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Definitions:

▶ Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.

▶ Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.

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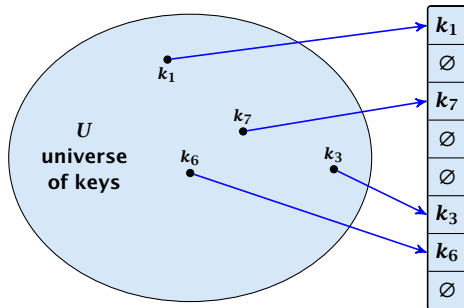
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Direct Addressing

Ideally the hash function maps all keys to different memory locations.

This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.
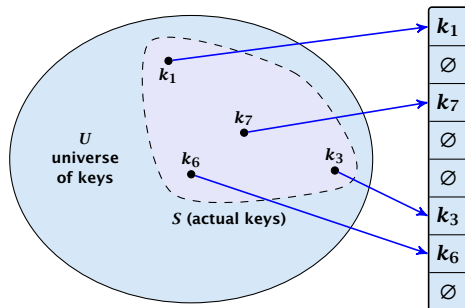
Such a hash function h is called a perfect hash function for set S.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions
Usually the universe U is much larger than the table-size n.

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

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Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 1

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

Uniform hashing:
Choose a hash function uniformly at random from all functions $f : U \rightarrow [0, \ldots, n-1]$.

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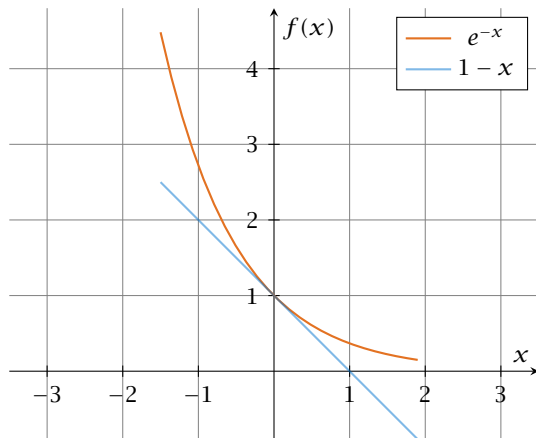### Proof.

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Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. $\qquad\square$

Collisions

The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

▶ **open addressing**, aka. closed hashing

▶ **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

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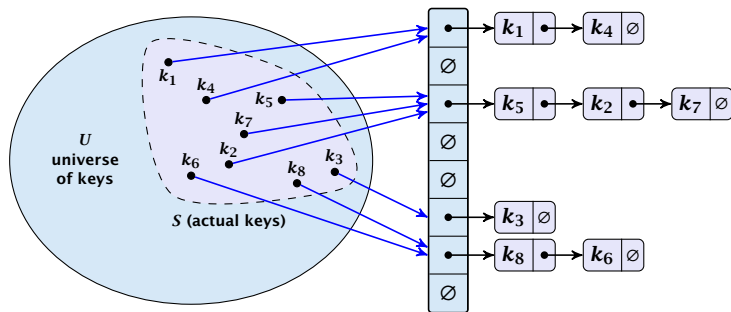The methods for dealing with collisions can be classified into the two main types

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Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute $h(x)$ and search list for $\text{key}[x]$.
- ▶ Insert: insert at the front of the list.

Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

▶ A^+ denotes the average time for a **successful** search when using A;

▶ A^- denotes the average time for an **unsuccessful** search when using A;

▶ We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called fill factor of the hash-table.

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$$A^- = 1 + \alpha \ .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

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Ernst Mayr, Harald Räcke

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Hence, the expected cost for a successful search is $A^+ \leq 1 + \frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

▶ pointers increase memory requirements

▶ pointers may lead to bad cache efficiency

Advantages:

▶ no à priori limit on the number of elements

▶ deletion can be implemented efficiently

▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k, 0), \ldots, h(k, n-1)$ must form a permutation of $0, \ldots, n-1$.

Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1), h(k, 2), \ldots$

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Define a function $h(k, j)$ that determines the table-position to be examined in the j-th step. The values $h(k, 0), \ldots, h(k, n-1)$ must form a permutation of $0, \ldots, n-1$.

Search(k): Try position $h(k, 0)$; if it is empty your search fails; otw. continue with $h(k, 1), h(k, 2), \ldots$.

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Choices for $h(k, j)$:

▶ **Linear probing:**
$h(k, i) = h(k) + i \mod n$
(sometimes: $h(k, i) = h(k) + c\,i \mod n$).

▶ Quadratic probing:
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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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- ▶ Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.

- ▶ Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 2

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1 - \alpha}\right)$$

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Open Addressing

Some values:

α	Linear Probing		Quadratic Probing		Double Hashing	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing
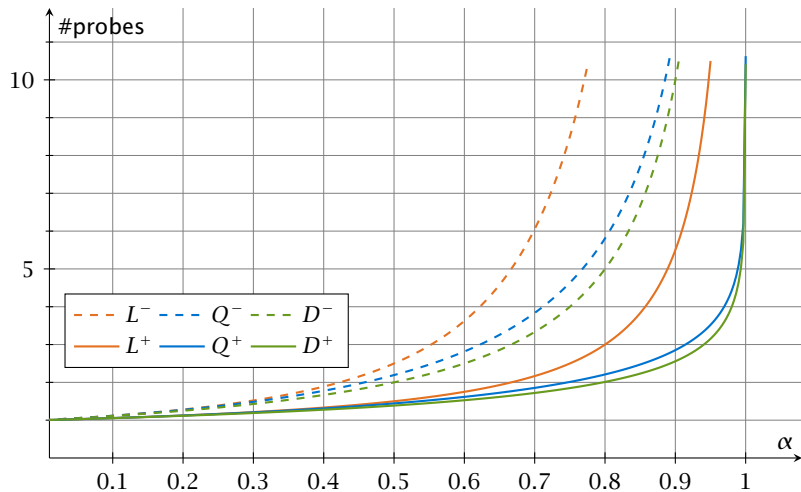
Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

▶ The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots$$

Analysis of Idealized Open Address Hashing

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$i = 1$

$$\sum_i i \Pr[X = i] = \sum_i \Pr[X \geq i]$$

Analysis of Idealized Open Address Hashing

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Analysis of Idealized Open Address Hashing

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Analysis of Idealized Open Address Hashing

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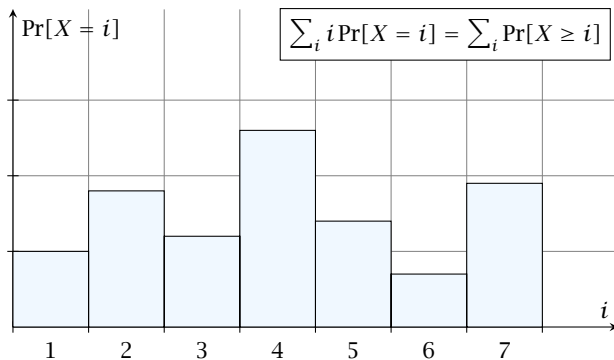
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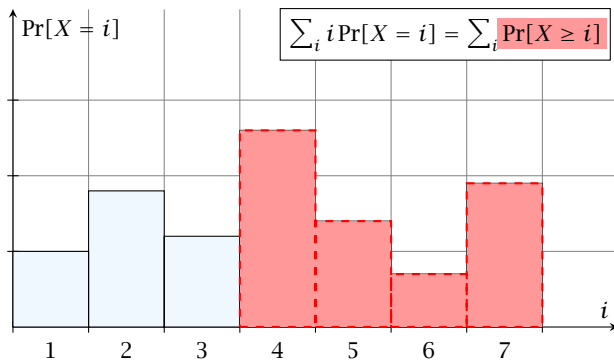# Analysis of Idealized Open Address Hashing

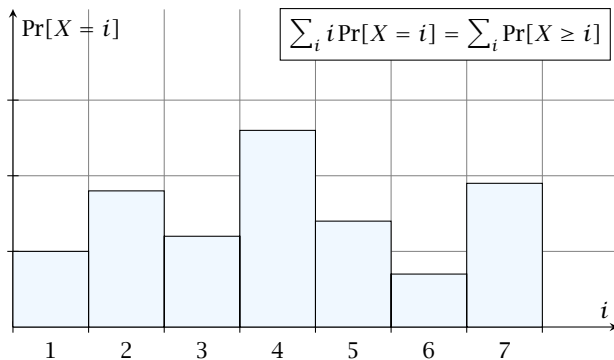$i = 2$
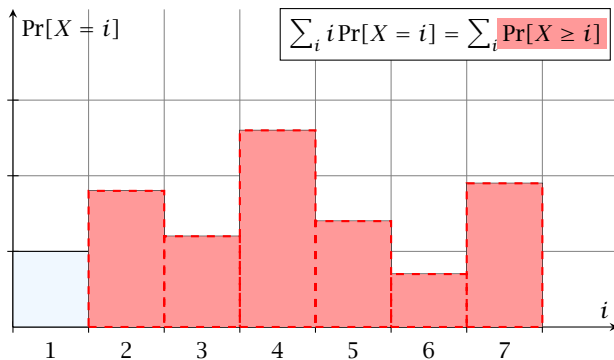
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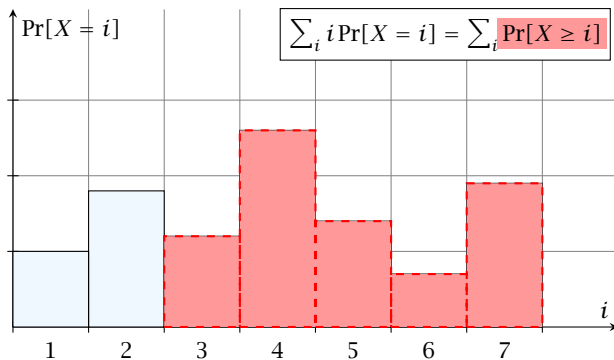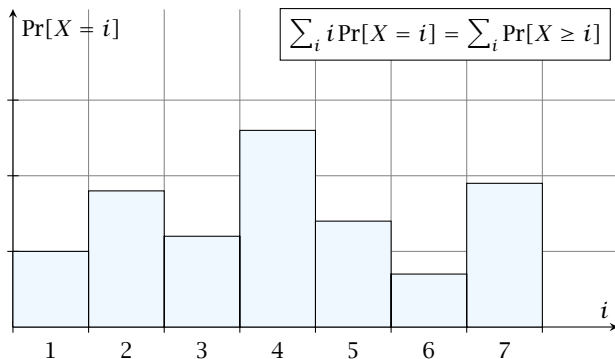
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The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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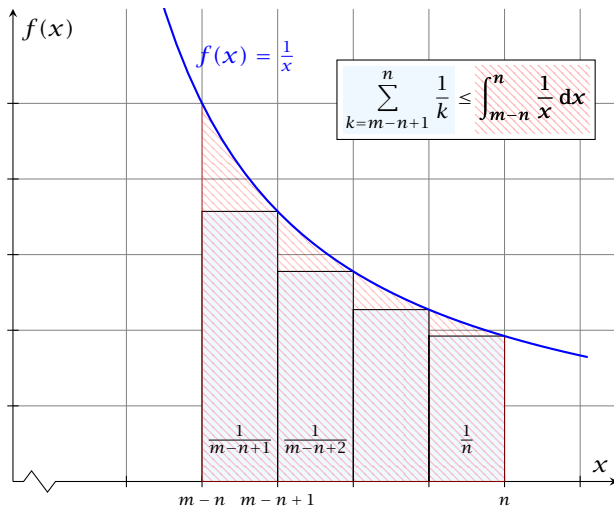$$\leq \frac{1}{\alpha} \int_{n-m}^{n} \frac{1}{x} \mathrm{d}x$$

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$$\sum_{k=m-n+1}^{n} \frac{1}{k} \leq \int_{m-n}^{n} \frac{1}{x}\,\mathrm{d}x$$

Deletions in Hashtables

How do we delete in a hash-table?

▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.

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▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.

▶ One can delete an element by replacing it with a deleted-marker.

▶ The table could fill up with deleted-markers leading to bad performance.

▶ If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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2: $p \leftarrow \text{succ}(p)$
3: **while** $T[p] \neq$ null **do**
4: $\quad\quad y \leftarrow T[p]$
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p is the index into the table-cell that contains the object to be deleted.

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Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

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Definition 5

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

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Definition 6

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

▶ For any key $u \in U$, and $t \in \{0, \ldots, n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.

▶ For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

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A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called k-independent if for any choice of $\ell \leq k$ distinct keys $u_1, \ldots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \ldots, t_\ell$:

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where the probability is w.r.t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 8

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0, \ldots, n-1\}$ is called (μ, k)-independent if for any choice of $\ell \leq k$ distinct keys $u_1, \ldots, u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1, \ldots, t_\ell$:

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Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

Define
$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 9

The class
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If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Multiplying with $a \not\equiv 0 \pmod{p}$ gives

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▶ The hash-function does not generate collisions before the $(\bmod\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$a \equiv (t_x - t_y)(x - y)^{-1} \qquad (\mod p)$$
$$b \equiv t_y - ay \qquad (\mod p)$$

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There is a one-to-one correspondence between hash-functions (pairs (a, b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the $\mod n$-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

What happens when we do the $\mod n$ operation?

Fix a value t_x. There are $p - 1$ possible values for choosing t_y.

From the range $0, \ldots, p - 1$ the values $t_x, t_x + n, t_x + 2n, \ldots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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$$\left\lceil \frac{p}{n} \right\rceil - 1 \leq \frac{p}{n} + \frac{n-1}{n} - 1 \leq \frac{p-1}{n}$$

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right]$$

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$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lceil \frac{p}{n} \right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 10

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$. Define for $x \in \{0, \ldots, q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \ldots, q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

$$f_{\bar{a}}(x) = \Big(\sum_{i=0}^{d} a_i x^i \Big) \bmod q$$

The polynomial is defined by $d + 1$ distinct points.

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Fix $\ell \le d + 1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\vec{a}} \in \mathcal{H} \mid h_{\vec{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$
Then
$$h_{\vec{a}} \in A^\ell \Leftrightarrow h_{\vec{a}} = f_{\vec{a}} \bmod n \text{ and}$$

$$f_{\vec{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.
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$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d + 1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.

We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Fix $\ell \leq d + 1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$
Then

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Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Therefore we have

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This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \ge (d+1)n$, and $\ell \le d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.

Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \ .$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

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$$
= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .
$$

Ernst Mayr, Harald Räcke

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

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- Two hash-tables $T_1[0, \ldots, n-1]$ and $T_2[0, \ldots, n-1]$, with hash-functions h_1, and h_2.

- An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.

- A search clearly takes constant time if the above constraint is met.

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Insert:

T_1 T_2

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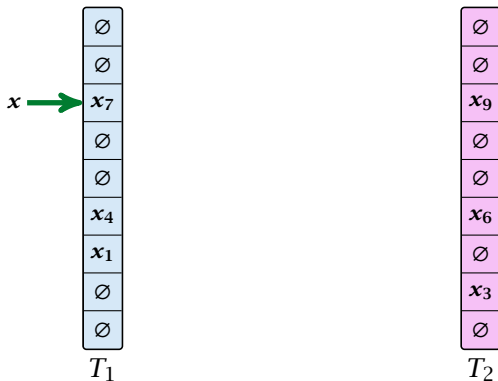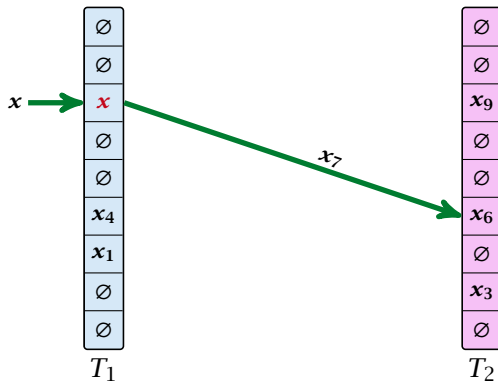
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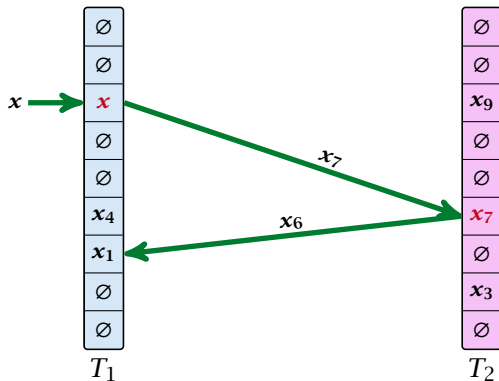
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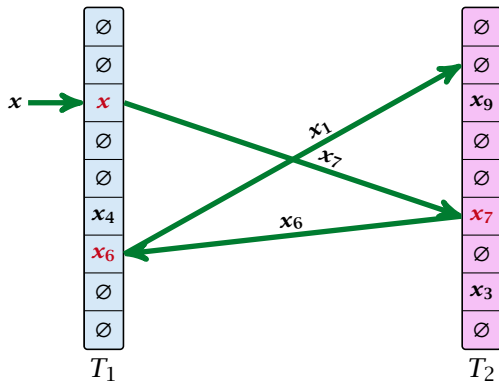
Algorithm 13 Cuckoo-Insert(x)

1: **if** $T_1[h_1(x)] = x \lor T_2[h_2(x)] = x$ **then return**
2: steps $\leftarrow 1$
3: **while** steps $\leq$ maxsteps **do**
4:      exchange x and $T_1[h_1(x)]$
5:      **if** $x =$ null **then return**
6:      exchange x and $T_2[h_2(x)]$
7:      **if** $x =$ null **then return**
8:      steps $\leftarrow$ steps $+1$
9: rehash() // change hash-functions; rehash everything
10: Cuckoo-Insert(x)

Cuckoo Hashing

▶ We call one iteration through the while-loop a step of the algorithm.

▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.

▶ We say a phase is successful if it is not terminated by the maxstep-condition, but the while loop is left because $x = $ null.

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What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

Formally what is the probability to enter an infinite loop that touches s different keys?

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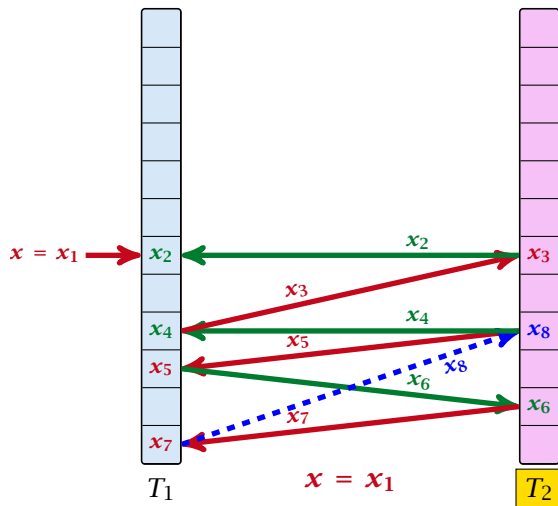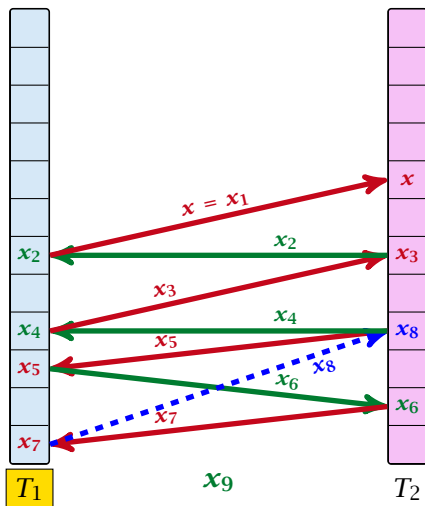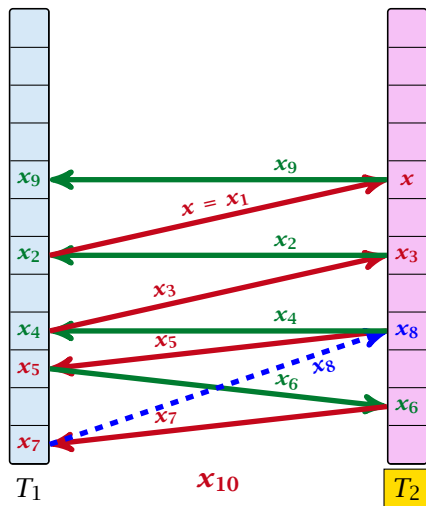
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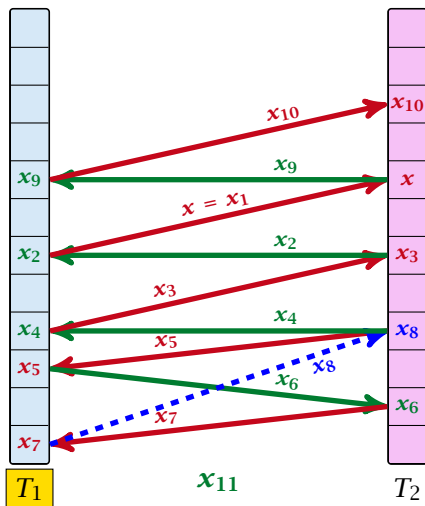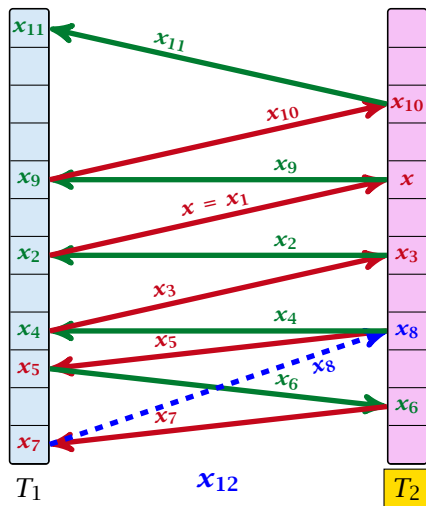
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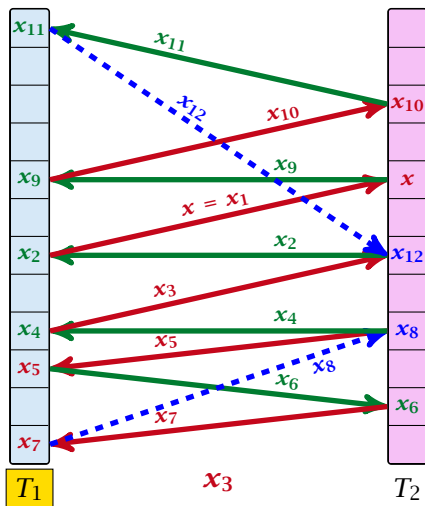# Cuckoo Hashing: Insert

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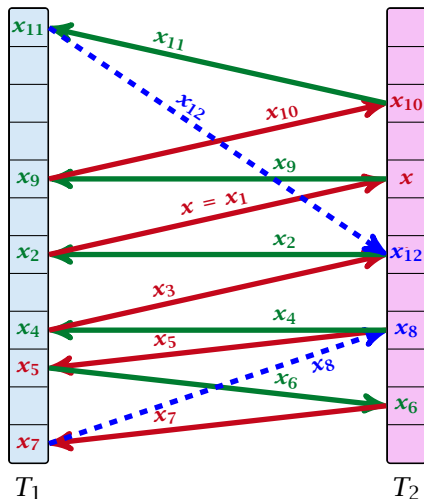
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▶ $s - 1$ different cells (alternating btw. cells from T_1 and T_2).

▶ s distinct keys $x = x_1, x_2, \ldots, x_s$, linking the cells.

▶ The leftmost cell is "linked forward" to some cell on the right.

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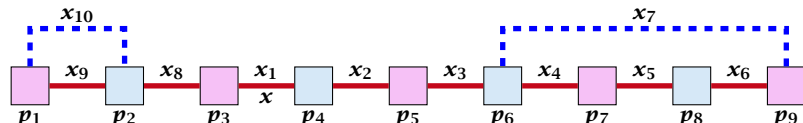
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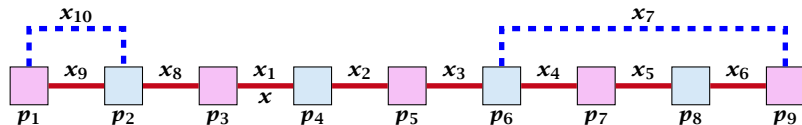
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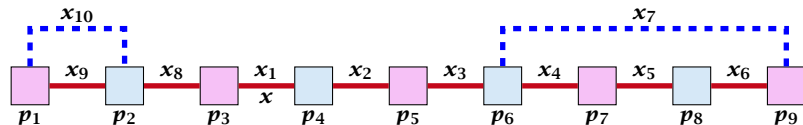
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A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If during a phase the insert-procedure runs into a cycle there
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What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

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These events are independent.

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What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

These events are independent.

Cuckoo Hashing

The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that there exists an active cycle structure of size s?

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Cuckoo Hashing

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- ▶ There are at most s^2 possibilities where to attach the forward and backward links.
- ▶ There are at most s possibilities to choose where to place key x.
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Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n} \right)^s$$

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Here we used the fact that $(1+\epsilon)m \leq n$.

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$$

Here we used the fact that $(1 + \epsilon)m \leq n$.

Hence,

$$
\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2} \right) \ .
$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing
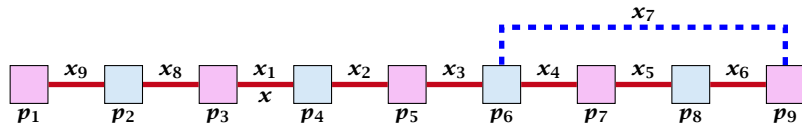
Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 11
If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

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Lemma 11
If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$$

As $r \le i - 1$ the length p of the sequence is

$$p = i + r + (j - i) \le i + j - 1 \ .$$

Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. $\qquad \square$

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Cuckoo Hashing

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Cuckoo Hashing

A path-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t + 2)/3$.

Cuckoo Hashing

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by choosing $\ell \geq \log\left(\frac{1}{2\mu^2 m^2}\right) / \log\left(\frac{1}{1+\epsilon}\right) = \log\left(2\mu^2 m^2\right) / \log\left(1+\epsilon\right)$

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This gives $\text{maxsteps} = \Theta(\log m)$.

Cuckoo Hashing

So far we estimated

$$\Pr[\text{cycle}] \le \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

and

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Observe that

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for a suitable constant $c > 0$.

Cuckoo Hashing

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The expected number of complete steps in the successful phase of an insert operation is:

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$$\leq \frac{1}{c} \sum_{t \geq 1} 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3}$$

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Hence,

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Hence,

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Hence,

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$$= \frac{2\mu^2}{c(1+\epsilon)^{1/3}} \sum_{t \geq 0} \left(\frac{1}{(1+\epsilon)^{2/3}}\right)^t = \mathcal{O}(1) \ .$$

This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $q = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching maxsteps without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is
$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.

Therefore the expected cost for re-hashes is
$\mathcal{O}(m) \cdot \mathcal{O}(p) = \mathcal{O}(1)$.

Cuckoo Hashing

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A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is
$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.

Therefore the expected cost for re-hashes is
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The expected cost for all rehashes is

$$\mathrm{E}\left[\sum_i \sum_s Z_i X_i^s\right]$$

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What kind of hash-functions do we need?

Since maxsteps is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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How do we make sure that $n \geq (1 + \epsilon)m$?

▶ Let $\alpha := 1/(1 + \epsilon)$.

▶ Keep track of the number of elements in the table. When $m \geq \alpha n$ we double n and do a complete re-hash (table-expand).

▶ Whenever m drops below $\alpha n/4$ we divide n by 2 and do a rehash (table-shrink).

▶ Note that right after a change in table-size we have $m = \alpha n/2$. In order for a table-expand to occur at least $\alpha n/2$ insertions are required. Similar, for a table-shrink at least $\alpha n/4$ deletions must occur.

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Lemma 12

Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most $\frac{1}{2(1+\epsilon)}$.

The $1/(2(1+\epsilon))$ fill-factor comes from the fact that the total hash-table is of size $2n$ (because we have two tables of size n); moreover $m \leq (1+\epsilon)n$.

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