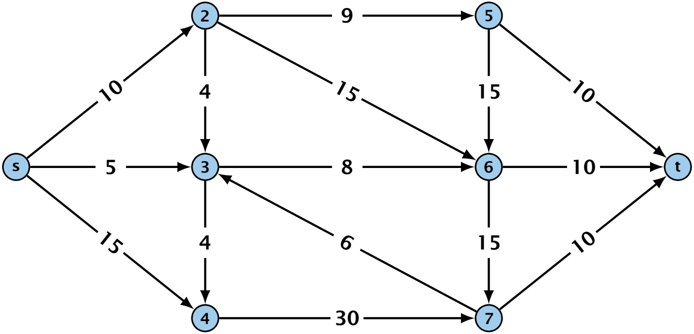
# 10 Introduction

## Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source $s$; target $t$;
- no edges entering $s$ or leaving $t$;
- at least for now: no parallel edges;



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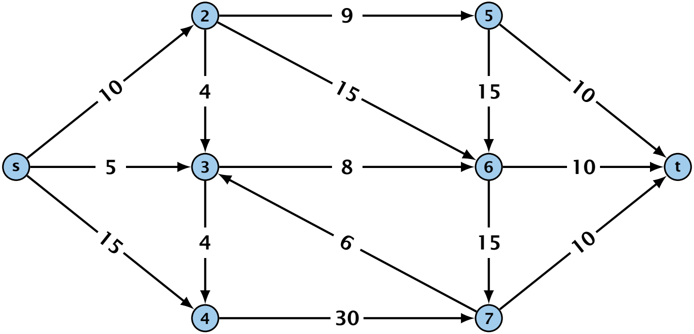

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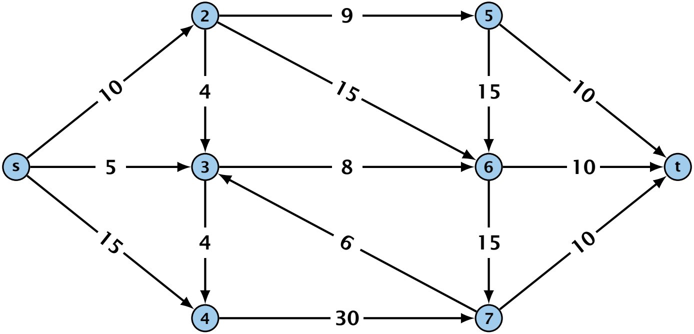

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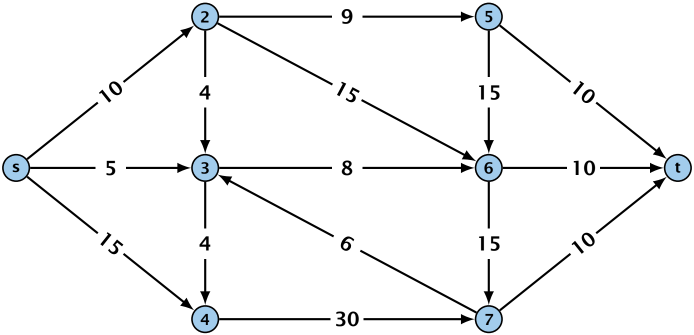

# Cuts

**Definition 1**

An $(s, t)$-cut in the graph $G$ is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

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The capacity of a cut $A$ is defined as

$$\mathrm{cap}(A, V \setminus A) := \sum_{e \in \mathrm{out}(A)} c(e) \ ,$$

where $\mathrm{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving $A$).

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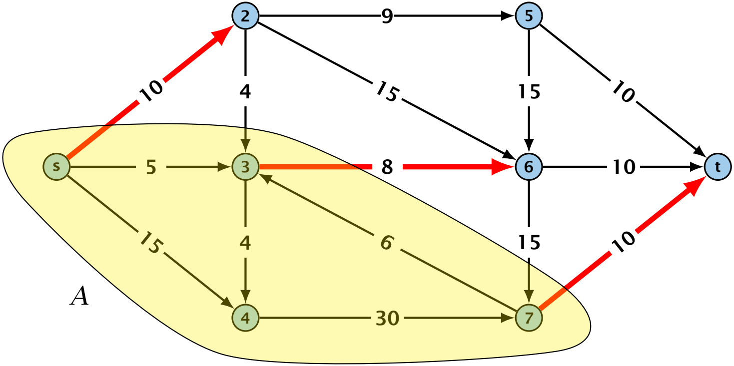
**Minimum Cut Problem:** Find an $(s,t)$-cut with minimum capacity.

# Cuts

## Example 3




The capacity of the cut is $\mathrm{cap}(A, V \setminus A) = 28$.

# Flows

**Definition 4**
An $(s,t)$-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge $e$

$$0 \le f(e) \le c(e) \ .$$

(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_{e \in \mathrm{out}(v)} f(e) = \sum_{e \in \mathrm{into}(v)} f(e) \ .$$

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# Flows

**Definition 5**
The value of an $(s,t)$-flow $f$ is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) \enspace .$$

**Maximum Flow Problem:** Find an $(s,t)$-flow with maximum value.

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# Flows

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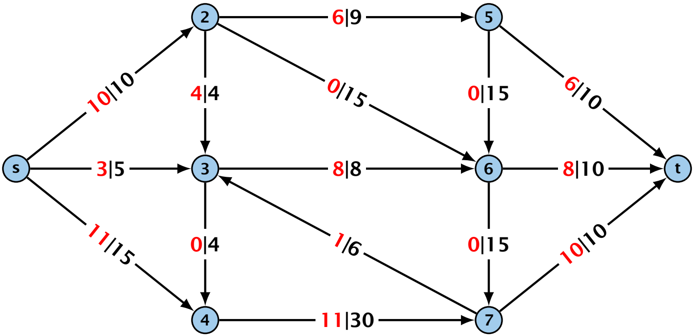


The value of the flow is $\text{val}(f) = 24$.

# Flows

## Lemma 7 (Flow value lemma)

*Let $f$ be a flow, and let $A \subseteq V$ be an $(s,t)$-cut. Then the net-flow across the cut is equal to the amount of flow leaving $s$, i.e.,*

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e) \enspace .$$

**Proof.**

$\mathrm{val}(f)$

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$$\begin{aligned}\mathrm{val}(f) &= \sum_{e\in\mathrm{out}(s)} f(e)\\ &= \sum_{e\in\mathrm{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}}\Bigg(\sum_{e\in\mathrm{out}(v)} f(e) - \sum_{e\in\mathrm{in}(v)} f(e)\Bigg)\end{aligned}$$

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$$
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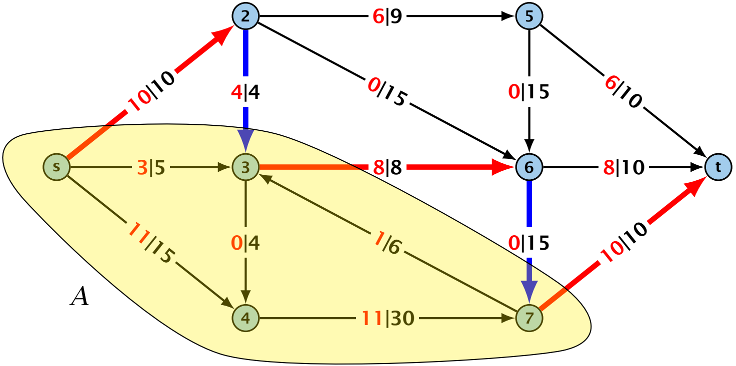
The last equality holds since every edge with both end-points in $A$ contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering $A$. □

## Example 8



2
5
3
6
4
7
s
t
6|9
10|10
4|4
0/15
0|15
6/10
3|5
8|8
8|10
11/15
0|4
1/6
0|15
10|10
11|30
A

## Corollary 9

*Let* $f$ *be an* $(s,t)$*-flow and let* $A$ *be an* $(s,t)$*-cut, such that*

$$\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A).$$

*Then* $f$ *is a maximum flow.*

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Suppose that there is a flow $f'$ with larger value. Then

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Suppose that there is a flow $f'$ with larger value. Then

$$\mathrm{cap}(A, V \setminus A) < \mathrm{val}(f')$$

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Suppose that there is a flow $f'$ with larger value. Then

$$\begin{aligned}\mathrm{cap}(A, V \setminus A) &< \mathrm{val}(f') \\ &= \sum_{e \in \mathrm{out}(A)} f'(e) - \sum_{e \in \mathrm{into}(A)} f'(e)\end{aligned}$$

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&\leq \mathrm{cap}(A, V \setminus A)
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□