There are many practically important optimization problems that are NP-hard.

What can we do?

- Heuristics.
- Exploit special structure of instances occurring in practise.
- Consider algorithms that do not compute the optimal solution but provide solutions that are close to optimum.

Definition 2

An α -approximation for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of α of the value of an optimal solution.

10 Introduction to Approximation

Harald Räcke

17. Apr. 2018 219/234

Why approximation algorithms?

- We need algorithms for hard problems.
- It gives a rigorous mathematical base for studying heuristics.
- It provides a metric to compare the difficulty of various optimization problems.
- Proving theorems may give a deeper theoretical understanding which in turn leads to new algorithmic approaches.

Why not?

Sometimes the results are very pessimistic due to the fact that an algorithm has to provide a close-to-optimum solution on every instance.



17. Apr. 2018 221/234

Definition 3

Harald Räcke

An optimization problem P = (1, sol, m, goal) is in **NPO** if

- $x \in \mathcal{I}$ can be decided in polynomial time
- $y \in sol(\mathcal{I})$ can be verified in polynomial time
- *m* can be computed in polynomial time
- ▶ goal \in {min, max}

In other words: the decision problem is there a solution y with m(x, y) at most/at least z is in NP.



10 Introduction to Approximation

17. Apr. 2018

220/234

- $\blacktriangleright x$ is problem instance
- > y is candidate solution
- $m^*(x)$ cost/profit of an optimal solution

Definition 4 (Performance Ratio)

$$R(x, y) := \max\left\{\frac{m(x, y)}{m^*(x)}, \frac{m^*(x)}{m(x, y)}\right\}$$

החוחר	10 Introduction to Approximation	17. Apr. 2018
Harald Räcke		223/234

Definition 6 (PTAS)

A PTAS for a problem *P* from NPO is an algorithm that takes as input $x \in I$ and $\epsilon > 0$ and produces a solution γ for x with

$R(x,y) \leq 1 + \epsilon$.

The running time is polynomial in |x|.

approximation with arbitrary good factor... fast?

	(<i>r</i> -approximation)
ri aigorithm	A is an r -approximation algorithm iff
	$\forall x \in \mathcal{I} : R(x, A(x)) \leq r ,$
nd A runs ir	n polynomial time.

Harald Räcke

10 Introduction to Approximation

Problems that have a PTAS

Scheduling. Given m jobs with known processing times; schedule the jobs on n machines such that the MAKESPAN is minimized.



17. Apr. 2018 225/234



17. Apr. 2018

224/234

Definition 7 (FPTAS)

An FPTAS for a problem *P* from NPO is an algorithm that takes as input $x \in \mathcal{I}$ and $\epsilon > 0$ and produces a solution \mathcal{Y} for x with

$R(x,y) \leq 1 + \epsilon$.

The running time is polynomial in |x| and $1/\epsilon$.

approximation with arbitrary good factor... fast!

10 Introduction to Harald Räcke

10 Introduction to Approximation

Definition 8 (APX – approximable)

A problem *P* from NPO is in APX if there exist a constant $r \ge 1$ and an r-approximation algorithm for *P*.

constant factor approximation...

Problems that have an FPTAS

KNAPSACK. Given a set of items with profits and weights choose a subset of total weight at most W s.t. the profit is maximized.

Harald Räcke

10 Introduction to Approximation

17. Apr. 2018 228/234

Problems that are in APX

MAXCUT. Given a graph G = (V, E); partition V into two disjoint pieces A and B s.t. the number of edges between both pieces is maximized.

MAX-3SAT. Given a 3CNF-formula. Find an assignment to the variables that satisfies the maximum number of clauses.



17. Apr. 2018 229/234

17. Apr. 2018

227/234



Problems with polylogarithmic approximation guarantees

- Set Cover
- Minimum Multicut
- Sparsest Cut
- Minimum Bisection

There is an r-approximation with $r \leq O(\log^{c}(|x|))$ for some constant c.

Note that only for some of the above problem a matching lower bound is known.

10 Introduction to Approximation

17. Apr. 2018 231/234

There are weird problems!

Asymmetric *k*-Center admits an $O(\log^* n)$ -approximation.

There is no $o(\log^* n)$ -approximation to Asymmetric *k*-Center unless $NP \subseteq DTIME(n^{\log \log \log n})$.

There are real	ly difficult	problems!
----------------	--------------	-----------

Theorem 9

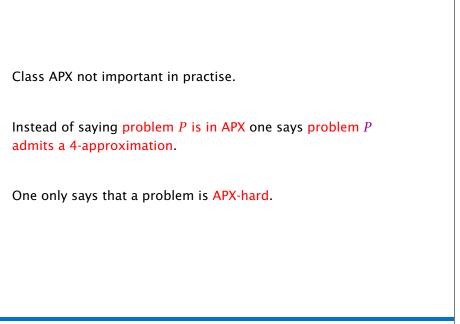
For any constant $\epsilon > 0$ there does not exist an $\Omega(n^{1-\epsilon})$ -approximation algorithm for the maximum clique problem on a given graph *G* with *n* nodes unless P = NP.

Note that an *n*-approximation is trivial.

Harald Räcke

10 Introduction to Approximation

17. Apr. 2018 232/234





17. Apr. 2018 233/234