#### **Degeneracy Revisited**

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

#### Idea:

Change LP :=  $\max\{c^T x, Ax = b; x \ge 0\}$  into LP' :=  $\max\{c^T x, Ax = b', x \ge 0\}$  such that

- I. LP is feasible
- **II.** If a set *B* of basis variables corresponds to an infeasible basis (i.e.  $A_B^{-1}b \neq 0$ ) then *B* corresponds to an infeasible basis in LP' (note that columns in  $A_B$  are linearly independent).
- III. LP has no degenerate basic solutions

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#### **Degeneracy Revisited**

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

#### Idea:

Given feasible LP :=  $\max\{c^T x, Ax = b; x \ge 0\}$ . Change it into LP' :=  $\max\{c^T x, Ax = b', x \ge 0\}$  such that

I. LP' is feasible

- II. If a set *B* of basis variables corresponds to an infeasible basis (i.e.  $A_B^{-1}b \neq 0$ ) then *B* corresponds to an infeasible basis in LP' (note that columns in  $A_B$  are linearly independent).
- III. LP' has no degenerate basic solutions

**Degenerate Example** max 13a + 23bs.t.  $5a + 15b + s_c$ = 480 $\frac{80}{17} \cdot a + 4b$  $+ S_h$ = 16035a + 20b $+ s_m = 1190$ a, b,  $s_c$ ,  $s_h$ ,  $s_m \ge 0$ {a. b. beer  $\{s_c, s_h, s_m\}$ ale

## Perturbation Let *B* be index set of some basis with basic solution $x_B^* = A_B^{-1}b \ge 0, x_N^* = 0 \quad (i.e. \ B \ is \ feasible)$ Fix $b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \text{ for } \varepsilon > 0 \ .$ This is the perturbation that we are using.

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#### **Property I**

The new LP is feasible because the set B of basis variables provides a feasible basis:

$$A_B^{-1}\left(b+A_B\begin{pmatrix}\varepsilon\\\vdots\\\varepsilon^m\end{pmatrix}\right)=x_B^*+\begin{pmatrix}\varepsilon\\\vdots\\\varepsilon^m\end{pmatrix}\geq 0.$$

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#### **Property III**

Let  $\tilde{B}$  be a basis. It has an associated solution

 $x_{\tilde{B}}^* = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$ 

in the perturbed instance.

We can view each component of the vector as a polynom with variable  $\varepsilon$  of degree at most m.

 $A_{\tilde{B}}^{-1}A_B$  has rank *m*. Therefore no polynom is 0.

A polynom of degree at most m has at most m roots (Nullstellen).

Hence,  $\epsilon > 0$  small enough gives that no component of the above vector is 0. Hence, no degeneracies.

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#### Property II

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Let  $\tilde{B}$  be a non-feasible basis. This means  $(A_{\tilde{B}}^{-1}b)_i < 0$  for some row i.

Then for small enough  $\epsilon > 0$ 

 $\left(A_{\tilde{B}}^{-1}\left(b+A_{B}\begin{pmatrix}\varepsilon\\\vdots\\\varepsilon^{m}\end{pmatrix}\right)\right)_{i} = (A_{\tilde{B}}^{-1}b)_{i} + \left(A_{\tilde{B}}^{-1}A_{B}\begin{pmatrix}\varepsilon\\\vdots\\\varepsilon^{m}\end{pmatrix}\right)_{i} < 0$ 

Hence,  $\tilde{B}$  is not feasible.

Since, there are no degeneracies Simplex will terminate when run on  $LP^\prime.$ 

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If it terminates because the reduced cost vector fulfills

 $\tilde{c} = (c^T - c_B^T A_B^{-1} A) \le 0$ 

then we have found an optimal basis. Note that this basis is also optimal for LP, as the above constraint does not depend on b.

If it terminates because it finds a variable x<sub>j</sub> with c̃<sub>j</sub> > 0 for which the *j*-th basis direction *d*, fulfills *d* ≥ 0 we know that LP' is unbounded. The basis direction does not depend on *b*. Hence, we also know that LP is unbounded.

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#### **Lexicographic Pivoting**

Doing calculations with perturbed instances may be costly. Also the right choice of  $\varepsilon$  is difficult.

Idea:

Simulate behaviour of LP' without explicitly doing a perturbation.

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**Lexicographic Pivoting** 

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## In the following we assume that $b \ge 0$ . This can be obtained by replacing the initial system $(A \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where *B* is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm). Then the perturbed instance is $b' = b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$

#### Lexicographic Pivoting

We choose the entering variable arbitrarily as before ( $\tilde{c}_e > 0$ , of course).

If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

Otherwise we have to be careful.

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### Matrix View Let our linear program be $c_B^T x_B + c_N^T x_N = Z$ $A_B x_B + A_N x_N = b$ $x_B , \quad x_N \ge 0$ The simplex tableaux for basis *B* is $(c_N^T - c_B^T A_B^{-1} A_N) x_N = Z - c_B^T A_B^{-1} b$ $I x_B + A_B^{-1} A_N x_N = A_B^{-1} b$ $I x_B , \quad x_N \ge 0$ The BFS is given by $x_N = 0, x_B = A_B^{-1} b$ . If $(c_N^T - c_B^T A_B^{-1} A_N) \le 0$ we know that we have an optimum solution.

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#### **Lexicographic Pivoting**

LP chooses an arbitrary leaving variable that has  $\hat{A}_{\ell e} > 0$  and minimizes

$$\theta_{\ell} = \frac{b_{\ell}}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_{\ell}}{(A_B^{-1}A_{*e})_{\ell}} \ .$$

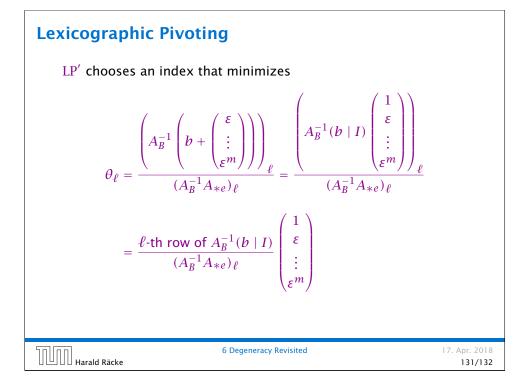
 $\ell$  is the index of a leaving variable within *B*. This means if e.g.  $B = \{1, 3, 7, 14\}$  and leaving variable is 3 then  $\ell = 2$ .

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#### Lexicographic Pivoting

#### **Definition 2**

 $u \leq_{\text{lex}} v$  if and only if the first component in which u and v differ fulfills  $u_i \leq v_i$ .

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# **Lexicographic Pivoting** This means you can choose the variable/row $\ell$ for which the vector $\frac{\ell \cdot \text{th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_{\ell}}$ is lexicographically minimal. Of course only including rows with $(A_B^{-1}A_{*e})_{\ell} > 0$ . This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

