There are many practically important optimization problems that are NP-hard.

#### What can we do?

- Heuristics.
- Exploit special structure of instances occurring in practise.
- Consider algorithms that do not compute the optimal solution but provide solutions that are close to optimum.

## **Definition 2**

An  $\alpha$ -approximation for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of  $\alpha$  of the value of an optimal solution.

10 Introduction to Approximation

Harald Räcke

17. Apr. 2018 219/234

## Why approximation algorithms?

- We need algorithms for hard problems.
- It gives a rigorous mathematical base for studying heuristics.
- It provides a metric to compare the difficulty of various optimization problems.
- Proving theorems may give a deeper theoretical understanding which in turn leads to new algorithmic approaches.

## Why not?

Sometimes the results are very pessimistic due to the fact that an algorithm has to provide a close-to-optimum solution on every instance.



17. Apr. 2018 221/234

## **Definition 3**

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An optimization problem P = (1, sol, m, goal) is in **NPO** if

- $x \in \mathcal{I}$  can be decided in polynomial time
- $y \in sol(\mathcal{I})$  can be verified in polynomial time
- *m* can be computed in polynomial time
- ▶ goal  $\in$  {min, max}

In other words: the decision problem is there a solution y with m(x, y) at most/at least z is in NP.



10 Introduction to Approximation

17. Apr. 2018

220/234

- $\blacktriangleright x$  is problem instance
- > y is candidate solution
- $m^*(x)$  cost/profit of an optimal solution

# **Definition 4 (Performance Ratio)**

$$R(x, y) := \max\left\{\frac{m(x, y)}{m^*(x)}, \frac{m^*(x)}{m(x, y)}\right\}$$

החוחר	10 Introduction to Approximation	17. Apr. 2018
Harald Räcke		223/234

# **Definition 6 (PTAS)**

A PTAS for a problem *P* from NPO is an algorithm that takes as input  $x \in I$  and  $\epsilon > 0$  and produces a solution  $\gamma$  for x with

# $R(x,y) \leq 1 + \epsilon$ .

The running time is polynomial in |x|.

approximation with arbitrary good factor... fast?

	( <i>r</i> -approximation)
ri aigorithm	A is an $r$ -approximation algorithm iff
	$\forall x \in \mathcal{I} : R(x, A(x)) \leq r ,$
nd A runs ir	n polynomial time.

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10 Introduction to Approximation

## Problems that have a PTAS

**Scheduling**. Given m jobs with known processing times; schedule the jobs on n machines such that the MAKESPAN is minimized.



17. Apr. 2018 225/234



17. Apr. 2018

224/234

#### **Definition 7 (FPTAS)**

An FPTAS for a problem *P* from NPO is an algorithm that takes as input  $x \in \mathcal{I}$  and  $\epsilon > 0$  and produces a solution  $\mathcal{Y}$  for x with

# $R(x,y) \leq 1 + \epsilon$ .

The running time is polynomial in |x| and  $1/\epsilon$ .

approximation with arbitrary good factor... fast!

10 Introduction to Harald Räcke

10 Introduction to Approximation

## **Definition 8 (APX – approximable)**

A problem *P* from NPO is in APX if there exist a constant  $r \ge 1$ and an r-approximation algorithm for *P*.

constant factor approximation...

# Problems that have an FPTAS

**KNAPSACK**. Given a set of items with profits and weights choose a subset of total weight at most W s.t. the profit is maximized.

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10 Introduction to Approximation

17. Apr. 2018 228/234

## Problems that are in APX

**MAXCUT.** Given a graph G = (V, E); partition V into two disjoint pieces A and B s.t. the number of edges between both pieces is maximized.

**MAX-3SAT.** Given a 3CNF-formula. Find an assignment to the variables that satisfies the maximum number of clauses.



17. Apr. 2018 229/234

17. Apr. 2018

227/234



Problems with polylogarithmic approximation guarantees

- Set Cover
- Minimum Multicut
- Sparsest Cut
- Minimum Bisection

There is an r-approximation with  $r \leq O(\log^{c}(|x|))$  for some constant c.

Note that only for some of the above problem a matching lower bound is known.

10 Introduction to Approximation

17. Apr. 2018 231/234

There are weird problems!

Asymmetric *k*-Center admits an  $O(\log^* n)$ -approximation.

There is no  $o(\log^* n)$ -approximation to Asymmetric *k*-Center unless  $NP \subseteq DTIME(n^{\log \log \log n})$ .

There are real	ly difficult	problems!
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## **Theorem 9**

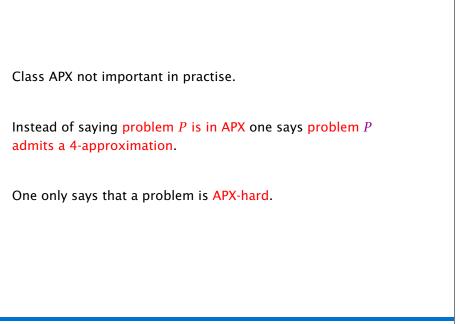
For any constant  $\epsilon > 0$  there does not exist an  $\Omega(n^{1-\epsilon})$ -approximation algorithm for the maximum clique problem on a given graph *G* with *n* nodes unless P = NP.

Note that an *n*-approximation is trivial.

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10 Introduction to Approximation

17. Apr. 2018 232/234





17. Apr. 2018 233/234