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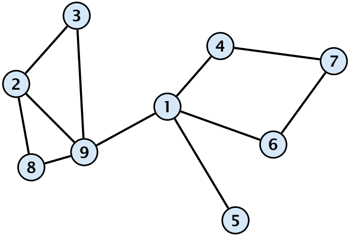
Given an undirected, capacitated graph $G = (V, E, c)$ find a partition of V into two non-empty sets $S, V \setminus S$ s.t. the capacity of edges between both sets is minimized.



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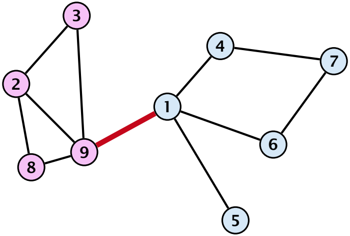

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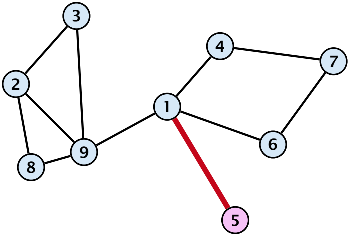

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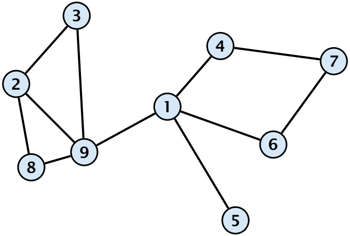

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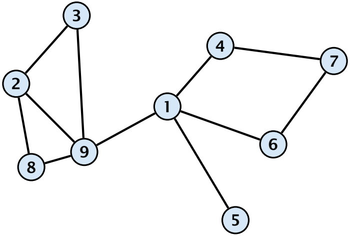

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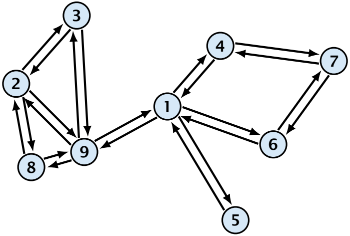

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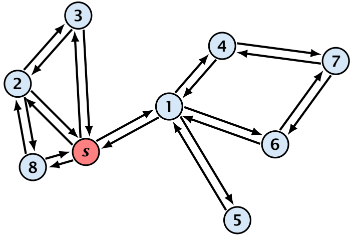

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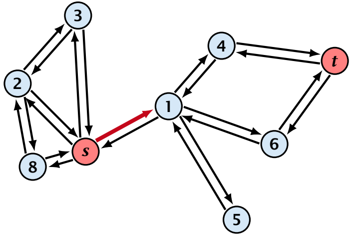
- Let $(S, V \setminus S)$ be a minimum global mincut. The above algorithm will output a cut of capacity $\text{cap}(S, V \setminus S)$ whenever $|\{s, t\} \cap S| = 1$.



Edge Contractions

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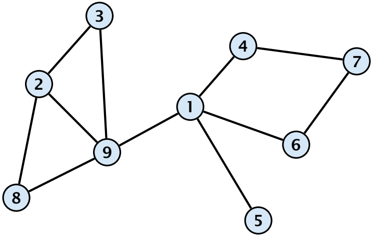

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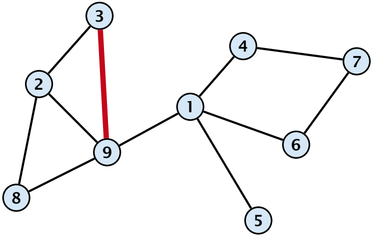

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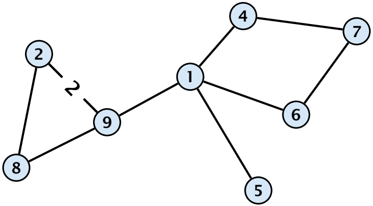

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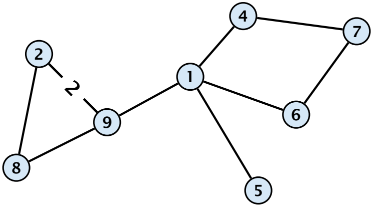

- Edge-contractions do no decrease the size of the mincut.

Edge Contractions

We can perform an edge-contraction in time $\mathcal{O}(n)$.

Randomized Mincut Algorithm

Algorithm 20 KargerMincut$(G = (V, E, c))$

1: **for** $i = 1 \to n - 2$ **do**
2: choose $e \in E$ randomly with probability $c(e)/c(E)$
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- What is the probability that this algorithm returns a mincut?

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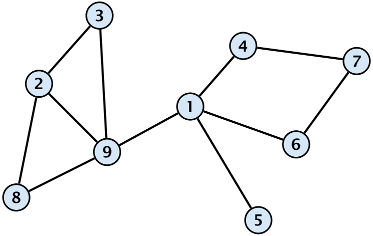

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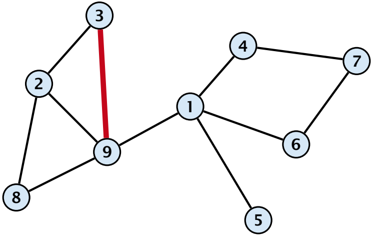

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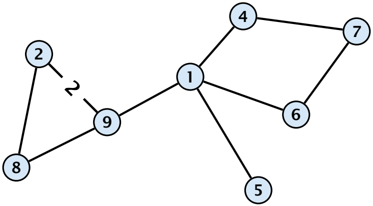

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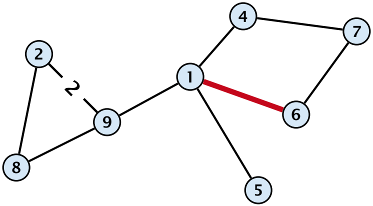

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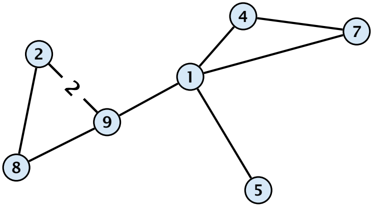

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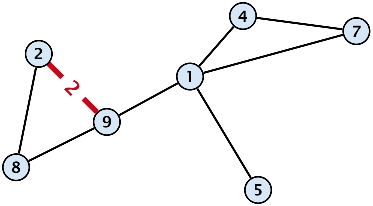

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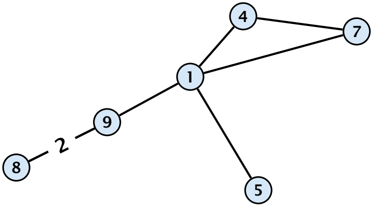

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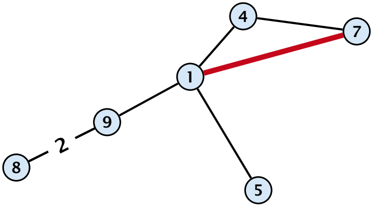

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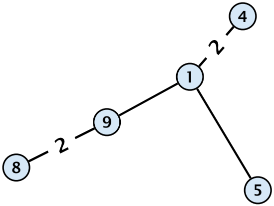

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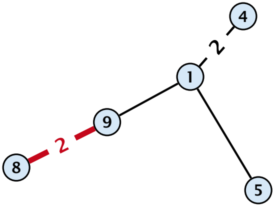

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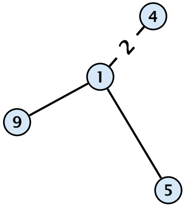

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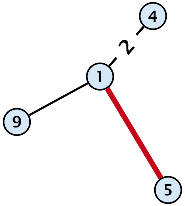

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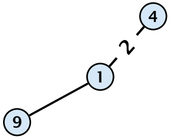

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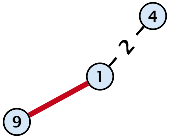

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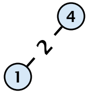

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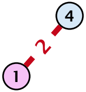

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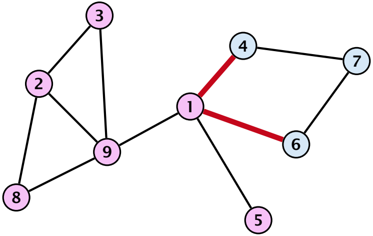

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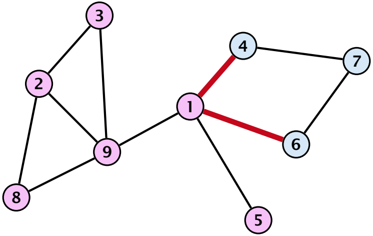

What is the probability that this algorithm returns a mincut?

Analysis

What is the probability that a given mincut A is still possible after round i?

- It is still possible to obtain cut A in the end if so far no edge in $(A, V \setminus A)$ has been contracted.

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$$2c(E) = 2\sum_{e \in E} c(e) = \sum_{v \in V} \text{cap}(v) \geq (n - i + 1) \cdot \min$$

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- Hence, the probability of choosing an edge from the cut is at most $\min / c(E) \leq 2/(n - i + 1)$.

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Choosing $t = 2$ gives that with probability $1/\binom{n}{2}$ the algorithm computes a mincut.

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Theorem 90

The randomized mincut algorithm computes an optimal cut with high probability. The total running time is $\mathcal{O}(n^4 \log n)$.

Improved Algorithm

Algorithm 21 RecursiveMincut($G = (V, E, c)$)

```
for i = 1 → n − n/√2 do
    choose e ∈ E randomly with probability c(e)/c(E)
    G ← G/e
if |V| = 2 return cut-value;
cuta ← RecursiveMincut(G);
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return min{cuta, cutb}
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Running time:

- $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + \mathcal{O}(n^2)$
- This gives $T(n) = \mathcal{O}(n^2 \log n)$.

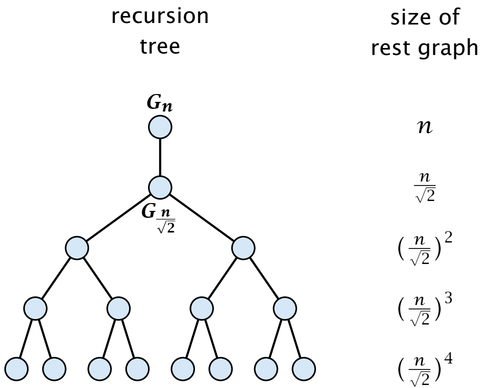
Probability of Success

The probability of contracting an edge from the mincut during one iteration through the for-loop is only

$$\frac{t(t-1)}{n(n-1)} \leq \frac{t^2}{n^2} = \frac{1}{2} \ ,$$

as $t = \frac{n}{\sqrt{2}}$.

Probability of Success

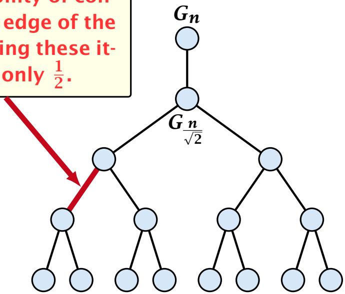


Probability of Success

recursion tree

size of rest graph

The probability of contracting an edge of the mincut during these iterations is only $\frac{1}{2}$.



n

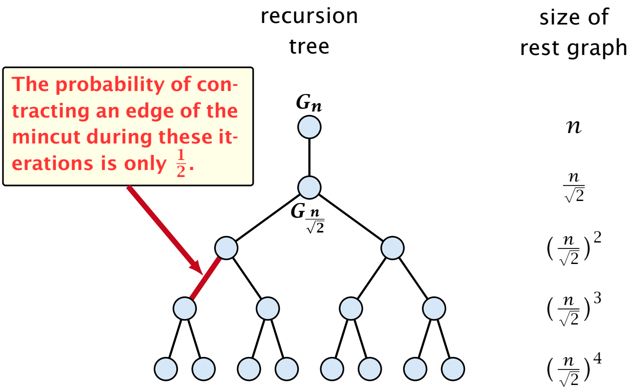
$\frac{n}{\sqrt{2}}$

$\left(\frac{n}{\sqrt{2}}\right)^2$

$\left(\frac{n}{\sqrt{2}}\right)^3$

$\left(\frac{n}{\sqrt{2}}\right)^4$

Probability of Success



We can estimate the success probability by using the following game on the recursion tree. Delete every edge with probability $\frac{1}{2}$. If in the end you have a path from the root to at least one leaf node you are successful.

Probability of Success

Let for an edge e in the recursion tree, $h(e)$ denote the height (distance to leaf level) of the parent-node of e (end-point that is higher up in the tree). Let h denote the height of the root node.

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Call an edge e alive if there exists a path from the parent-node of e to a descendant leaf, after we randomly deleted edges. Note that an edge can only be alive if it hasn't been deleted.

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Lemma 91

The probability that an edge e *is alive is at least* $\frac{1}{h(e)+1}$.

Probability of Success

Proof.

- An edge e with $h(e) = 1$ is alive if and only if it is not deleted. Hence, it is alive with proability at least $\frac{1}{2}$.

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15 Global Mincut

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One run of the algorithm can be performed in time $\mathcal{O}(n^2 \log n)$ *and has a success probability of* $\Omega(\frac{1}{\log n})$.

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Doing $\Theta(\log^2 n)$ *runs gives that the algorithm succeeds with high probability. The total running time is* $\mathcal{O}(n^2 \log^3 n)$.