4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

4 Simplex Algorithm

$$\begin{aligned}
\max \quad & 13a + 23b \\
\text{s.t.} \quad & 5a + 15b + s_c && = 480 \\
& 4a + 4b && + s_h && = 160 \\
& 35a + 20b && + s_m && = 1190 \\
& a, \quad b, \quad s_c, \quad s_h, \quad s_m && \geq 0
\end{aligned}$$

$$\begin{aligned}
\max Z \\
13a + 23b && - Z && = 0 \\
5a + 15b + s_c && && = 480 \\
4a + 4b && + s_h && = 160 \\
35a + 20b && + s_m && = 1190 \\
a, \quad b, \quad s_c, \quad s_h, \quad s_m && \geq 0
\end{aligned}$$

basis = $\{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Pivoting Step

$$\max Z$$
$$13a + 23b \qquad\qquad\qquad - Z = 0$$
$$5a + 15b + s_c \qquad\qquad = 480$$
$$4a + 4b \qquad + s_h \qquad = 160$$
$$35a + 20b \qquad\qquad + s_m = 1190$$
$$a , \quad b , s_c , s_h , s_m \qquad \geq 0$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

▶ choose variable to bring into the basis
▶ chosen variable should have positive coefficient in objective function
▶ apply min-ratio test to find out by how much the variable can be increased
▶ pivot on row found by min-ratio test
▶ the existing basis variable in this row leaves the basis

$$\begin{aligned}
\max Z & & & \\
13a + 23\boldsymbol{b} & & - Z &= 0 \\
5a + 15\boldsymbol{b} + \boldsymbol{s_c} & & &= 480 \\
4a + 4\boldsymbol{b} + s_h & & &= 160 \\
35a + 20\boldsymbol{b} + s_m & & &= 1190 \\
a, \quad \boldsymbol{b}, s_c, s_h, s_m & & &\geq 0
\end{aligned}$$

$\text{basis} = \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

▶ Choose variable with coefficient > 0 as entering variable.

▶ If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

▶ For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.

▶ Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.

▶ The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

max Z

$$13a + 23\boldsymbol{b} \qquad\qquad - Z = 0$$
$$5a + 15\boldsymbol{b} + s_c \qquad\qquad = 480$$
$$4a + 4\boldsymbol{b} \qquad + s_h \qquad = 160$$
$$35a + 20\boldsymbol{b} \qquad\qquad + s_m = 1190$$
$$a\ ,\quad \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0$$

basis = $\{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

max Z

$$\frac{16}{3}a \qquad - \frac{23}{15}s_c \qquad - Z = -736$$
$$\frac{1}{3}a + b + \frac{1}{15}s_c \qquad\qquad = 32$$
$$\frac{8}{3}a \qquad - \frac{4}{15}s_c + s_h \qquad = 32$$
$$\frac{85}{3}a \qquad - \frac{4}{3}s_c \qquad + s_m = 550$$
$$a\ ,\ b\ ,\qquad s_c\ ,\ s_h\ ,\ s_m \qquad \geq 0$$

basis = $\{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

$$\max Z$$
$$\frac{16}{3}a \qquad - \frac{23}{15}s_c \qquad\qquad - Z = -736$$
$$\frac{1}{3}a + b + \frac{1}{15}s_c \qquad\qquad = 32$$
$$\frac{8}{3}a \qquad - \frac{4}{15}s_c + s_h \qquad = 32$$
$$\frac{85}{3}a \qquad - \frac{4}{3}s_c \qquad + s_m = 550$$
$$a \ , \ b \ , \qquad s_c \ , \ s_h \ , \ s_m \qquad \geq 0$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\max Z$$
$$- s_c - 2s_h \qquad - Z = -800$$
$$b + \frac{1}{10}s_c - \frac{1}{8}s_h \qquad = 28$$
$$a \qquad - \frac{1}{10}s_c + \frac{3}{8}s_h \qquad = 12$$
$$\frac{3}{2}s_c - \frac{85}{8}s_h + s_m = 210$$
$$a \ , \ b \ , \qquad s_c \ , \qquad s_h \ , \ s_m \qquad \geq 0$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$
$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

▶ any feasible solution satisfies all equations in the tableaux

▶ in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$

▶ hence optimum solution value is at most 800

▶ the current solution has value 800

Matrix View

Let our linear program be

$$
\begin{array}{ccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

$$
\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting

$$\begin{aligned}
\max \quad & 13a + 23b \\
\text{s.t.} \quad & 5a + 15b + s_c && = 480 \\
& 4a + 4b && + s_h && = 160 \\
& 35a + 20b && + s_m = 1190 \\
& a \,, \quad b \,, \; s_c \,, \; s_h \,, \; s_m \geq 0
\end{aligned}$$
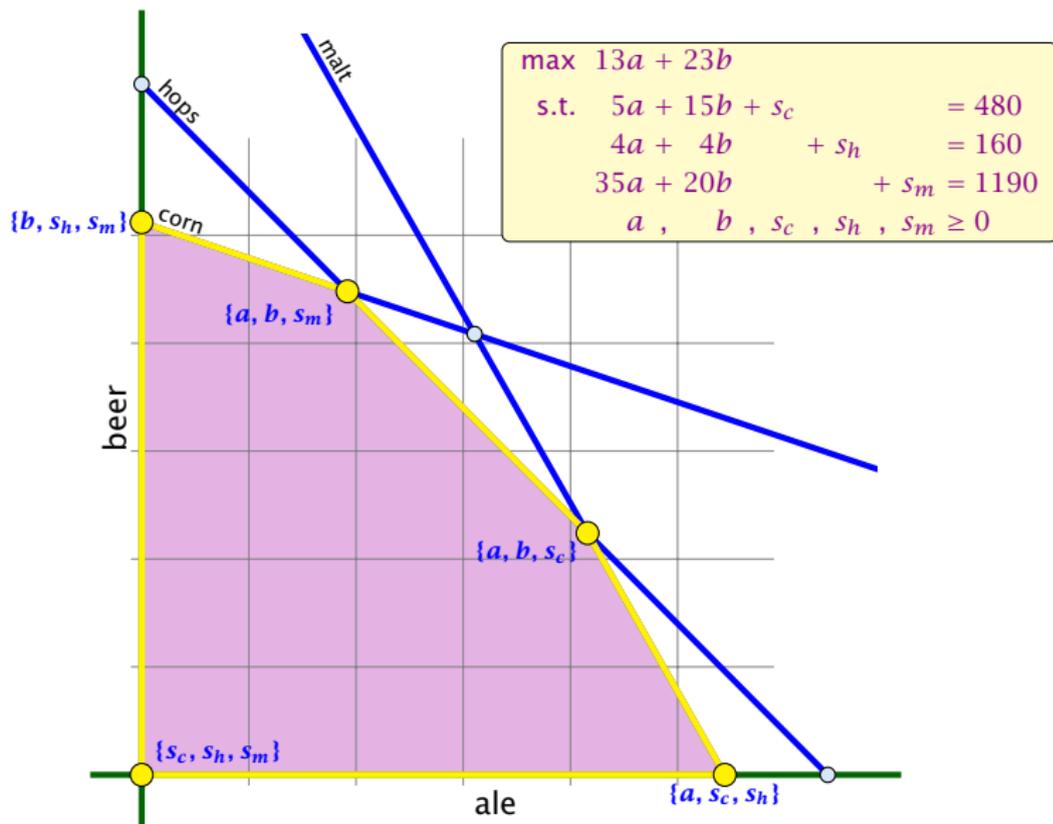
Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
  - Other non-basis variables should stay at 0.
  - Basis variables change to maintain feasibility.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

- $d_j = 1$ (normalization)
- $d_\ell = 0$, $\ell \notin B$, $\ell \neq j$
- $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

Definition 2 (j-th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1} A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

Algebraic Definition of Pivoting

Definition 3 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the reduced cost for variable x_j.

Note that this is defined for every j. If $j \in B$ then the above term is 0.

Algebraic Definition of Pivoting

Let our linear program be

$$
\begin{array}{ccccc}
c_B^T x_B & + & c_N^T x_N & = & Z \\
A_B x_B & + & A_N x_N & = & b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The simplex tableaux for basis B is

$$
\begin{array}{ccccc}
 & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\
I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\
x_B & , & x_N & \geq & 0
\end{array}
$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Questions:

- What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- How do we find the initial basic feasible solution?
- Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 \ ?$$

  Then we can terminate because we know that the solution is optimal.
- If yes how do we make sure that we reach such a basis?

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

Termination

The objective function may not increase!

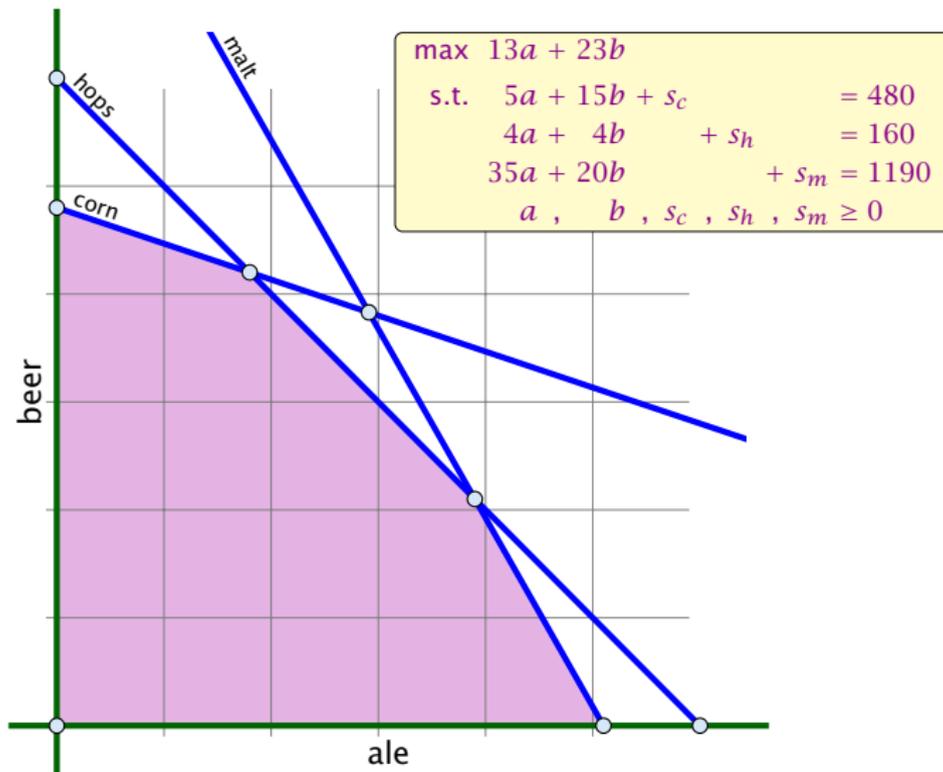Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is degenerate (also the basis is degenerate).
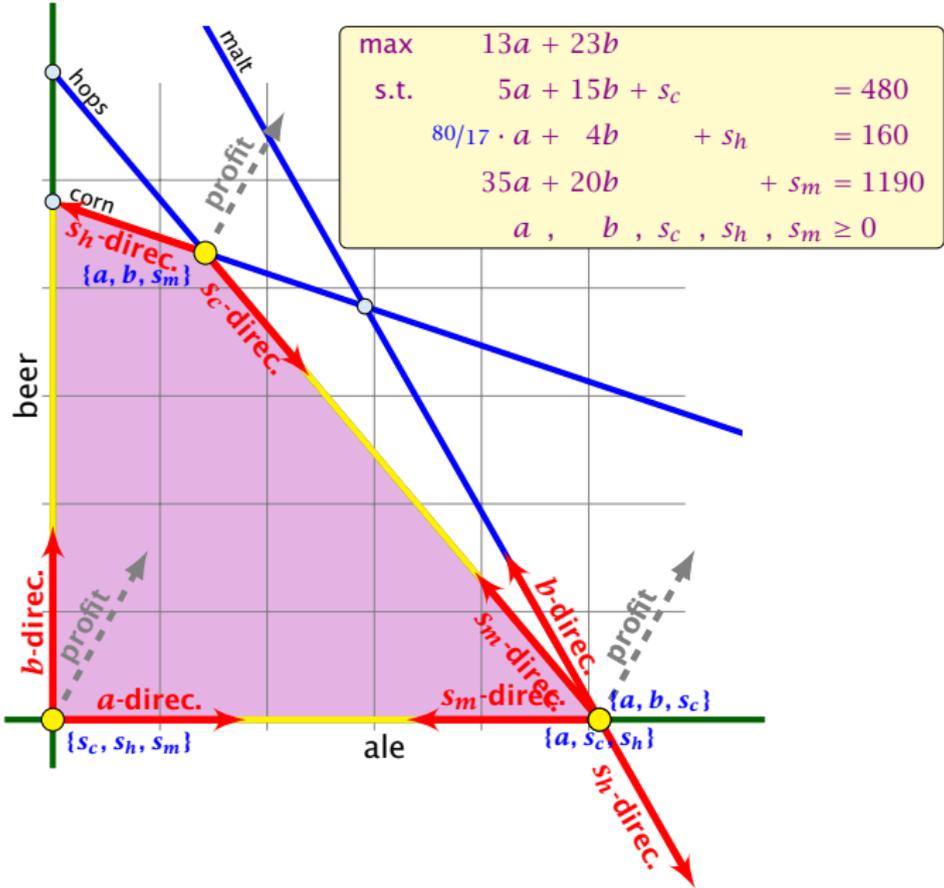
Definition 4 (Degeneracy)

A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

Non Degenerate Example

max $13a + 23b$

s.t.
$$\begin{aligned}
5a + 15b + s_c &&&= 480 \\
4a + 4b &+ s_h&& = 160 \\
35a + 20b &&+ s_m &= 1190 \\
a \,, \quad b \,, \; s_c \,, \; s_h \,, \; s_m &\geq 0
\end{aligned}$$

Degenerate Example

$$\max \quad 13a + 23b$$

$$\text{s.t.} \quad 5a + 15b + s_c \qquad\qquad\qquad = 480$$

$$80/17 \cdot a + \; 4b \qquad\quad + s_h \qquad\quad = 160$$

$$35a + 20b \qquad\qquad\qquad + s_m = 1190$$

$$a \; , \quad b \; , \; s_c \; , \; s_h \; , \; s_m \geq 0$$

Summary: How to choose pivot-elements

▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).

▶ The standard choice is the column that maximizes $\tilde{c}_e$.

▶ If $A_{ie} \leq 0$ for all $i \in \{1, \ldots, m\}$ then the maximum is not bounded.

▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.

▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.

▶ Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Termination

What do we have so far?
Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

How do we come up with an initial solution?

▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.

▶ The standard slack form for this problem is
  $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of
  slack variables.

▶ Then $s = b$, $x = 0$ is a basic feasible solution (how?).

▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary
problem?

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1.
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b$, $x \geq 0$, $v \geq 0$ using Simplex. $x = 0$, $v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Optimality

Lemma 5

Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.