Duality

How do we get an upper bound to a maximization LP?

$$
\begin{aligned}
\max \quad & 13a \;+\; 23b \\
\text{s.t.} \quad & 5a \;+\; 15b \;\le\; 480 \\
& 4a \;+\; 4b \;\le\; 160 \\
& 35a \;+\; 20b \;\le\; 1190 \\
& a, b \;\ge\; 0
\end{aligned}
$$

Note that a lower bound is easy to derive. Every choice of $a, b \ge 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \ge 0$) such that $\sum_i y_i a_{ij} \ge c_j$ then $\sum_i y_i b_i$ will be an upper bound.

Duality

How do we get an upper bound to a maximization LP?

$$
\begin{aligned}
\max \quad 13a \;+\; &\; 23b \\
\text{s.t.} \quad 5a \;+\; &\; 15b \;\le\; 480 \\
4a \;+\; &\; 4b \;\le\; 160 \\
35a \;+\; &\; 20b \;\le\; 1190 \\
&\; a, b \;\ge\; 0
\end{aligned}
$$

Note that a lower bound is easy to derive. Every choice of $a, b \ge 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \ge 0$) such that $\sum_i y_i a_{ij} \ge c_j$ then $\sum_i y_i b_i$ will be an upper bound.

Duality

How do we get an upper bound to a maximization LP?

$$
\begin{aligned}
\max \quad 13a \;+\; 23b \\
\text{s.t.} \quad 5a \;+\; 15b \;&\le\; 480 \\
4a \;+\; 4b \;&\le\; 160 \\
35a \;+\; 20b \;&\le\; 1190 \\
a, b \;&\ge\; 0
\end{aligned}
$$

Note that a lower bound is easy to derive. Every choice of $a, b \ge 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \ge 0$) such that $\sum_i y_i a_{ij} \ge c_j$ then $\sum_i y_i b_i$ will be an upper bound.

Duality

Definition 2

Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the dual problem.

Duality

Lemma 3

The dual of the dual problem is the primal problem.

Proof:

The dual problem is

Duality

Lemma 3

The dual of the dual problem is the primal problem.

Proof:

▶ $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$

▶ $w = -\max\{-b^T y \mid -A^T y \leq -c, y \geq 0\}$

The dual problem is

Duality

Lemma 3
The dual of the dual problem is the primal problem.

Proof:
- $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$
- $w = -\max\{-b^T y \mid -A^T y \leq -c, y \geq 0\}$

The dual problem is

Duality

Lemma 3
The dual of the dual problem is the primal problem.

Proof:
- $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$
- $w = -\max\{-b^T y \mid -A^T y \leq -c, y \geq 0\}$

The dual problem is
- $z = -\min\{-c^T x \mid -Ax \geq -b, x \geq 0\}$
- $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$

Duality

Lemma 3

The dual of the dual problem is the primal problem.

Proof:

▶ $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$

▶ $w = -\max\{-b^T y \mid -A^T y \leq -c, y \geq 0\}$

The dual problem is

▶ $z = -\min\{-c^T x \mid -Ax \geq -b, x \geq 0\}$

▶ $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$

Weak Duality

Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ and
$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \leq b, x \geq 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \geq c, y \geq 0\}$.

Theorem 4 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T \hat{x} \leq z \leq w \leq b^T \hat{y} .$$

Weak Duality

Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ and
$w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \le b, x \ge 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \ge c, y \ge 0\}$.

Theorem 4 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T \hat{x} \le z \le w \le b^T \hat{y} \ .$$

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \; (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b \; (\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A\hat{x} \leq b^T \hat{y} \; .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c$ $(\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b$ $(\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A\hat{x} \leq b^T \hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \; (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b \; (\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A\hat{x} \leq b^T \hat{y} \; .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \; (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b \; (\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A\hat{x} \leq b^T \hat{y} \; .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \; (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b \; (\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A\hat{x} \leq b^T \hat{y} \; .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \ (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b \ (\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A\hat{x} \leq b^T \hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \ (\hat{x} \geq 0)$

$A \hat{x} \leq b \Rightarrow y^T A \hat{x} \leq \hat{y}^T b \ (\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A \hat{x} \leq b^T \hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \ (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A \hat{x} \leq \hat{y}^T b \ (\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A \hat{x} \leq b^T \hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \; (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b \; (\hat{y} \geq 0)$

This gives

$$c^T \hat{x} \leq \hat{y}^T A\hat{x} \leq b^T \hat{y} \; .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.