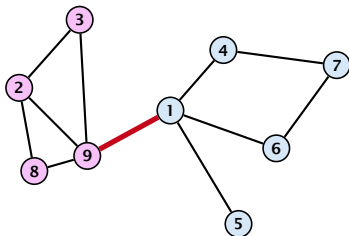


15 Global Mincut

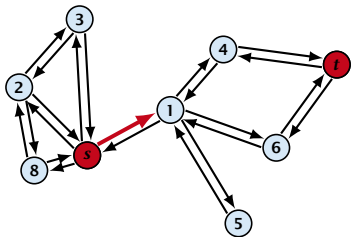
Given an **undirected, capacitated graph** $G = (V, E, c)$ find a partition of V into two non-empty sets $S, V \setminus S$ s.t. the capacity of edges between both sets is minimized.



15 Global Mincut

We can solve this problem using standard maxflow/mincut.

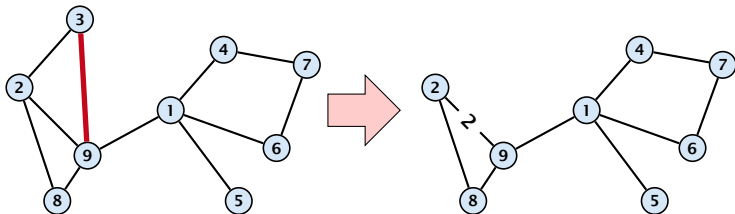
- ▶ Construct a directed graph $G' = (V, E')$ that has edges (u, v) and (v, u) for every edge $\{u, v\} \in E$.
- ▶ Fix an arbitrary node $s \in V$ as source. Compute a minimum s - t cut for all possible choices $t \in V, t \neq s$. (Time: $\mathcal{O}(n^4)$)
- ▶ Let $(S, V \setminus S)$ be a minimum global mincut. The above algorithm will output a cut of capacity $\text{cap}(S, V \setminus S)$ whenever $|\{s, t\} \cap S| = 1$.



Edge Contractions

- ▶ Given a graph $G = (V, E)$ and an edge $e = \{u, v\}$.
- ▶ The graph G/e is obtained by “identifying” u and v to form a new node.
- ▶ Resulting parallel edges are replaced by a single edge, whose capacity equals the sum of capacities of the parallel edges.

Example 1



- ▶ Edge-contractions do not decrease the size of the mincut.

Edge Contractions

We can perform an edge-contraction in time $\mathcal{O}(n)$.

Randomized Mincut Algorithm

Algorithm 7 KargerMincut($G = (V, E, c)$)

```
1: for  $i = 1 \rightarrow n - 2$  do
2:     choose  $e \in E$  randomly with probability  $c(e)/c(E)$ 
3:      $G \leftarrow G/e$ 
4: return only cut in  $G$ 
```

- ▶ Let G_t denote the graph after the $(n - t)$ -th iteration, when t nodes are left.
- ▶ Note that the final graph G_2 only contains a single edge.
- ▶ The cut in G_2 corresponds to a cut in the original graph G with the same capacity.
- ▶ What is the probability that this algorithm returns a mincut?

Example: Randomized Mincut Algorithm



What is the probability that a given mincut A is still possible after round i ?

- ▶ It is still possible to obtain cut A in the end if so far **no** edge in $(A, V \setminus A)$ has been contracted.

Analysis

What is the probability that we select an edge from A in iteration i ?

- ▶ Let $\min = \text{cap}(A, V \setminus A)$ denote the capacity of a mincut.
- ▶ Let $\text{cap}(v)$ be capacity of edges incident to vertex $v \in V_{n-i+1}$.
- ▶ Clearly, $\text{cap}(v) \geq \min$.
- ▶ Summing $\text{cap}(v)$ over all edges gives

$$2c(E) = 2 \sum_{e \in E} c(e) = \sum_{v \in V} \text{cap}(v) \geq (n - i + 1) \cdot \min$$

- ▶ Hence, the probability of choosing an edge from the cut is at most $\frac{\min}{2c(E)} \leq \frac{2}{n - i + 1}$

$n - i + 1$ is the number of nodes in graph $G_{n-i+1} = (V_{n-i+1}, E_{n-i+1})$, the graph at the start of iteration i .

Analysis

The probability that we do **not** choose an edge from the cut in iteration i is

$$1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1} .$$

The probability that the cut is alive after iteration $n-t$ (after which t nodes are left) is

$$\prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1} = \frac{t(t-1)}{n(n-1)} .$$

Choosing $t = 2$ gives that with probability $1/\binom{n}{2}$ the algorithm computes a mincut.

Analysis

Repeating the algorithm $c \ln n \binom{n}{2}$ times gives that the probability that we are never successful is

$$\left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2} c \ln n} \leq \left(e^{-1/\binom{n}{2}}\right)^{\binom{n}{2} c \ln n} \leq n^{-c},$$

where we used $1 - x \leq e^{-x}$.

Theorem 2

The randomized mincut algorithm computes an optimal cut with high probability. The total running time is $\mathcal{O}(n^4 \log n)$.

Improved Algorithm

Algorithm 8 RecursiveMincut($G = (V, E, c)$)

```
1: for  $i = 1 \rightarrow n - n/\sqrt{2}$  do
2:   choose  $e \in E$  randomly with probability  $c(e)/c(E)$ 
3:    $G \leftarrow G/e$ 
4: if  $|V| = 2$  return cut-value;
5:  $cuta \leftarrow$  RecursiveMincut( $G$ );
6:  $cutb \leftarrow$  RecursiveMincut( $G$ );
7: return  $\min\{cuta, cutb\}$ 
```

Running time:

- ▶ $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + \mathcal{O}(n^2)$
- ▶ This gives $T(n) = \mathcal{O}(n^2 \log n)$.

Note that the above implementation only works for very special values of n .

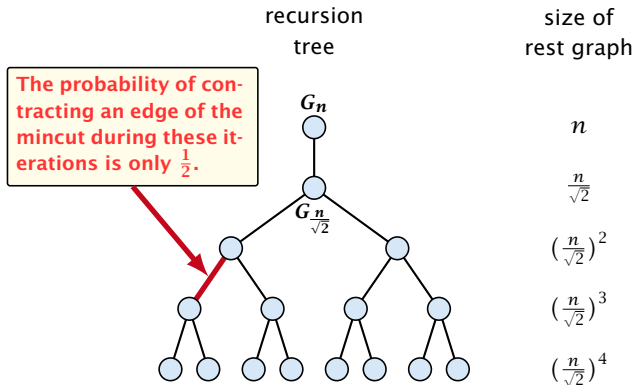
Probability of Success

The probability of contracting an edge from the mincut during one iteration through the for-loop is only

$$\frac{t(t-1)}{n(n-1)} \leq \frac{t^2}{n^2} = \frac{1}{2} ,$$

as $t = \frac{n}{\sqrt{2}}$.

Probability of Success



We can estimate the success probability by using the following game on the recursion tree. Delete every edge with probability $\frac{1}{2}$. If in the end you have a path from the root to **at least one** leaf node you are successful.

Probability of Success

Let for an edge e in the recursion tree, $h(e)$ denote the height (distance to leaf level) of the parent-node of e (end-point that is higher up in the tree). Let h denote the height of the root node.

Call an edge e **alive** if there exists a path from the parent-node of e to a descendant leaf, after we randomly deleted edges. Note that an edge can only be alive if it hasn't been deleted.

Lemma 3

The probability that an edge e is alive is at least $\frac{1}{h(e)+1}$.

Probability of Success

Proof.

- ▶ An edge e with $h(e) = 1$ is alive if and only if it is not deleted. Hence, it is alive with probability at least $\frac{1}{2}$.
- ▶ Let p_d be the probability that an edge e with $h(e) = d$ is alive. For $d > 1$ this happens for edge $e = \{c, p\}$ if it is not deleted **and** if one of the child-edges connecting to c is alive.
- ▶ This happens with probability

$$p_d = \frac{1}{2} (2p_{d-1} - p_{d-1}^2) \quad \Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$

$$= p_{d-1} - \frac{p_{d-1}^2}{2}$$

$$\geq \frac{1}{d} - \frac{1}{2d^2} \geq \frac{1}{d} - \frac{1}{d(d+1)} = \frac{1}{d+1} .$$

$x - x^2/2$ is monotonically increasing for $x \in [0, 1]$

Lemma 4

One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ and has a success probability of $\Omega(\frac{1}{\log n})$.

Doing $\Theta(\log^2 n)$ runs gives that the algorithm succeeds with high probability. The total running time is $\mathcal{O}(n^2 \log^3 n)$.