A Fast Matching Algorithm

Algorithm 52 Bimatch-Hopcroft-Karp(G)

$M \leftarrow \emptyset$
repeat
  let $\mathcal{P} = \{P_1, \dots, P_k\}$ be maximal set of
  vertex-disjoint, shortest augmenting path w.r.t. M.
  $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
until $\mathcal{P} = \emptyset$
return M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis Hopcroft-Karp

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Given a matching M and a matching M^* with $|M^*| - |M| \geq 0$. There exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

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- Similar to the proof that a matching is optimal iff it does not contain an augmenting path.

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- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.

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- The connected components of G are cycles and paths.

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- The connected components of G are cycles and paths.
- The graph contains $k \stackrel{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least k components that form a path starting and ending with a red edge. These are augmenting paths w.r.t. M.

Analysis Hopcroft-Karp

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Lemma 7

The set $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \cdots \cup P_k) \oplus P$ *contains at least* $(k+1)\ell$ *edges.*

Analysis Hopcroft-Karp

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- Hence, the set contains at least $k+1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.
- Each of these paths is of length at least ℓ.

Analysis Hopcroft-Karp

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- Hence, $|A| \leq k\ell + |P| - 1$.

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- Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \dots, P_k\}$.
- This edge is not contained in A.
- Hence, $|A| \leq k\ell + |P| - 1$.
- The lower bound on $|A|$ gives $(k+1)\ell \leq |A| \leq k\ell + |P| - 1$, and hence $|P| \geq \ell + 1$.

Analysis Hopcroft-Karp

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Proof.
The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

Analysis Hopcroft-Karp

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The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ *phases.*

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Proof.

- After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|} + 1) \leq \sqrt{|V|}$ additional augmentations.

Analysis Hopcroft-Karp

Lemma 10

One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

construct a "level graph" G':

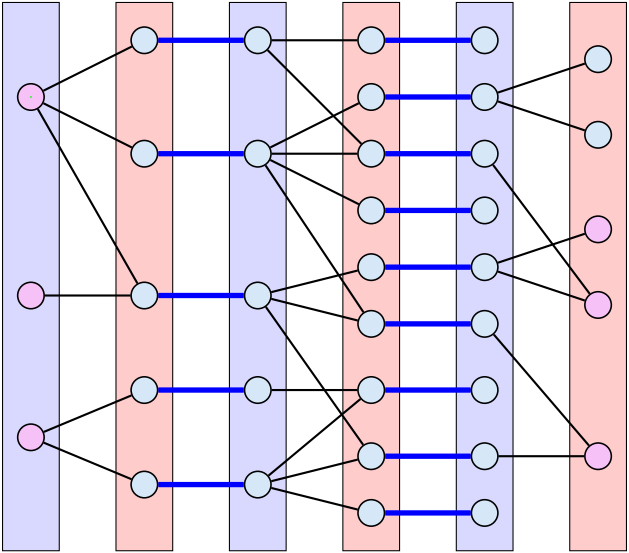
- construct Level 0 that includes all free vertices on left side L
- construct Level 1 containing all neighbors of Level 0
- construct Level 2 containing matching neighbors of Level 1
- construct Level 3 containing all neighbors of Level 2
- ...
- stop when a level (apart from Level 0) contains a free vertex

can be done in time $\mathcal{O}(m)$ by a modified BFS

Analysis Hopcroft-Karp

- a shortest augmenting path must go from Level 0 to the last layer constructed
- it can only use edges between layers
- construct a maxim**al** set of vertex disjoint augmenting path connecting the layers
- for this, go forward until you either reach a free vertex or you reach a "dead end" v
- if you reach a free vertex delete the augmenting path and all incident edges from the graph
- if you reach a dead end backtrack and delete v together with its incident edges

Analysis Hopcroft-Karp



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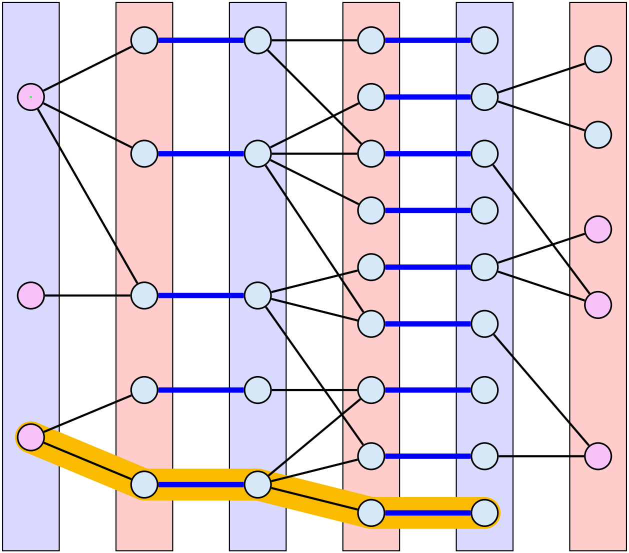

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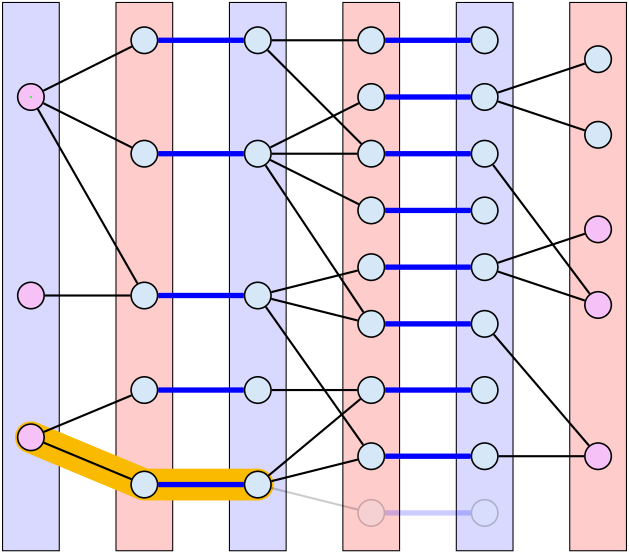

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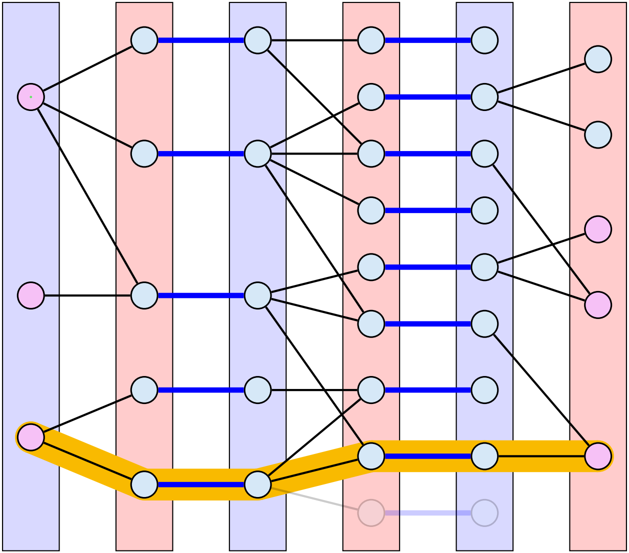

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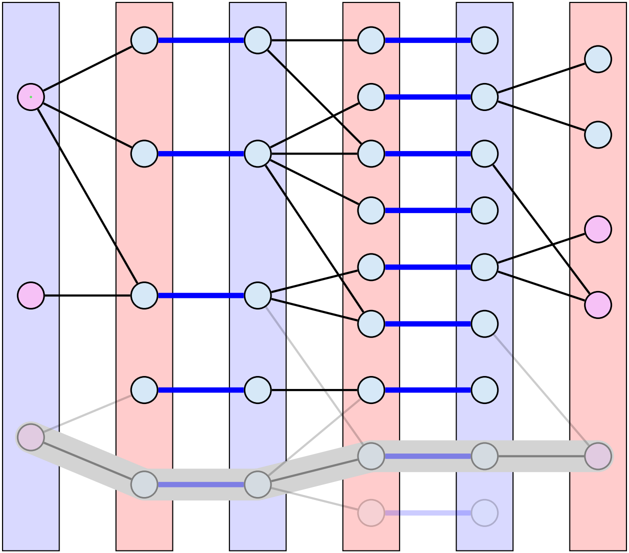

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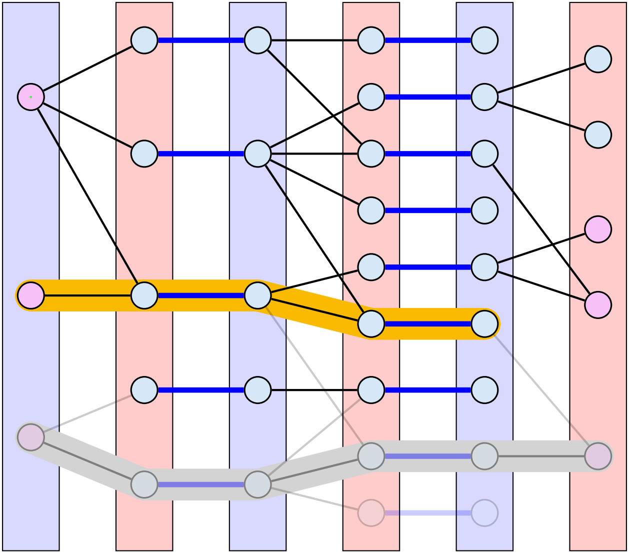

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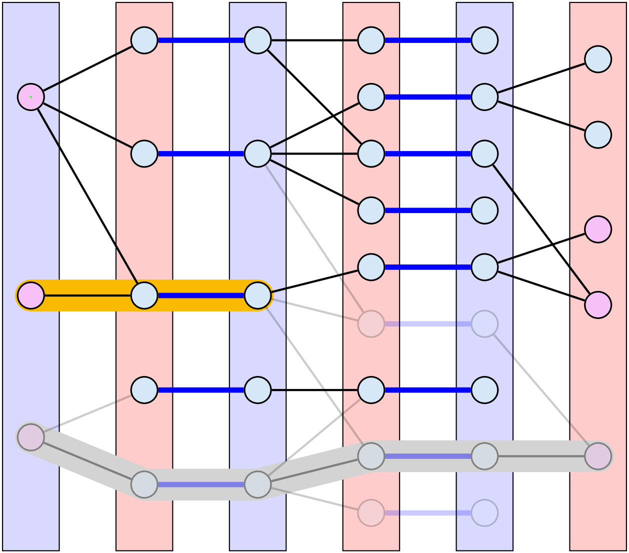

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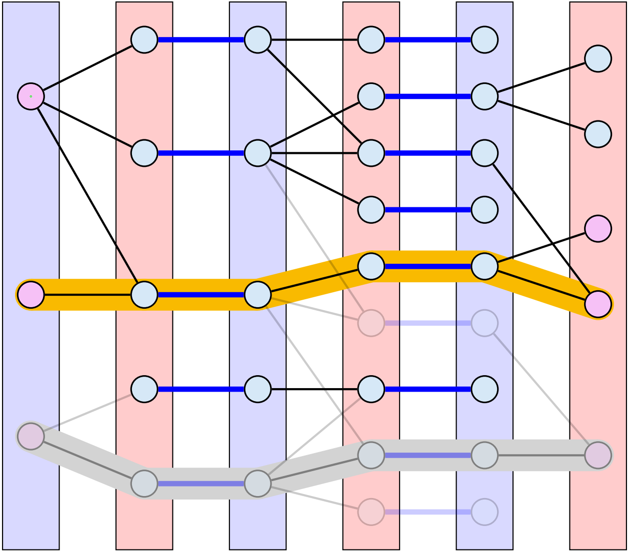

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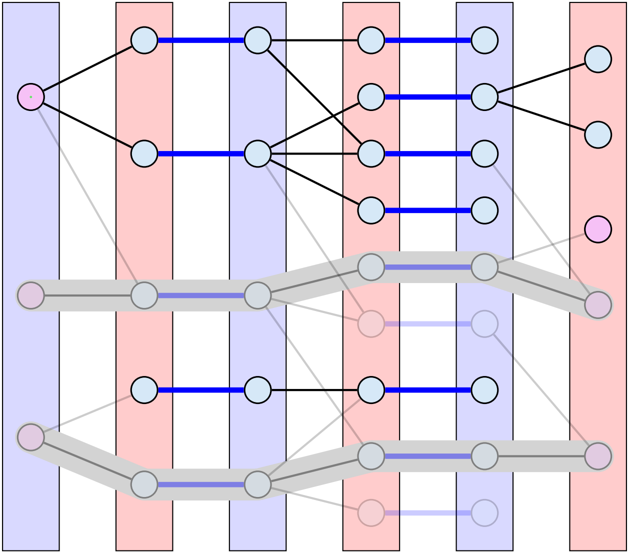

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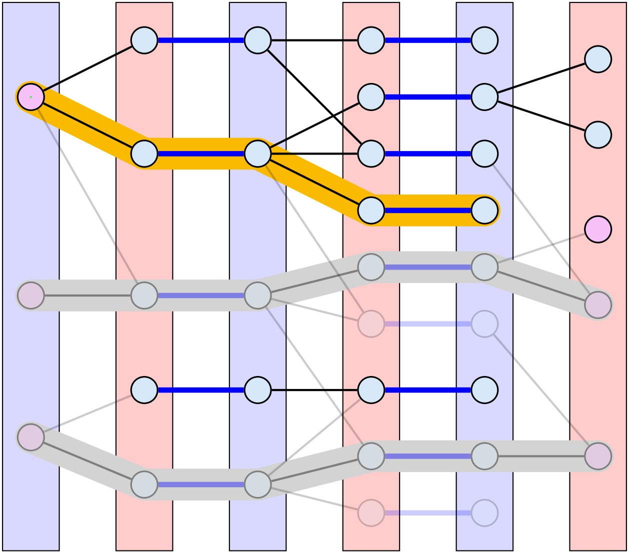

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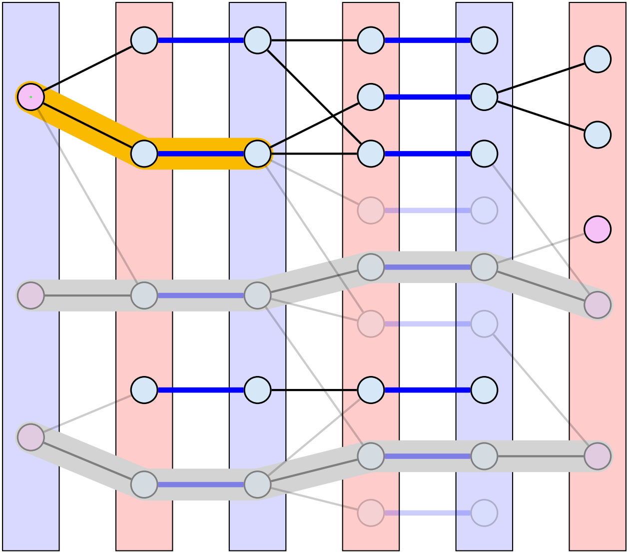

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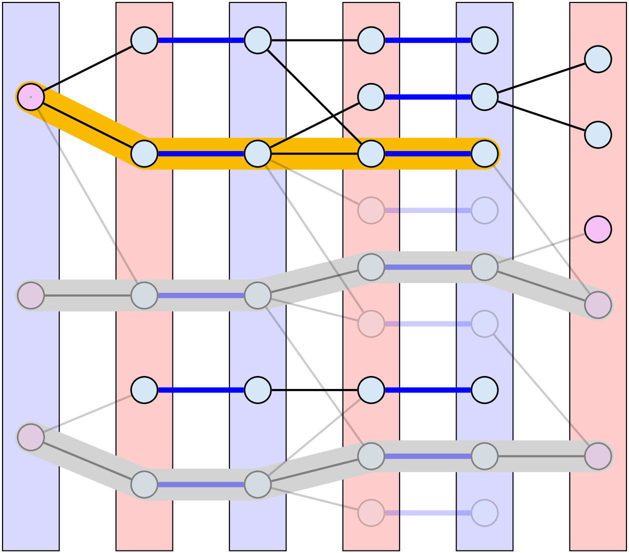

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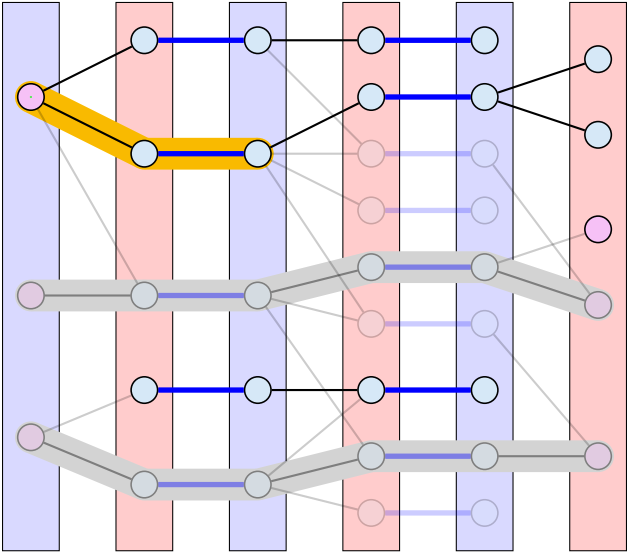

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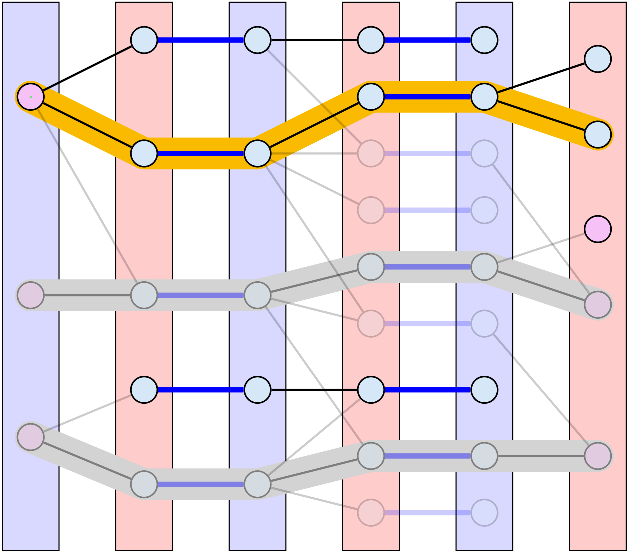

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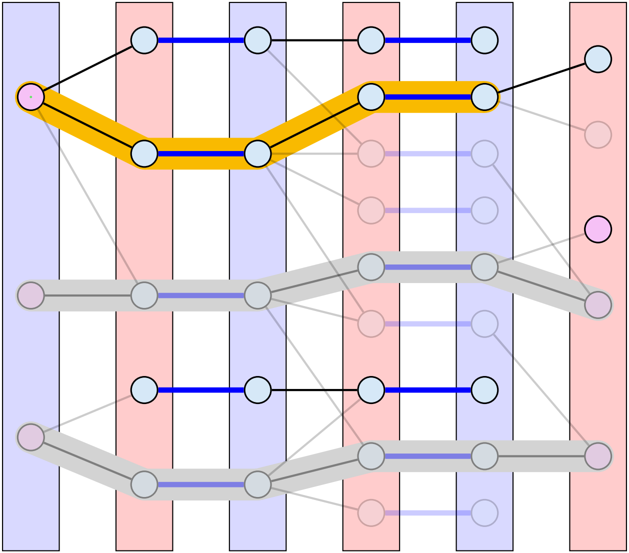

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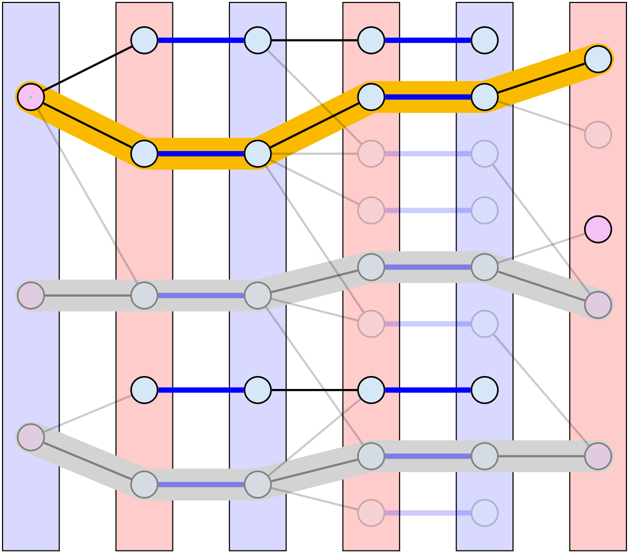

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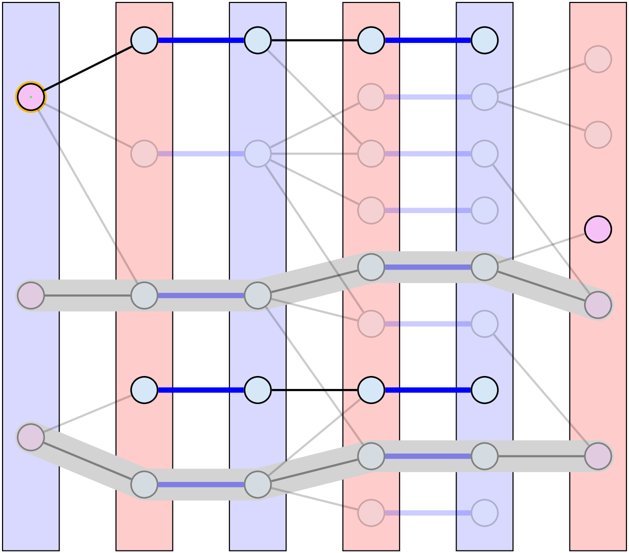

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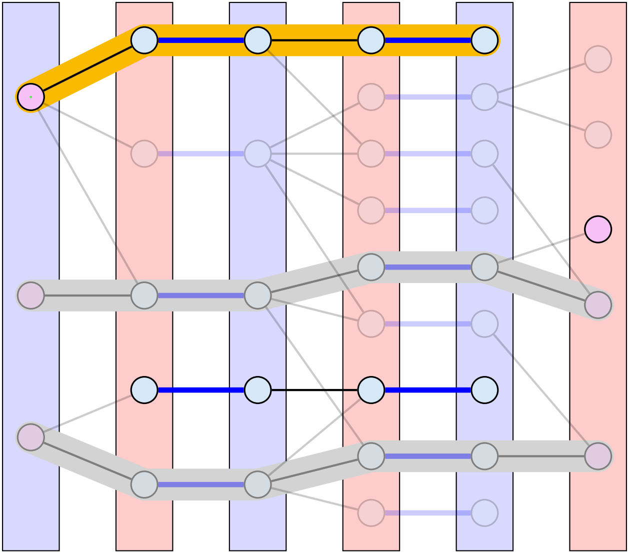

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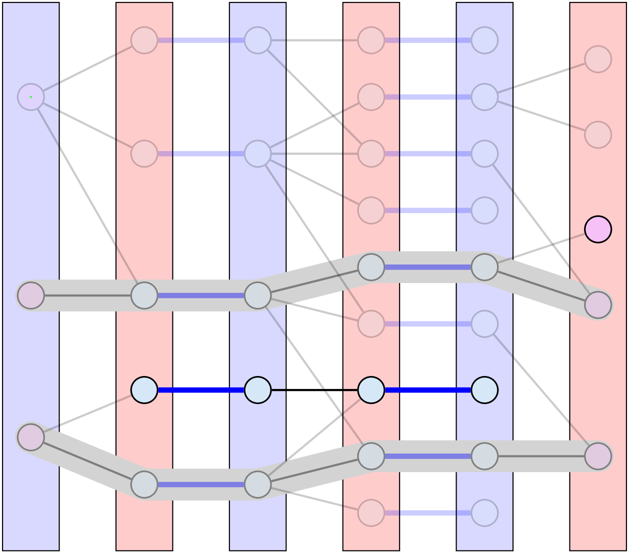

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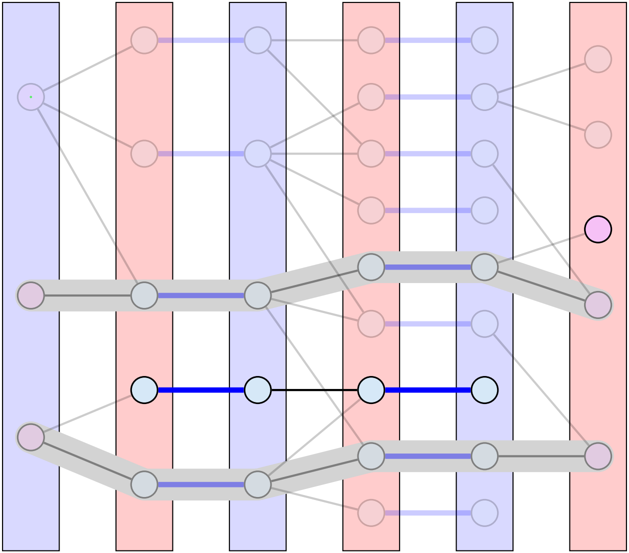

Analysis: Shortest Augmenting Path for Flows

cost for searches during a phase is $\mathcal{O}(mn)$

- a search (successful or unsuccessful) takes time $\mathcal{O}(n)$
- a search deletes at least one edge from the level graph

there are at most n phases

Time: $\mathcal{O}(mn^2)$.

Analysis for Unit-capacity Simple Networks

cost for searches during a phase is $\mathcal{O}(m)$

- an edge/vertex is traversed at most twice

need at most $\mathcal{O}(\sqrt{n})$ phases

- after $\sqrt{n}$ phases there is a cut of size at most $\sqrt{n}$ in the residual graph
- hence at most $\sqrt{n}$ additional augmentations required

Time: $\mathcal{O}(m\sqrt{n})$.