

5 Asymptotic Notation

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- ▶ A linear speed-up (i.e., by a constant factor) is always possible by e.g. implementing the algorithm on a faster machine.
- ▶ Running time should be expressed by simple functions.

Asymptotic Notation

Formal Definition

Let f, g denote functions from \mathbb{N} to \mathbb{R}^+ .

- ▶ $\mathcal{O}(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
(set of functions that asymptotically grow **not faster** than f)

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There is an equivalent definition using limes notation (**assuming that the respective limes exists**). f and g are functions from \mathbb{N}_0 to \mathbb{R}_0^+ .

$$\blacktriangleright g \in \mathcal{O}(f): 0 \leq \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} < \infty$$

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- There also exist versions for arbitrary functions, and for the case that the limes is not infinity.

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Abuse of notation

1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).

2. In this context $f(n)$ does **not** mean the function f evaluated at n , but instead it is a shorthand for the function itself (leaving out domain and codomain and only giving the rule of correspondence of the function).

3. This is particularly useful if you do not want to ignore constant factors. For example the median of n elements can be determined using $\frac{3}{2}n + o(n)$ comparisons.

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3. People write e.g. $h(n) = f(n) + o(g(n))$ when they mean that there exists a function $z : \mathbb{N} \rightarrow \mathbb{R}^+, n \mapsto z(n), z \in o(g)$ such that $h(n) = f(n) + z(n)$.

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4. People write $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$, when they mean $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$. Again this is not an equality.

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Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

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Here, $\Theta(n)$ stands for an **anonymous function** in the set $\Theta(n)$ that makes the expression true.

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Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

Asymptotic Notation in Equations

How do we interpret an expression like:

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How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function $f(n) \in \mathcal{O}(n)$ there is an anonymous function $g(n) \in \Theta(n^2)$ that makes the expression true.

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How do we interpret an expression like:

$$\sum_{i=1}^n \Theta(i) = \Theta(n^2)$$

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Careful!

Asymptotic Notation in Equations

The $\Theta(i)$ -symbol on the left represents **one** anonymous function $f : \mathbb{N} \rightarrow \mathbb{R}^+$, and then $\sum_i f(i)$ is computed.

How do we interpret an expression like:

$$\sum_{i=1}^n \Theta(i) = \Theta(n^2)$$

Careful!

“It is understood” that every occurrence of an Θ -symbol (or Ω, o, ω) on the left represents **one anonymous function**.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$$

$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$ does not really have a reasonable interpretation.

Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\{f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)\}$$

with $g(n) \in \mathcal{O}(n)$ and $h(n) \in \mathcal{O}(\log n)$

Recall that according to the previous slide e.g. the expressions $\sum_{i=1}^n \mathcal{O}(i)$ and $\sum_{i=1}^{n/2} \mathcal{O}(i) + \sum_{i=n/2+1}^n \mathcal{O}(i)$ generate different sets.

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containment btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Note that the equation does not hold.

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Lemma 1

Let f, g be functions with the property

$\exists n_0 > 0 \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- ▶ $c \cdot f(n) \in \Theta(f(n))$ for any constant c

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The expressions also hold for Ω . Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

Asymptotic Notation

Comments

- ▶ Do not use asymptotic notation within induction proofs.

Asymptotic Notation

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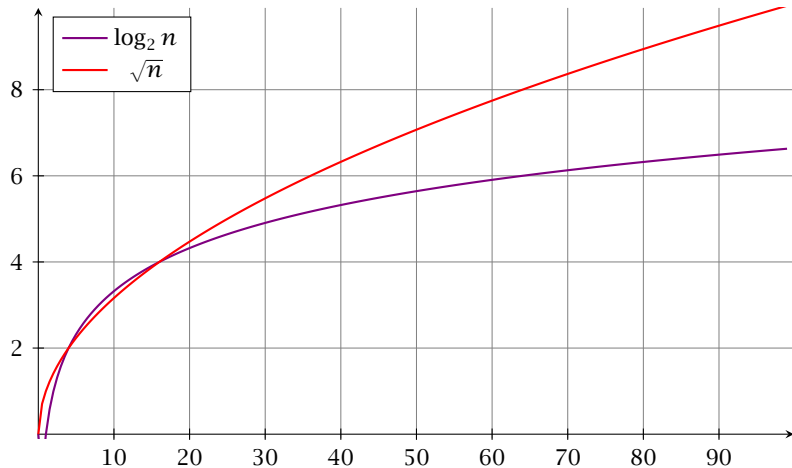
- ▶ Do not use asymptotic notation within induction proofs.
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Therefore, we will usually ignore the base of a logarithm within asymptotic notation.

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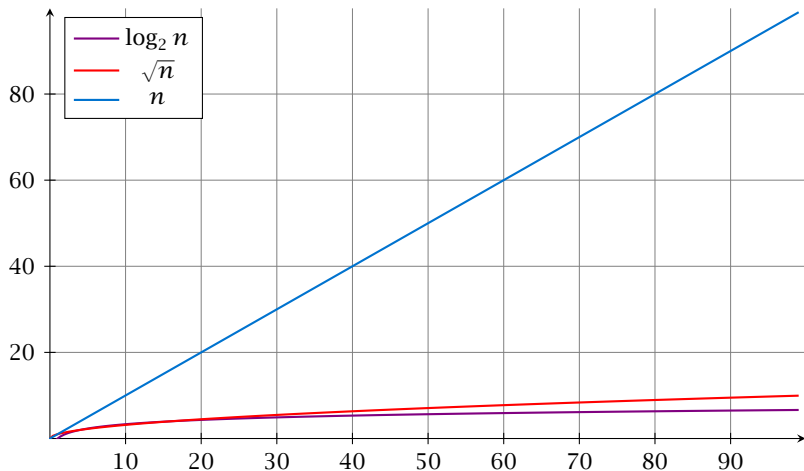
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Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
- ▶ In general $\log n = \log_2 n$, i.e., we use 2 as the default base for the logarithm.

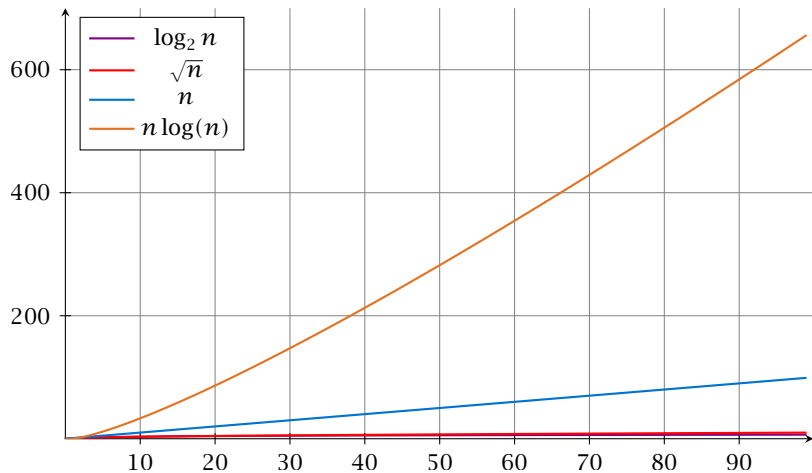
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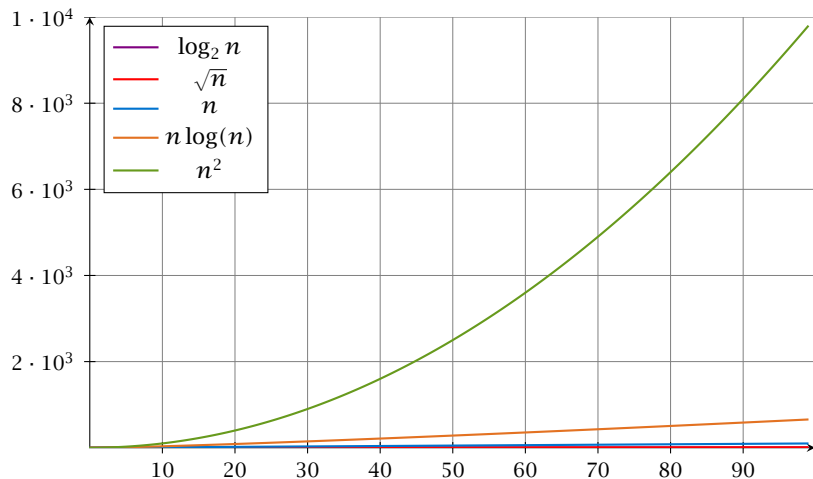
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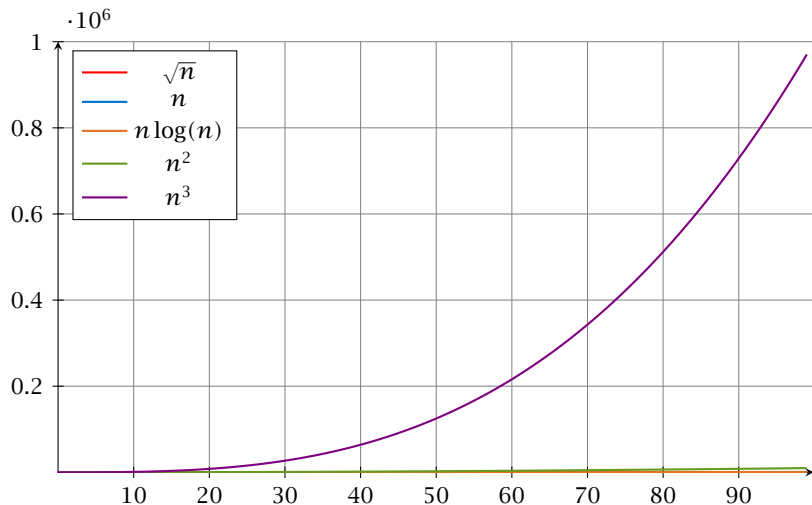
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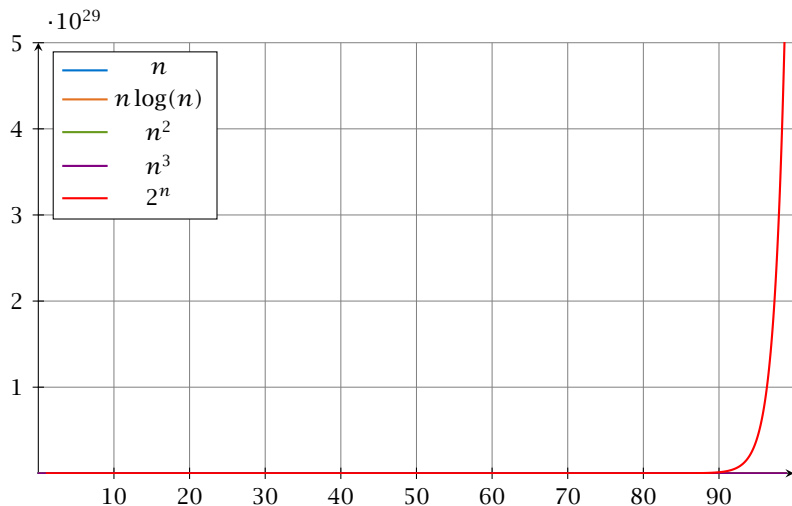
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Laufzeiten

Funktion	Eingabelänge n							
	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\log n$	33ns	66ns	0.1 μ s	0.1 μ s	0.2 μ s	0.2 μ s	0.2 μ s	0.3 μ s
\sqrt{n}	32ns	0.1 μ s	0.3 μ s	1 μ s	3.1 μ s	10 μ s	31 μ s	0.1ms
n	100ns	1 μ s	10 μ s	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3 μ s	6.6 μ s	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3 μ s	10 μ s	0.3ms	10ms	0.3s	10s	5.2min	2.7h
n^2	1 μ s	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
n^3	10 μ s	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$ y	
1.1^n	26ns	0.1ms	$7.8 \cdot 10^{25}$ y					
2^n	10 μ s	$4 \cdot 10^{14}$ y						
$n!$	36ms	$3 \cdot 10^{142}$ y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca. $13.8 \cdot 10^9$ y

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In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n .

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Clearly $f = o(g)$. However, as long as $\log n \leq 1000$ Algorithm B will be more efficient.

Multiple Variables in Asymptotic Notation

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Example 2

► $f : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $g(n, m) = n - 1$

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- ▶ $f : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $g(n, m) = n - 1$
then $f = \mathcal{O}(g)$ does not hold
- ▶ $f : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $f(n, m) = 1$ und $g : \mathbb{N} \rightarrow \mathbb{R}_0^+$, $g(n, m) = n$
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