

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]

Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

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Move from BFS to **adjacent** BFS, without decreasing objective function.

Two BFSs are called **adjacent** if the bases just differ in one variable.

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$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & Z \\ & 13a + 23b - Z = 0 \\ & 5a + 15b + s_c = 480 \\ & 4a + 4b + s_h = 160 \\ & 35a + 20b + s_m = 1190 \\ & a, b, s_c, s_h, s_m \geq 0 \end{aligned}$$

$$\begin{aligned} \text{basis} &= \{s_c, s_h, s_m\} \\ a &= b = 0 \\ Z &= 0 \\ s_c &= 480 \\ s_h &= 160 \\ s_m &= 1190 \end{aligned}$$

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Pivoting Step

max Z

$$13a + 23b \quad \quad \quad - Z = 0$$

$$5a + 15b + s_c \quad \quad \quad = 480$$

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- ▶ choose variable to bring into the basis
- ▶ chosen variable should have positive coefficient in objective function
- ▶ apply min-ratio test to find out by how much the variable can be increased
- ▶ pivot on row found by min-ratio test
- ▶ the existing basis variable in this row leaves the basis

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- ▶ Choose variable with coefficient > 0 as **entering variable**.
- ▶ If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

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- ▶ Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0 , and no variable goes negative.

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- ▶ The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the **leaving variable**.

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Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

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max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

$$\frac{1}{3}a + b + \frac{1}{15}s_c = 32$$

$$\frac{8}{3}a - \frac{4}{15}s_c + s_h = 32$$

$$\frac{85}{3}a - \frac{4}{3}s_c + s_m = 550$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{b, s_h, s_m\}$

$$a = s_c = 0$$

$$Z = 736$$

$$b = 32$$

$$s_h = 32$$

$$s_m = 550$$

max Z

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Choose variable a to bring into basis.

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Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.

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Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

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max Z

$$-s_c - 2s_h - Z = -800$$

$$b + \frac{1}{10}s_c - \frac{1}{8}s_h = 28$$

$$a - \frac{1}{10}s_c + \frac{3}{8}s_h = 12$$

$$\frac{3}{2}s_c - \frac{85}{8}s_h + s_m = 210$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{a, b, s_m\}$

$$s_c = s_h = 0$$

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4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all constraints in the problem
- the current solution is optimal if and only if all coefficients in the objective function are non-positive
- the current solution value is at most equal to the objective value of any feasible solution

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- ▶ any feasible solution satisfies all equations in the tableaux
- ▶ in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0$, $s_h \geq 0$
- ▶ hence optimum solution value is at most 800
- ▶ the current solution has value 800

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Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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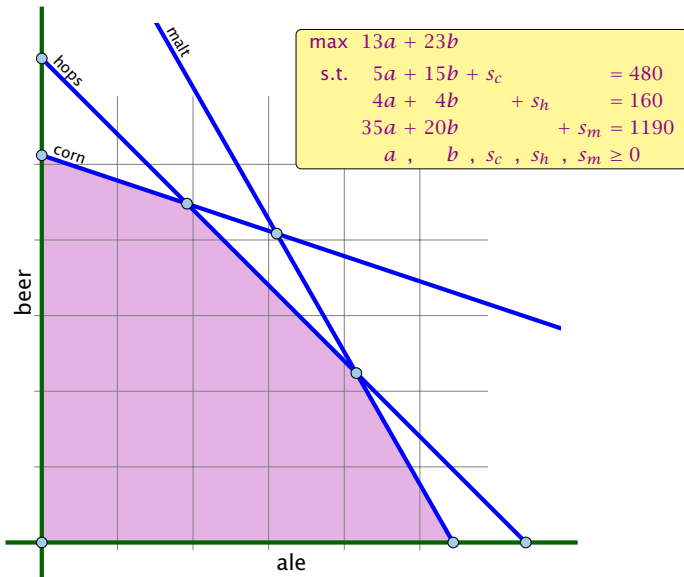
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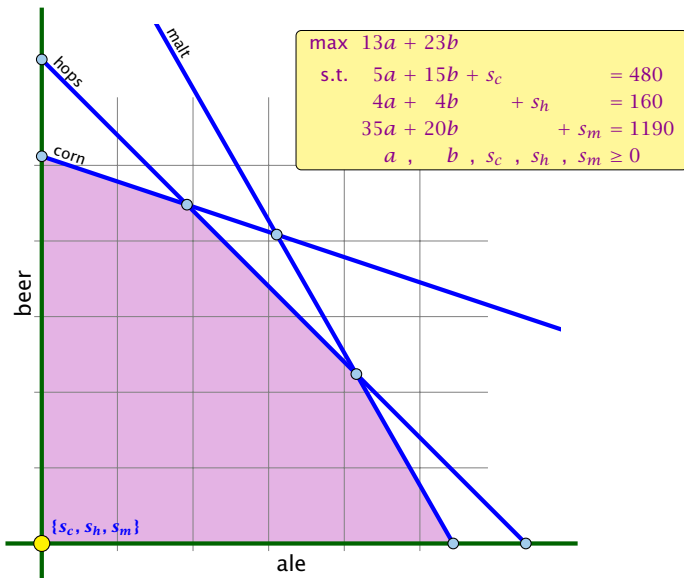
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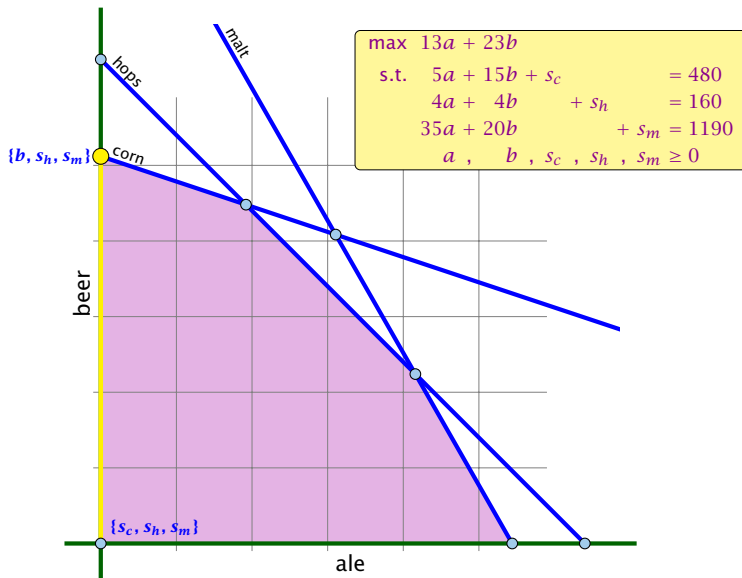
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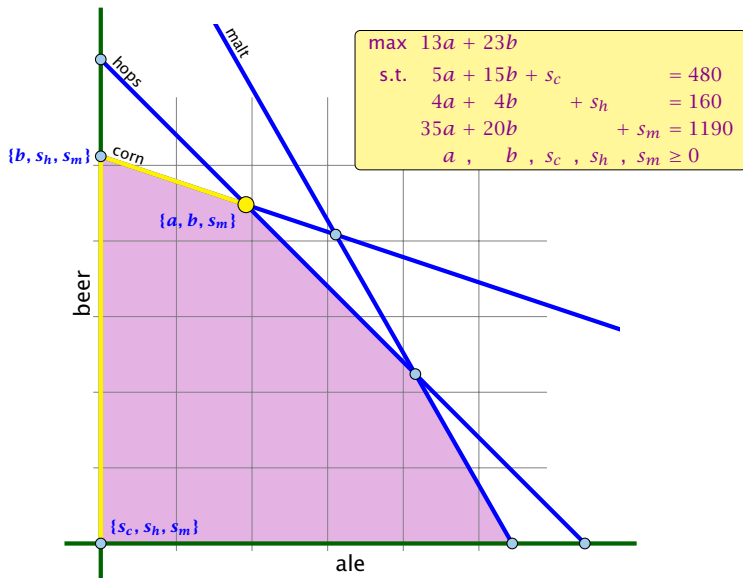
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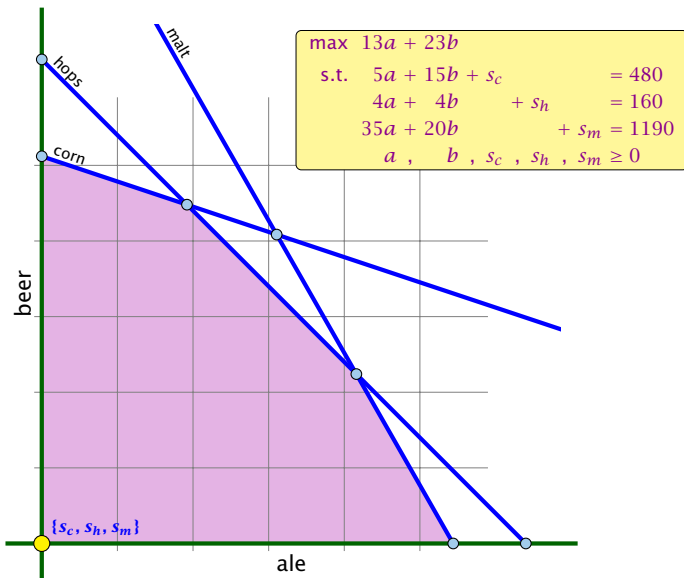
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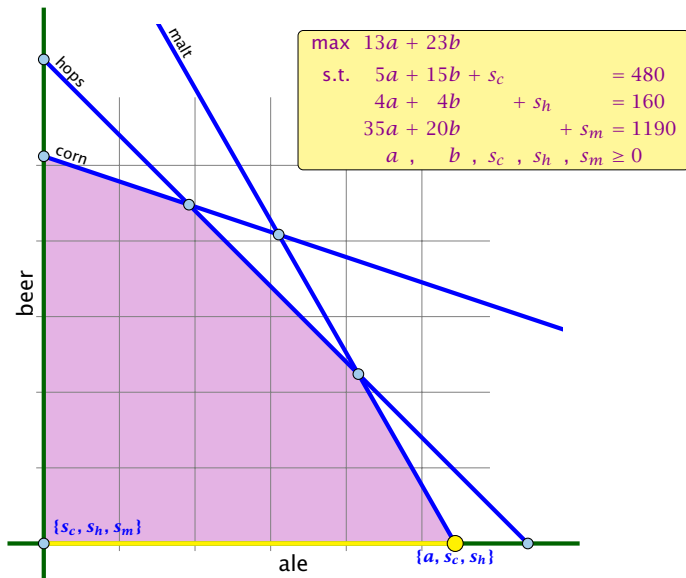
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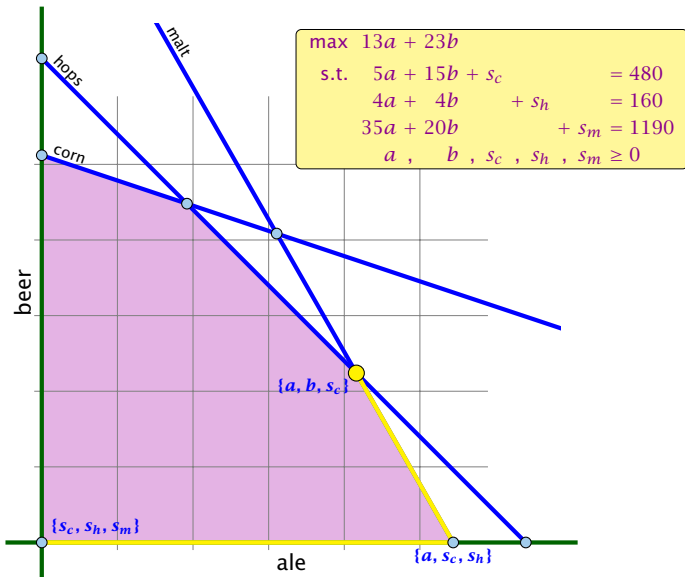
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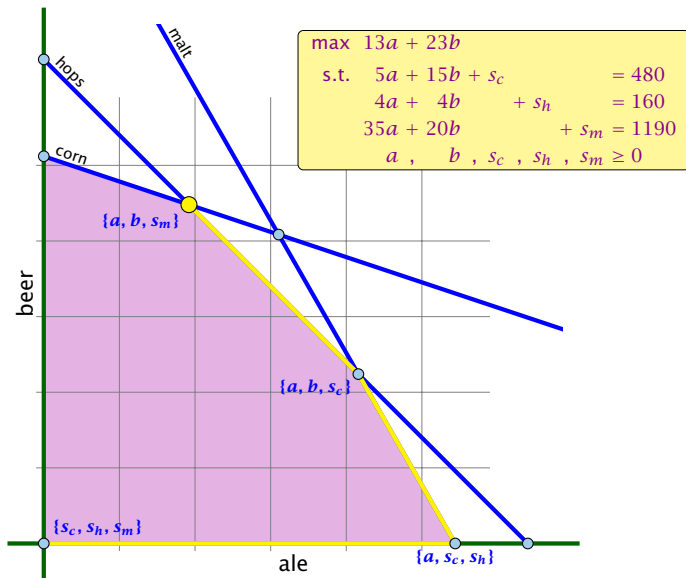
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Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

- ▶ $d_j = 1$ (normalized)
- ▶ $d_i = 0$ for $i \in B$
- ▶ $d_i \leq 0$ for $i \in B$ (must hold, hence $d_i = 0$)
- ▶ $d_i \geq 0$ for $i \notin B$ (must hold, hence $d_i = 0$)

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Requirements for d :

• $d_j = 1$ (normal vector)

• $d_B = -A^{-1} \cdot A_j$ (row i of $A^{-1} \cdot A_j$)

• $d_B \geq 0$ (row i of $A^{-1} \cdot A_j$)

• $d_B \leq 0$ (row i of $A^{-1} \cdot A_j$)

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Requirements for d :

- ▶ $d_j = 1$ (normalization)
- ▶ $d_\ell = 0, \ell \notin B, \ell \neq j$
- ▶ $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- ▶ Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

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Algebraic Definition of Pivoting

Definition 2 (j -th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1}A_{*j}$ is called the j -th basis direction for B .

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

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Definition 3 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the **reduced cost** for variable x_j .

Note that this is defined for every j . If $j \in B$ then the above term is 0.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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4 Simplex Algorithm

Questions:

- What happens if the min ratio test returns zero as a value?
- How can we deal with this situation?
- How can we safely increase the entering variable?
- How do we find the initial basic feasible solution?
- How do we always find a basis?
- How do we know when we are done?
- How do we know when we are stuck?
- How do we know when we are stuck with a basis?

4 Simplex Algorithm

Questions:

- ▶ What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- ▶ How do we find the initial basic feasible solution?
- ▶ Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 ?$$

Then we can terminate because we know that the solution is optimal.

- ▶ If yes how do we make sure that we reach such a basis?

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Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if all b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

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Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

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The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is **degenerate** (also the basis is degenerate).

Definition 4 (Degeneracy)

A BFS x^* is called **degenerate** if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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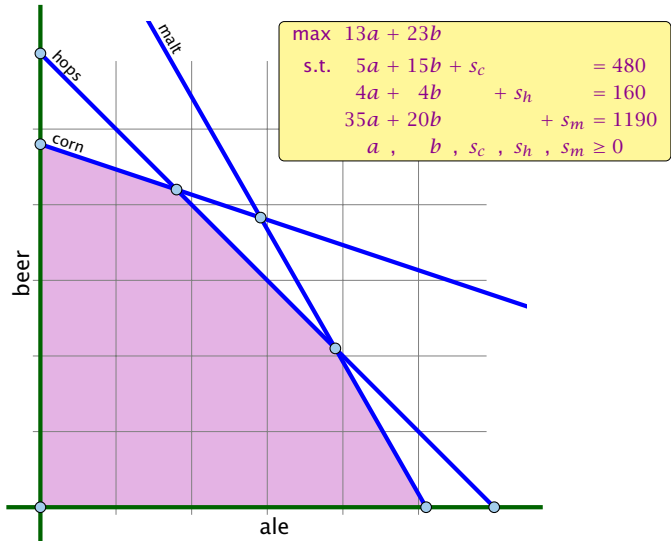
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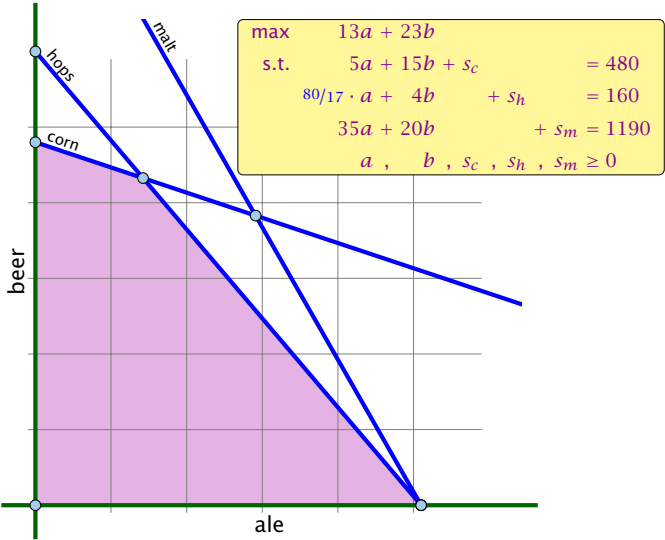
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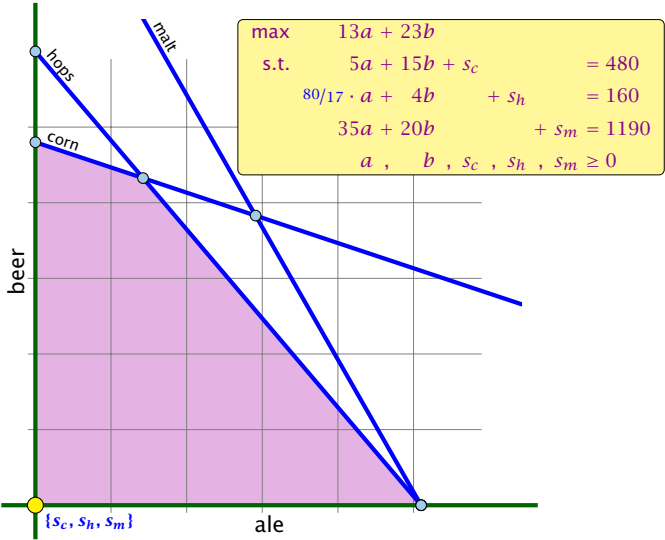
Non Degenerate Example



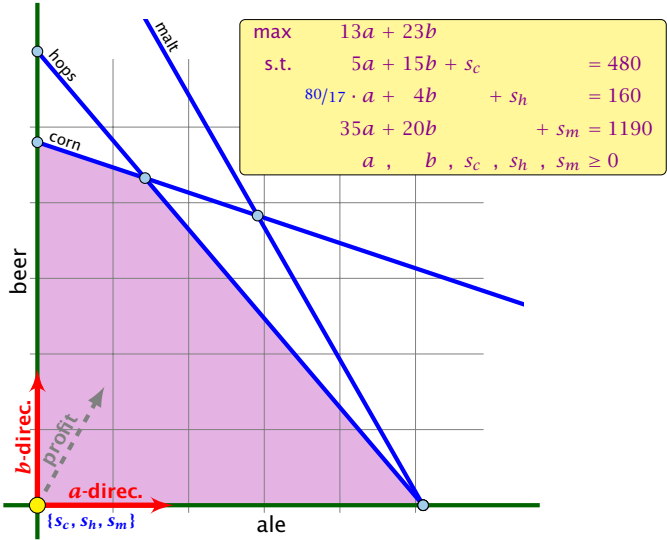
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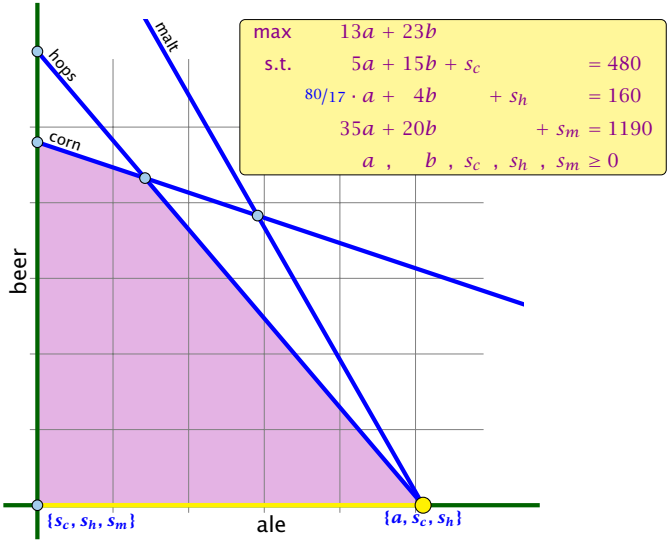
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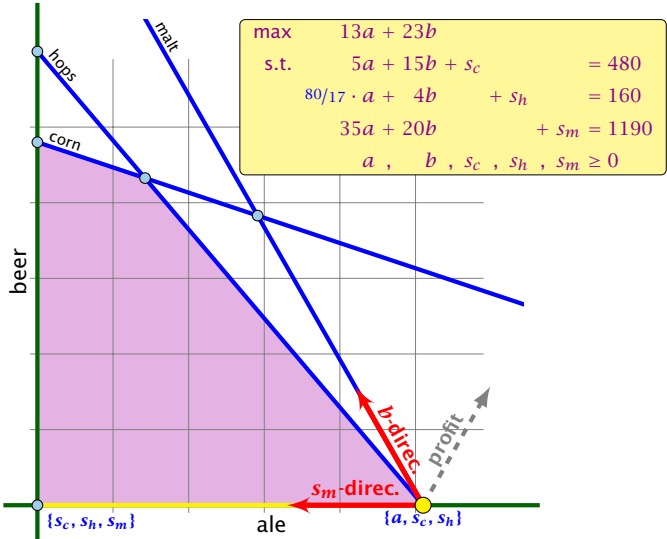
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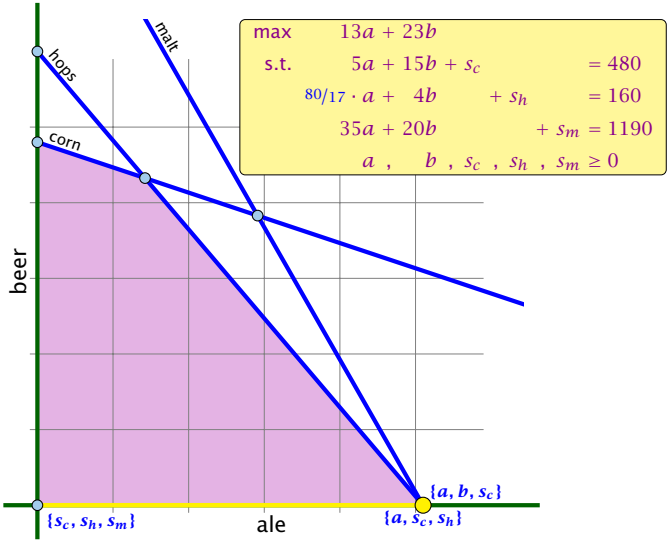
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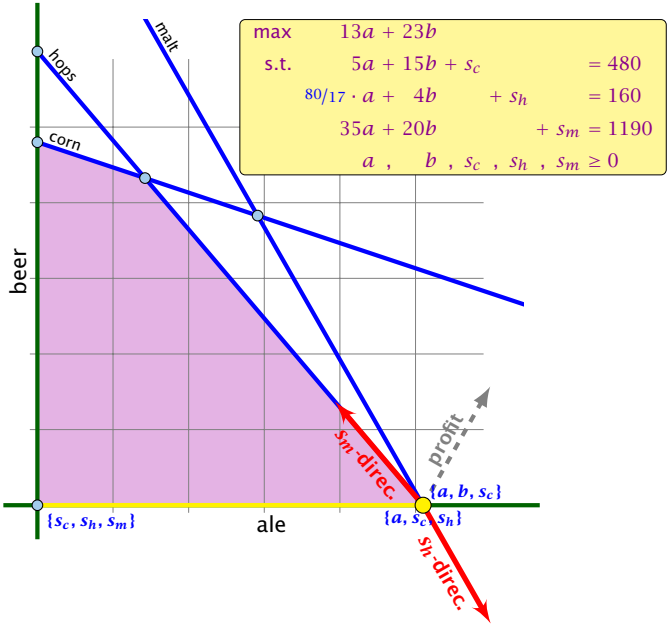
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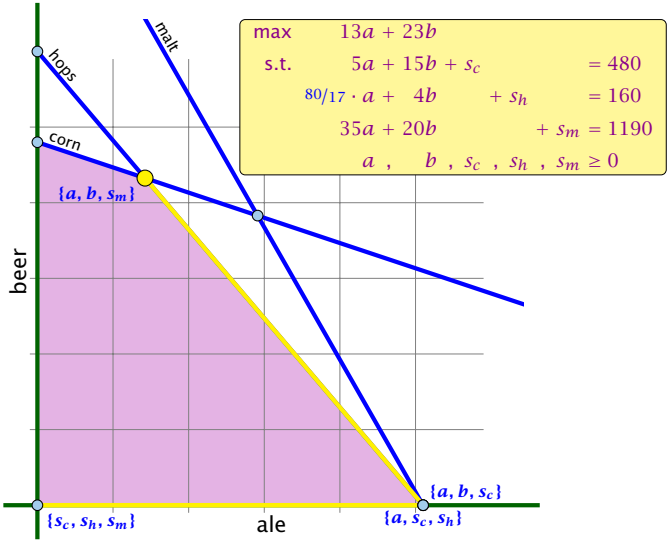


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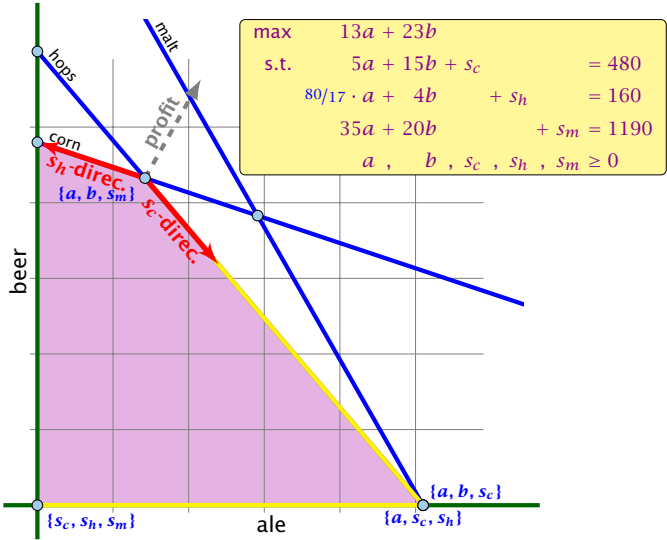


max	$13a + 23b$		
s.t.	$5a + 15b + s_c$	=	480
	$80/17 \cdot a + 4b + s_h$	=	160
	$35a + 20b + s_m$	=	1190
	$a, b, s_c, s_h, s_m \geq 0$		

Degenerate Example



Degenerate Example



Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ (\tilde{c}_e is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes \tilde{c}_e .
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
- ▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

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Termination

What do we have so far?

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is **unbounded**, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an **optimum solution**.

How do we come up with an initial solution?

- ▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.
- ▶ The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- ▶ Then $s = b, x = 0$ is a basic feasible solution (how?).
- ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

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How do we find an initial basic feasible solution for an arbitrary problem?

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

Multiply all rows with $b_i < 0$ by -1

maximize $c^T x$ s.t. $Ax = b, x \geq 0$ (with $b_i \geq 0$)

Simplex starts, phase is initial feasible

if $b_i = 0$ then the original problem is

feasible (you have $x = 0$ with $b_i = 0$)

if not you can get basic feasible solution

then you can start the simplex for the original problem

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

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Lemma 5

Let B be a basis and x^* a BFS corresponding to basis B . $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.