# 18 Weighted Bipartite Matching

**Weighted Bipartite Matching/Assignment**

- Input: undirected, bipartite graph $G = L \cup R, E$.
- an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

**Simplifying Assumptions (wlog [why?]):**

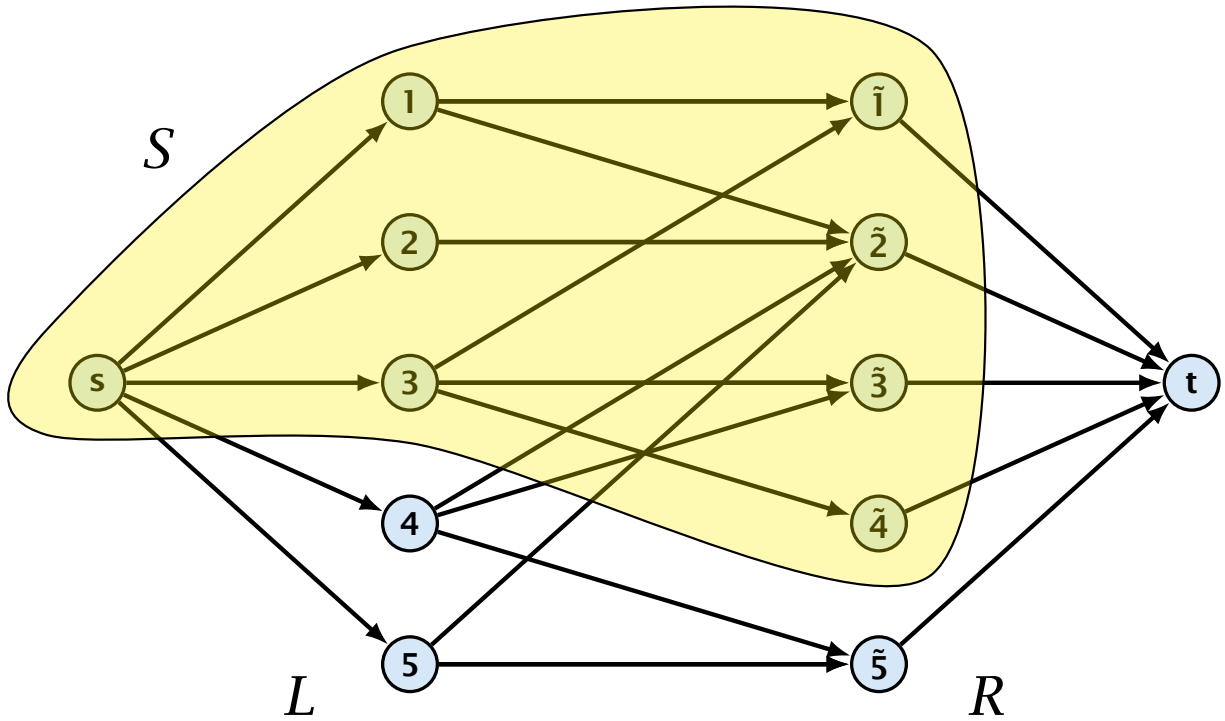
- assume that $|L| = |R| = n$
- assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$
- can assume goal is to construct maximum weight perfect matching

# Weighted Bipartite Matching

**Theorem 6 (Halls Theorem)**

*A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in $R$ that have a neighbour in $S$.*

# 18 Weighted Bipartite Matching



# Halls Theorem

**Proof:**

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in $S$ could be matched to different neigbhours.

# Halls Theorem

**Proof:**

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in $S$ could be matched to different neigbhours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph $G'$ is at least $|L|$.

# Halls Theorem

**Proof:**

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in $S$ could be matched to different neigbhours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph $G'$ is at least $|L|$.
  - Let $S$ denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of $S$ inside $L$ and $R$, respectively.

# Halls Theorem

**Proof:**

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in $S$ could be matched to different neigbhours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph $G'$ is at least $|L|$.
  - Let $S$ denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of $S$ inside $L$ and $R$, respectively.
  - Clearly, all neighbours of nodes in $L_S$ have to be in $S$, as otherwise we would cut an edge of infinite capacity.

# Halls Theorem

**Proof:**

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in $S$ could be matched to different neigbhours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph $G'$ is at least $|L|$.
  - Let $S$ denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of $S$ inside $L$ and $R$, respectively.
  - Clearly, all neighbours of nodes in $L_S$ have to be in $S$, as otherwise we would cut an edge of infinite capacity.
  - This gives $R_S \geq |\Gamma(L_S)|$.

# Halls Theorem

**Proof:**

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in $S$ could be matched to different neigbhours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph $G'$ is at least $|L|$.
  - Let $S$ denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of $S$ inside $L$ and $R$, respectively.
  - Clearly, all neighbours of nodes in $L_S$ have to be in $S$, as otherwise we would cut an edge of infinite capacity.
  - This gives $R_S \geq |\Gamma(L_S)|$.
  - The size of the cut is $|L| - |L_S| + |R_S|$.

# Halls Theorem

**Proof:**

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in $S$ could be matched to different neigbhours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph $G'$ is at least $|L|$.
  - Let $S$ denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of $S$ inside $L$ and $R$, respectively.
  - Clearly, all neighbours of nodes in $L_S$ have to be in $S$, as otherwise we would cut an edge of infinite capacity.
  - This gives $R_S \geq |\Gamma(L_S)|$.
  - The size of the cut is $|L| - |L_S| + |R_S|$.
  - Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

# Algorithm Outline

**Idea:**

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node $v$.

# Algorithm Outline

**Idea:**

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node $v$.

- Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

# Algorithm Outline

**Idea:**

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node $v$.

- Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

- Let $H(\vec{x})$ denote the subgraph of $G$ that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.

# Algorithm Outline

**Idea:**

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \in \mathbb{R}$ denote the weight of node $v$.

- Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

- Let $H(\vec{x})$ denote the subgraph of $G$ that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.
- Try to compute a perfect matching in the subgraph $H(\vec{x})$. If you are successful you found an optimal matching.

# Algorithm Outline

**Reason:**

- The weight of your matching $M^*$ is

$$\sum_{(u,v)\in M^*} w_{(u,v)} = \sum_{(u,v)\in M^*} (x_u + x_v) = \sum_v x_v \ .$$

- Any other perfect matching $M$ (in $G$, not necessarily in $H(\vec{x})$) has

$$\sum_{(u,v)\in M} w_{(u,v)} \leq \sum_{(u,v)\in M} (x_u + x_v) = \sum_v x_v \ .$$

# Algorithm Outline

**What if you don't find a perfect matching?**

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where $\Gamma$ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

# Algorithm Outline

**What if you don't find a perfect matching?**

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where $\Gamma$ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

**Idea:** reweight such that:

- the total weight assigned to nodes decreases
- the weight function still dominates the edge-weights

# Algorithm Outline

**What if you don't find a perfect matching?**

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where $\Gamma$ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

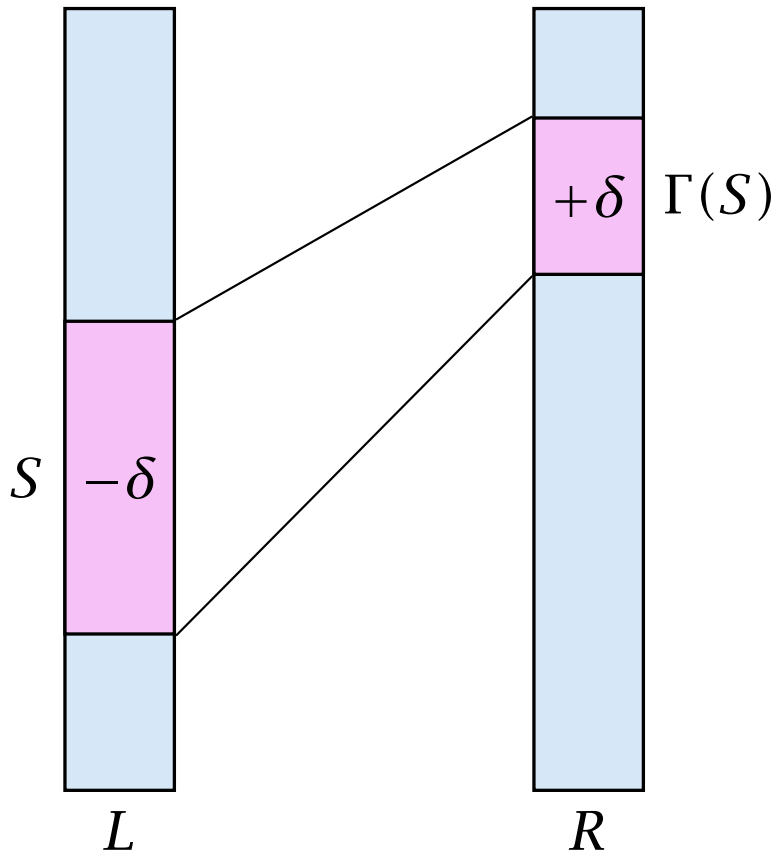
**Idea:** reweight such that:

- the total weight assigned to nodes decreases
- the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

# Changing Node Weights

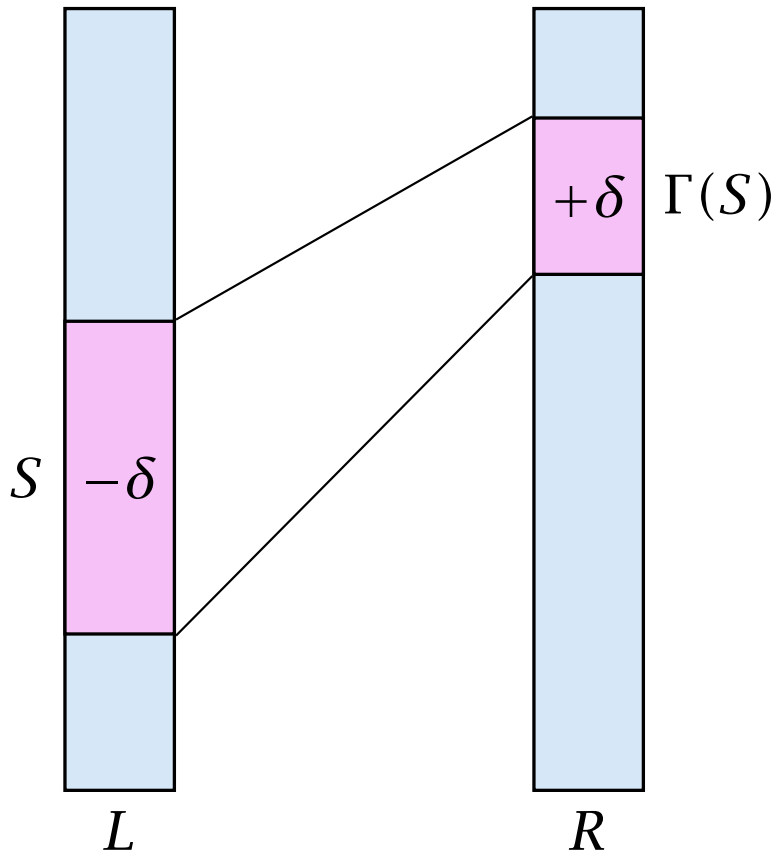
Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in $S$ by $-\delta$.



# Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in $S$ by $-\delta$.

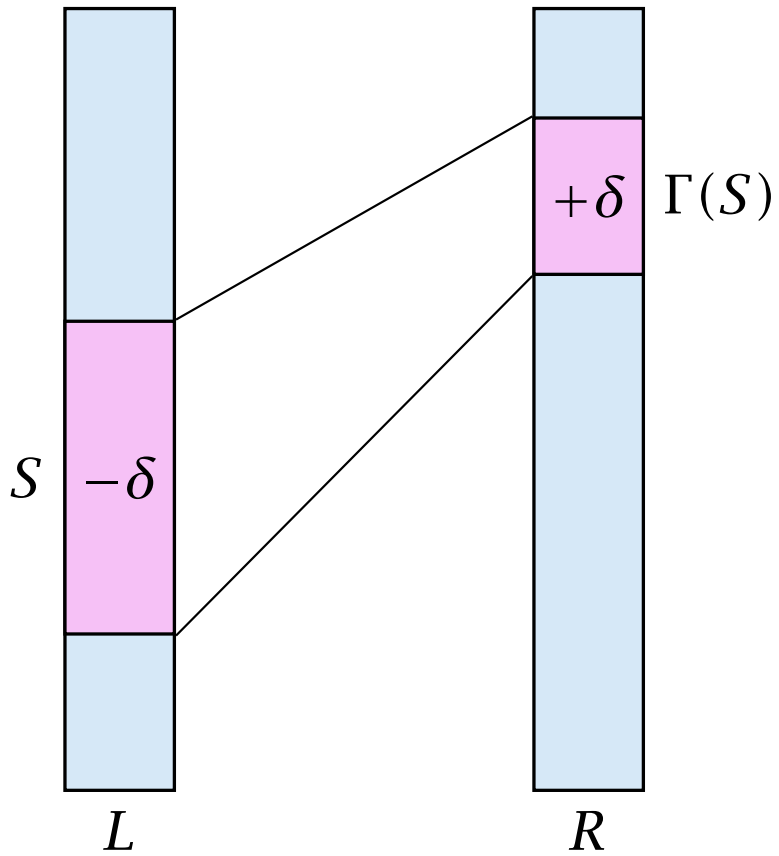
- Total node-weight decreases.



# Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in $S$ by $-\delta$.

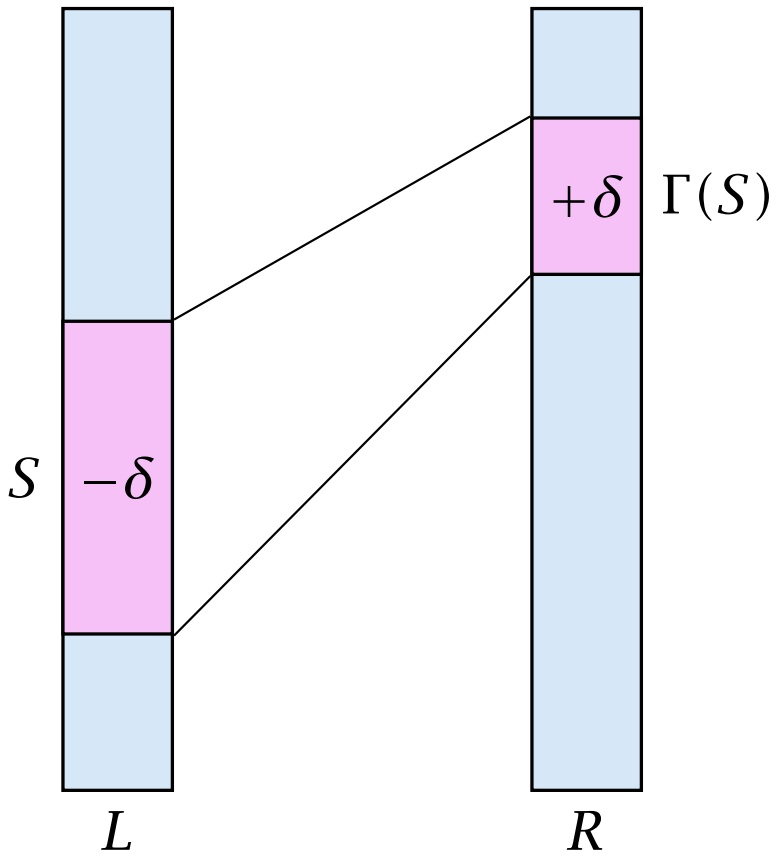
- Total node-weight decreases.
- Only edges from $S$ to $R - \Gamma(S)$ decrease in their weight.



# Changing Node Weights

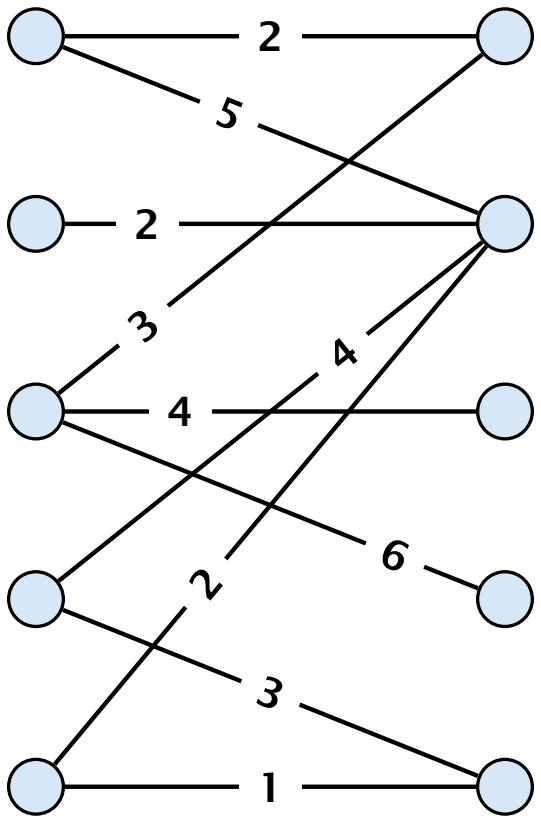
Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in $S$ by $-\delta$.

- Total node-weight decreases.
- Only edges from $S$ to $R - \Gamma(S)$ decrease in their weight.
- Since, none of these edges is tight (otw. the edge would be contained in $H(\vec{x})$, and hence would go between $S$ and $\Gamma(S)$) we can do this decrement for small enough $\delta > 0$ until a new edge gets tight.



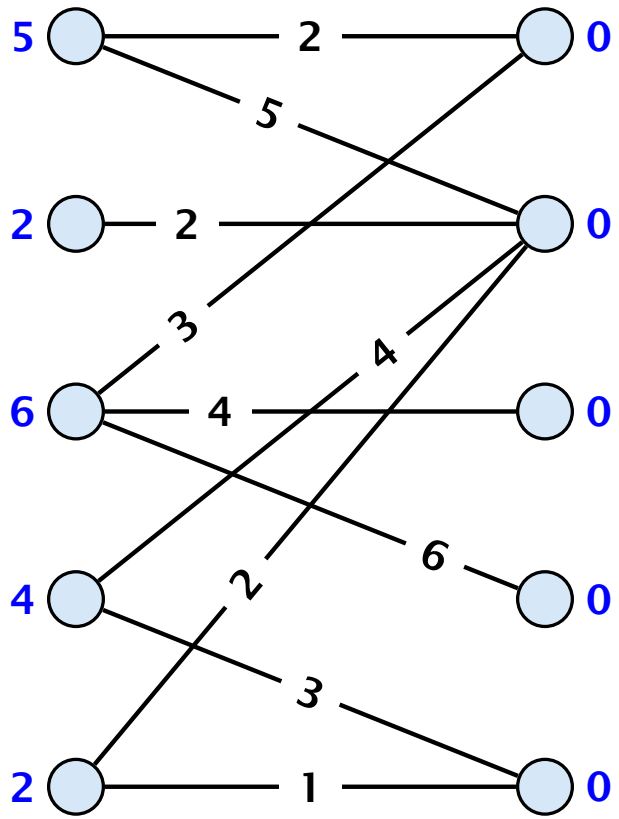
# Weighted Bipartite Matching

Edges not drawn have weight $0$.



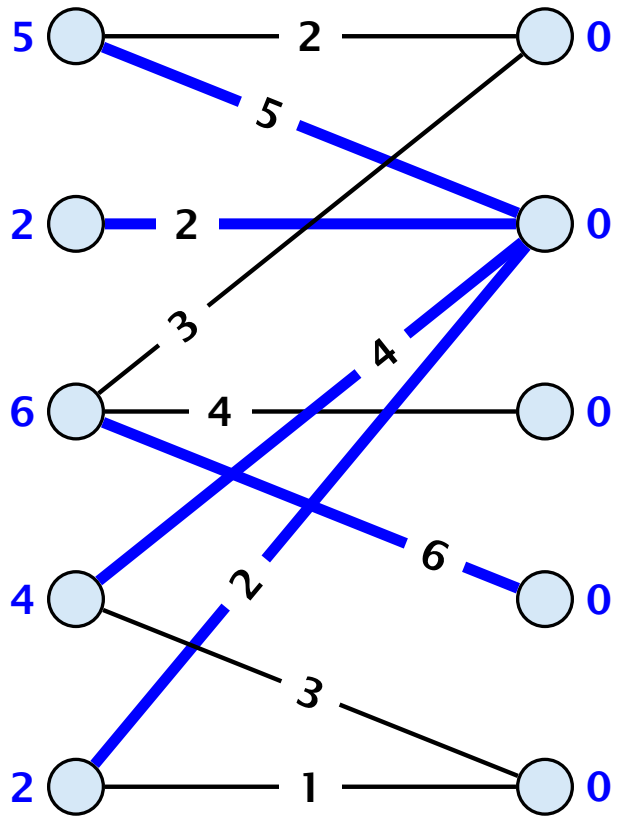
# Weighted Bipartite Matching

Edges not drawn have weight $0$.



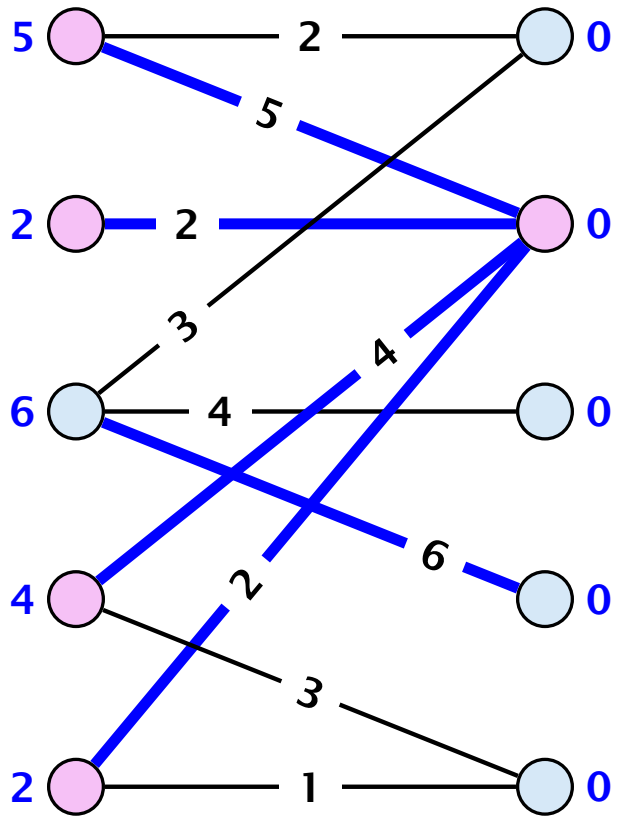
# Weighted Bipartite Matching

Edges not drawn have weight $0$.



# Weighted Bipartite Matching

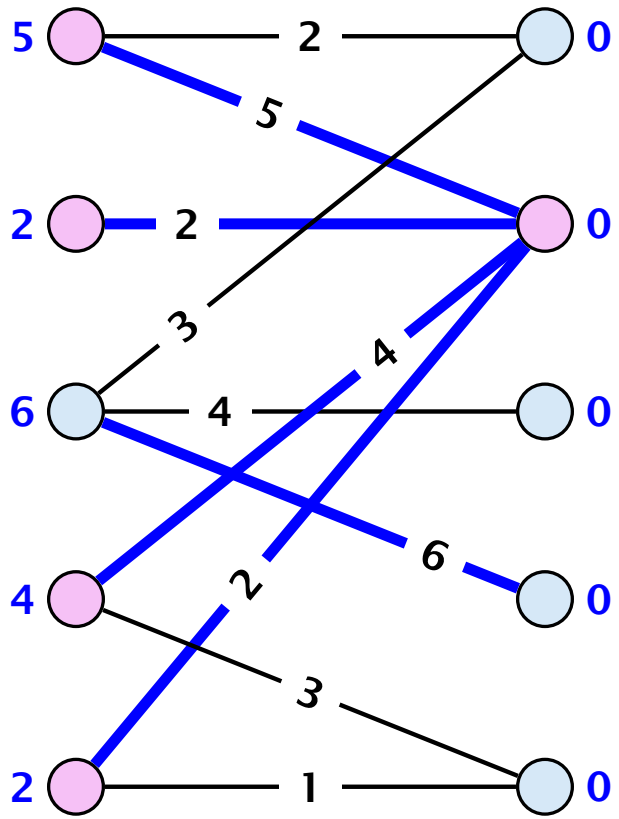
Edges not drawn have weight $0$.



# Weighted Bipartite Matching

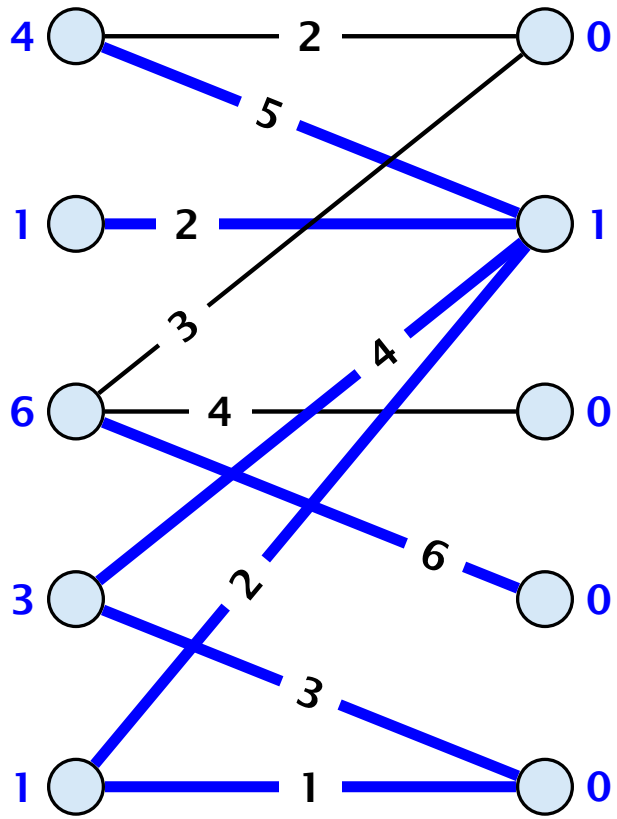
Edges not drawn have weight $0$.

$\delta = 1$



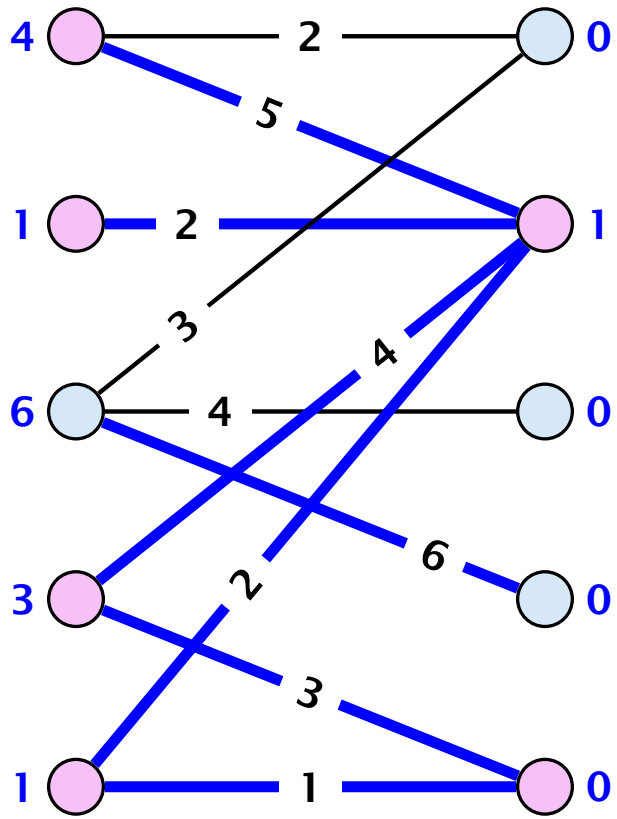
# Weighted Bipartite Matching

Edges not drawn have weight $0$.



# Weighted Bipartite Matching

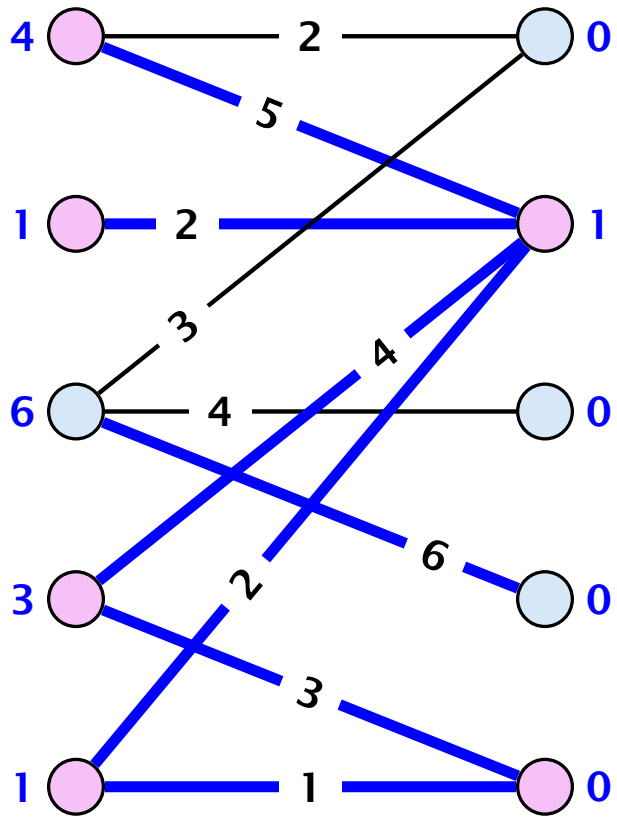
Edges not drawn have weight $0$.



# Weighted Bipartite Matching

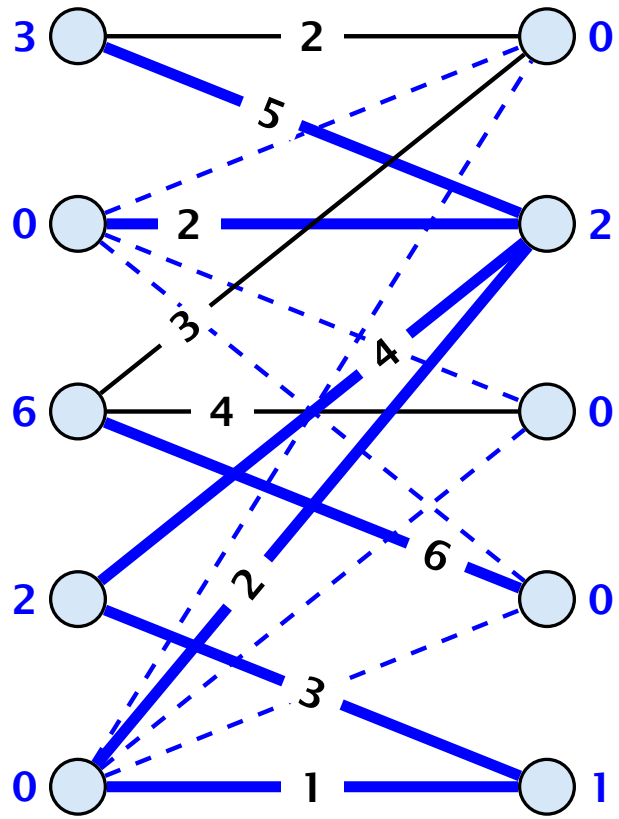
Edges not drawn have weight $0$.

$\delta = 1$



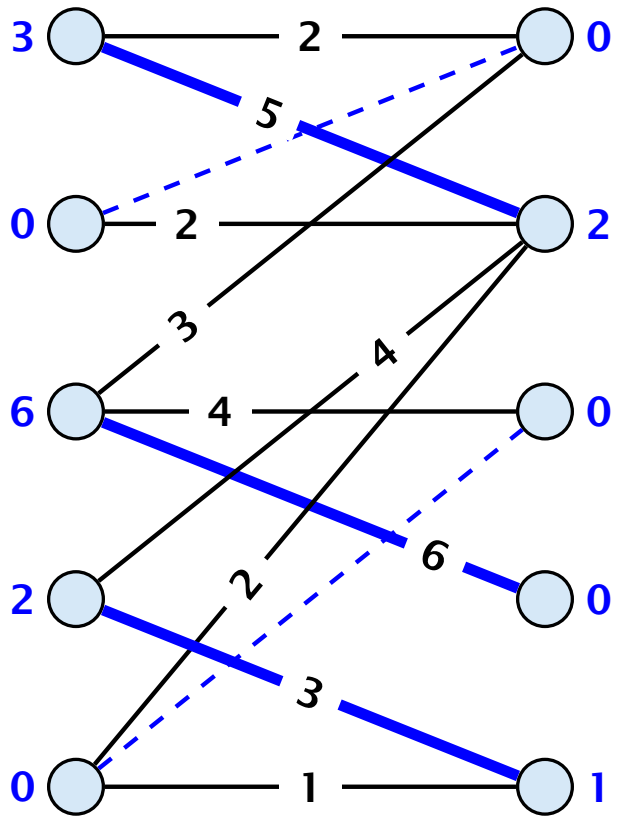
# Weighted Bipartite Matching

Edges not drawn have weight $0$.



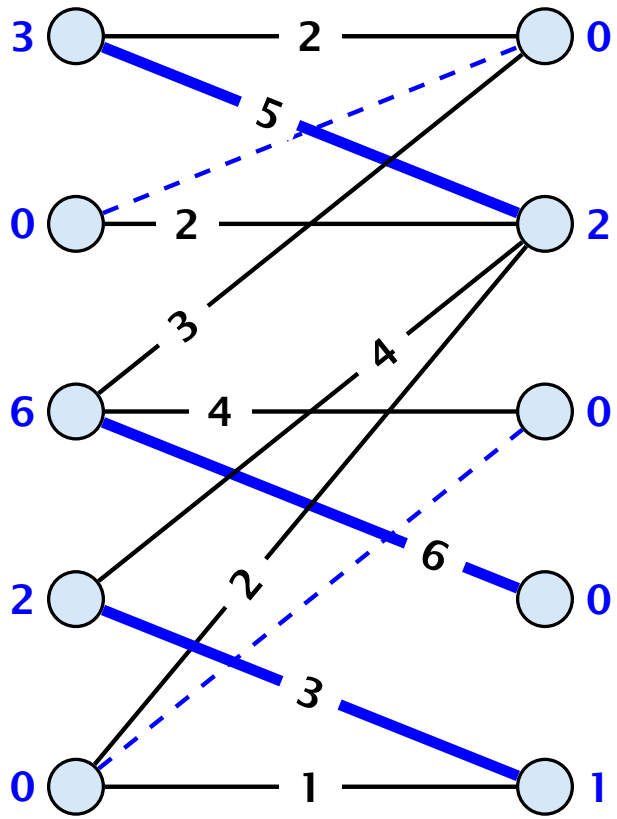
# Weighted Bipartite Matching

Edges not drawn have weight $0$.



# Weighted Bipartite Matching

Edges not drawn have weight $0$.



# Analysis

**How many iterations do we need?**

- One reweighting step increases the number of edges out of $S$ by at least one.

# Analysis

**How many iterations do we need?**

- One reweighting step increases the number of edges out of $S$ by at least one.
- Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in $S$ (we will show that we can always find $S$ and a matching such that this holds).

# Analysis

**How many iterations do we need?**

- One reweighting step increases the number of edges out of $S$ by at least one.
- Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in $S$ (we will show that we can always find $S$ and a matching such that this holds).
- This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and $S$ or between $L - S$ and $R - \Gamma(S)$.

# Analysis

**How many iterations do we need?**

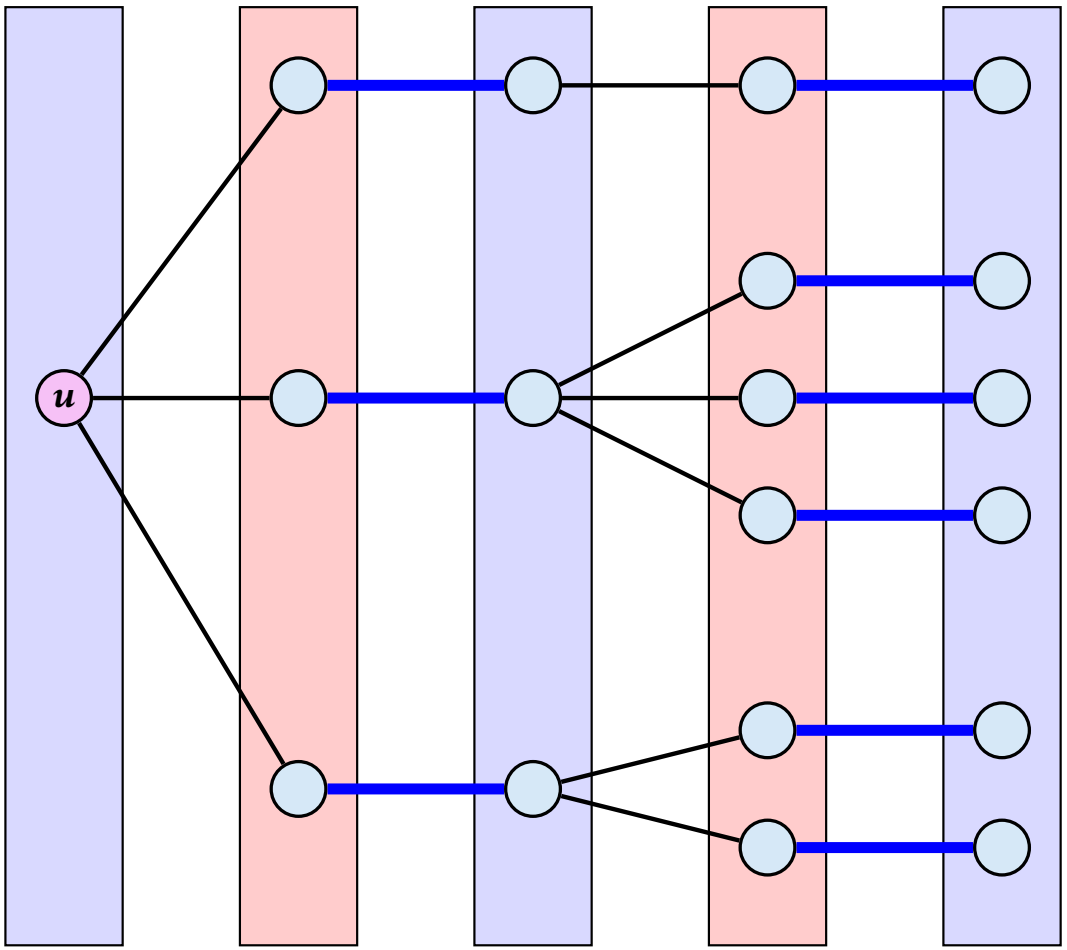
- One reweighting step increases the number of edges out of $S$ by at least one.
- Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in $S$ (we will show that we can always find $S$ and a matching such that this holds).
- This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and $S$ or between $L - S$ and $R - \Gamma(S)$.
- Hence, reweighting does not decrease the size of a maximum matching in the tight sub-graph.

# Analysis

- We will show that after at most $n$ reweighting steps the size of the maximum matching can be increased by finding an augmenting path.
- This gives a polynomial running time.

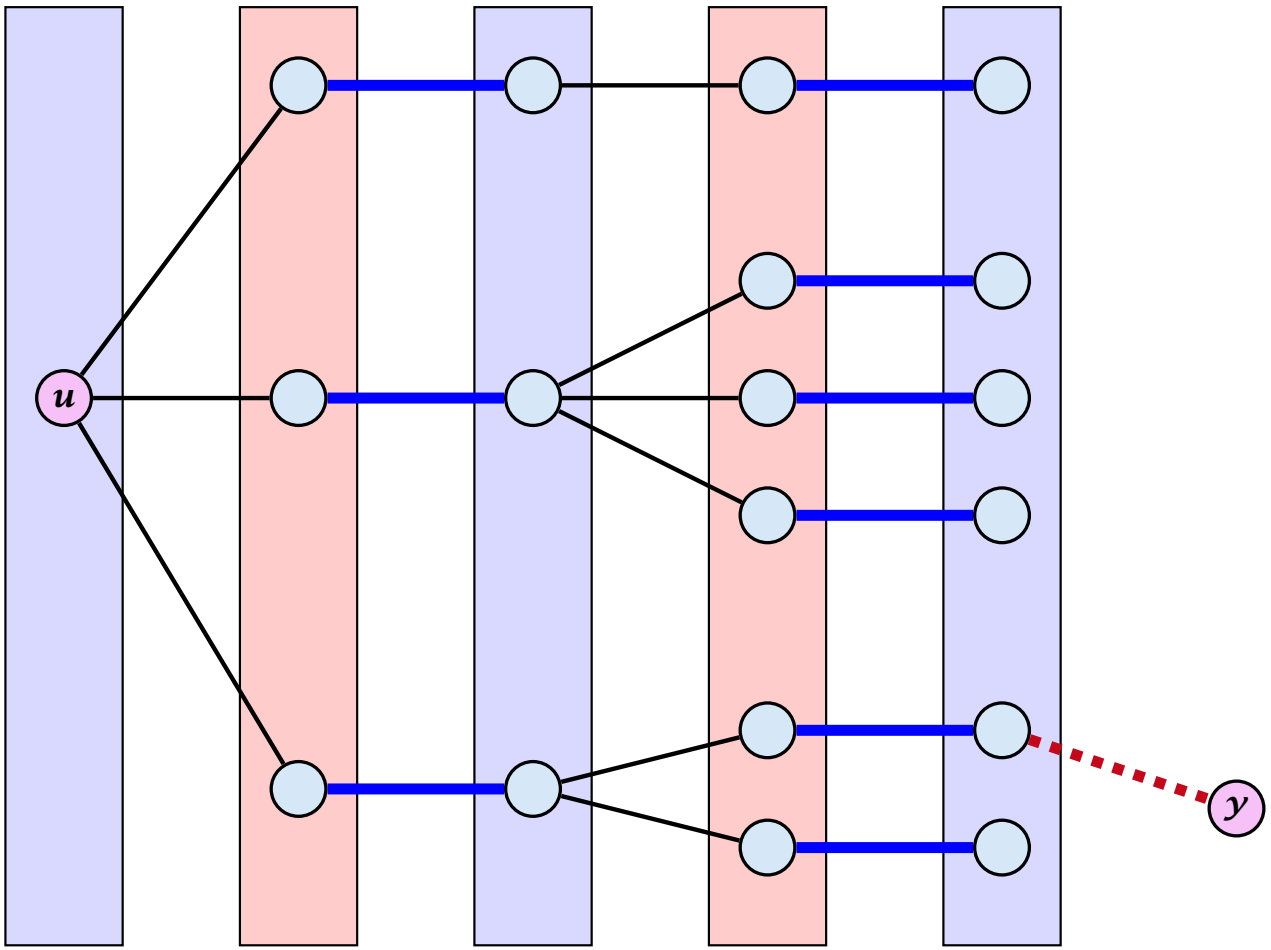
# How to find an augmenting path?

**Construct an alternating tree.**



# How to find an augmenting path?

**Construct an alternating tree.**



# Analysis

**How do we find $S$?**

- Start on the left and compute an alternating tree, starting at any free node $u$.

# Analysis

**How do we find $S$?**

- Start on the left and compute an alternating tree, starting at any free node $u$.
- If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at $u$).

# Analysis

**How do we find $S$?**

- Start on the left and compute an alternating tree, starting at any free node $u$.
- If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at $u$).
- The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.

# Analysis

**How do we find $S$?**

- Start on the left and compute an alternating tree, starting at any free node $u$.
- If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at $u$).
- The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.
- All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex $u$. Hence, $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

# Analysis

- The current matching does not have any edges from $V_{\text{odd}}$ to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).

# Analysis

- The current matching does not have any edges from $V_{\text{odd}}$ to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).
- After changing weights, there is at least one more edge connecting $V_{\text{even}}$ to a node outside of $V_{\text{odd}}$. After at most $n$ reweights we can do an augmentation.

# Analysis

- The current matching does not have any edges from $V_{\text{odd}}$ to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).
- After changing weights, there is at least one more edge connecting $V_{\text{even}}$ to a node outside of $V_{\text{odd}}$. After at most $n$ reweights we can do an augmentation.
- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).

# Analysis

- The current matching does not have any edges from $V_{\text{odd}}$ to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).
- After changing weights, there is at least one more edge connecting $V_{\text{even}}$ to a node outside of $V_{\text{odd}}$. After at most $n$ reweights we can do an augmentation.
- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).
- An augmentation takes at most $\mathcal{O}(n)$ time.

# Analysis

- The current matching does not have any edges from $V_{\text{odd}}$ to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).
- After changing weights, there is at least one more edge connecting $V_{\text{even}}$ to a node outside of $V_{\text{odd}}$. After at most $n$ reweights we can do an augmentation.
- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).
- An augmentation takes at most $\mathcal{O}(n)$ time.
- In total we obtain a running time of $\mathcal{O}(n^4)$.

# Analysis

- The current matching does not have any edges from $V_{\text{odd}}$ to $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).
- After changing weights, there is at least one more edge connecting $V_{\text{even}}$ to a node outside of $V_{\text{odd}}$. After at most $n$ reweights we can do an augmentation.
- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).
- An augmentation takes at most $\mathcal{O}(n)$ time.
- In total we obtain a running time of $\mathcal{O}(n^4)$.
- A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.