

# Preflows

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An  $(s, t)$ -preflow is a function  $f : E \mapsto \mathbb{R}^+$  that satisfies

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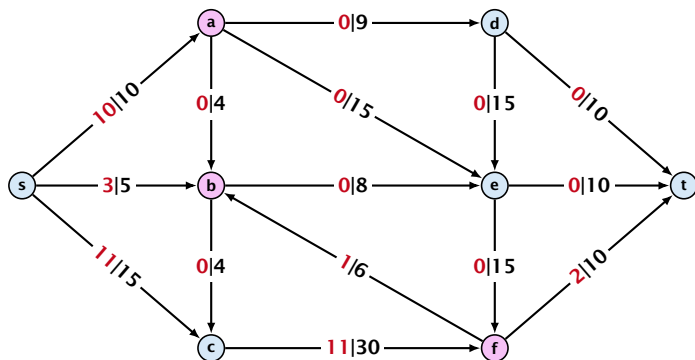
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2. For each  $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) \leq \sum_{e \in \text{into}(v)} f(e) .$$

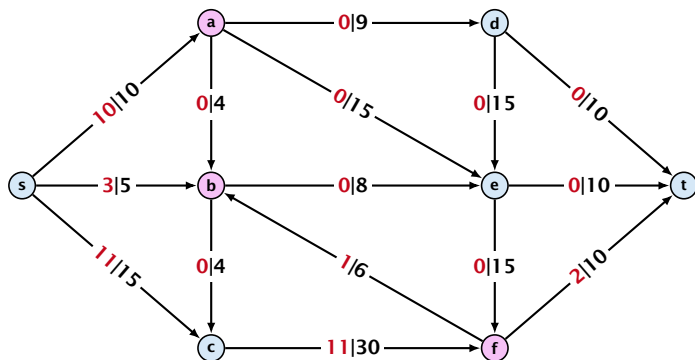
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A node that has  $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$  is called an **active node**.



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A **labelling** is a function  $\ell : V \rightarrow \mathbb{N}$ . It is **valid** for preflow  $f$  if

- ▶  $\ell(u) \leq \ell(v) + 1$  for all edges  $(u, v)$  in the residual graph  $G_f$   
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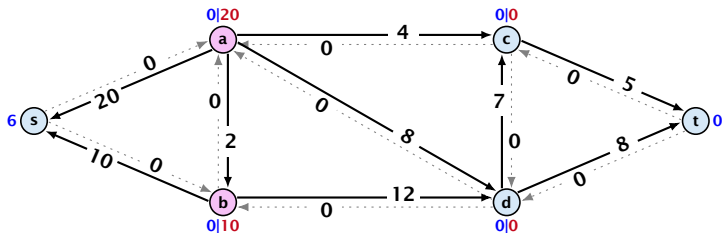
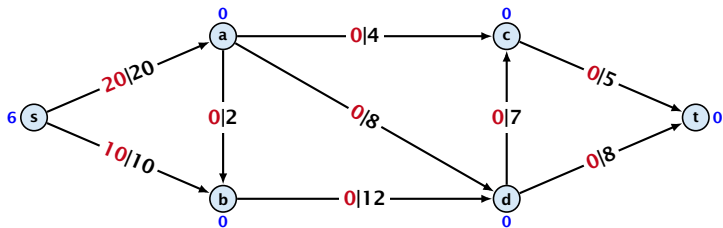
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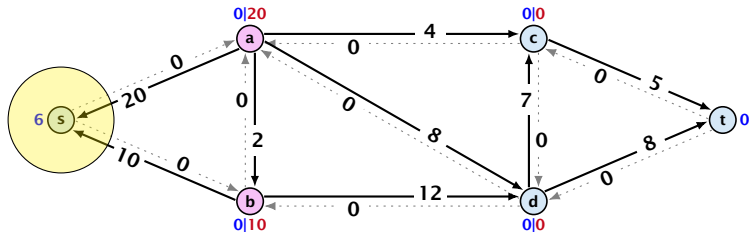
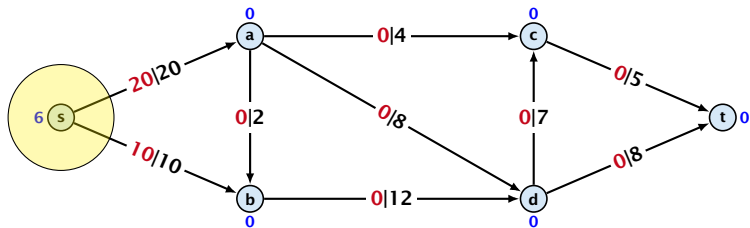
## Intuition:

The labelling can be viewed as a height function. Whenever the height from node  $u$  to node  $v$  decreases by more than 1 (i.e., it goes very steep downhill from  $u$  to  $v$ ), the corresponding edge must be saturated.

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## Lemma 8

A *flow* that has a valid labelling is a maximum flow.

# Push Relabel Algorithms

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## Idea:

- ▶ start with some preflow and some valid labelling

Note that this is somewhat dual to an augmenting path algorithm. The former maintains the property that it has a feasible flow. It successively changes this flow until it saturates some cut in which case we conclude that the flow is maximum. A preflow push algorithm maintains the property that it has a saturated cut. The preflow is changed iteratively until it fulfills conservation constraints in which case we can conclude that we have a maximum flow.

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- ▶ successively change the preflow while maintaining a valid labelling

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## Idea:

- ▶ start with some preflow and some valid labelling
- ▶ successively change the preflow while maintaining a valid labelling
- ▶ stop when you have a flow (i.e., no more active nodes)

Note that this is somewhat dual to an augmenting path algorithm. The former maintains the property that it has a feasible flow. It successively changes this flow until it saturates some cut in which case we conclude that the flow is maximum. A preflow push algorithm maintains the property that it has a saturated cut. The preflow is changed iteratively until it fulfills conservation constraints in which case we can conclude that we have a maximum flow.

## Changing a Preflow



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An arc  $(u, v)$  with  $c_f(u, v) > 0$  in the residual graph is **admissible** if  $\ell(u) = \ell(v) + 1$  (i.e., it goes downwards w.r.t. labelling  $\ell$ ).

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### The push operation

Consider an active node  $u$  with **excess flow**

$f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$  and suppose  $e = (u, v)$  is an admissible arc with residual capacity  $c_f(e)$ .

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We can send flow  $\min\{c_f(e), f(u)\}$  along  $e$  and obtain a new preflow. The old labelling is still valid (!!!).

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- ▶  **saturating push**:  $\min\{f(u), c_f(e)\} = c_f(e)$   
the arc  $e$  is deleted from the residual graph

Note that a push-operation may be saturating **and** deactivating at the same time.

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- ▶  **saturating push** :  $\min\{f(u), c_f(e)\} = c_f(e)$   
the arc  $e$  is deleted from the residual graph
- ▶  **deactivating push** :  $\min\{f(u), c_f(e)\} = f(u)$   
the node  $u$  becomes inactive

Note that a push-operation may be saturating **and** deactivating at the same time.

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Increasing the label of  $u$  by 1 results in a valid labelling.

- ▶ Edges  $(w, u)$  incoming to  $u$  still fulfill their constraint  $\ell(w) \leq \ell(u) + 1$ .
- ▶ An outgoing edge  $(u, w)$  had  $\ell(u) < \ell(w) + 1$  before since it was not admissible. Now:  $\ell(u) \leq \ell(w) + 1$ .

# Push Relabel Algorithms

## Intuition:

We want to send flow downwards, since the source has a height/label of  $n$  and the target a height/label of  $0$ . If we see an active node  $u$  with an admissible arc we push the flow at  $u$  towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into  $u$  it should roughly mean that the level/height/label of  $u$  should rise. (If we consider the flow to be water then this would be natural.)

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

# Reminder

- ▶ In a **preflow** nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- ▶ Such a node is called **active**.
- ▶ A labelling is **valid** if for every edge  $(u, v)$  in the residual graph  $\ell(u) \leq \ell(v) + 1$ .
- ▶ An arc  $(u, v)$  in residual graph is **admissible** if  $\ell(u) = \ell(v) + 1$ .
- ▶ A **saturating push** along  $e$  pushes an amount of  $c(e)$  flow along the edge, thereby saturating the edge (and making it disappear from the residual graph).
- ▶ A **deactivating push** along  $e = (u, v)$  pushes a flow of  $f(u)$ , where  $f(u)$  is the **excess flow** of  $u$ . This makes  $u$  inactive.

# Push Relabel Algorithms

**Algorithm 1**  $\text{maxflow}(G, s, t, c)$

```
1: find initial preflow  $f$ 
2: while there is active node  $u$  do
3:     if there is admiss. arc  $e$  out of  $u$  then
4:          $\text{push}(G, e, f, c)$ 
5:     else
6:          $\text{relabel}(u)$ 
7: return  $f$ 
```

# Push Relabel Algorithms

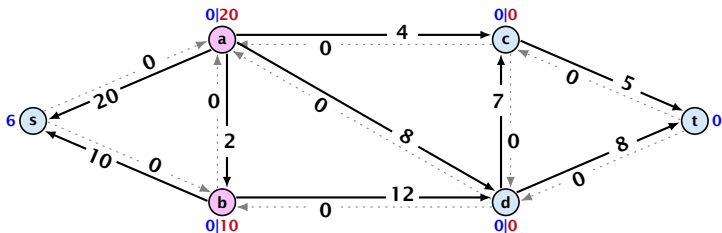
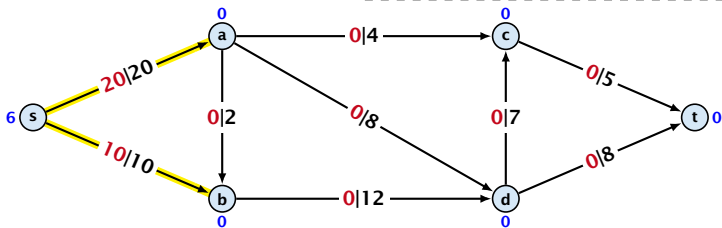
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In the following example we always stick to the same active node  $u$  until it becomes inactive but this is not required.

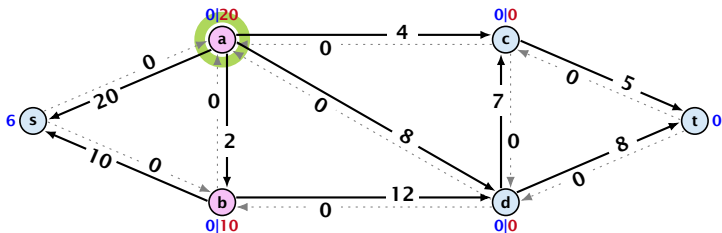
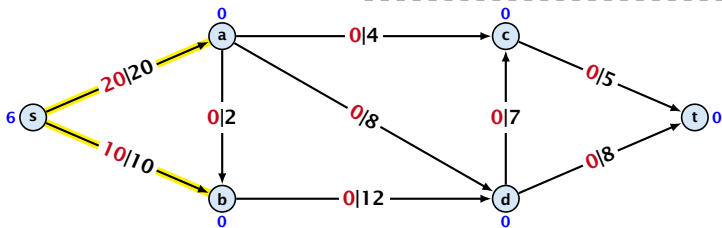
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The yellow edges indicate the cut that is introduced by the smallest missing label.



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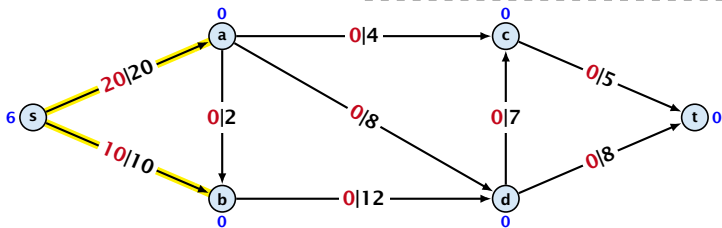
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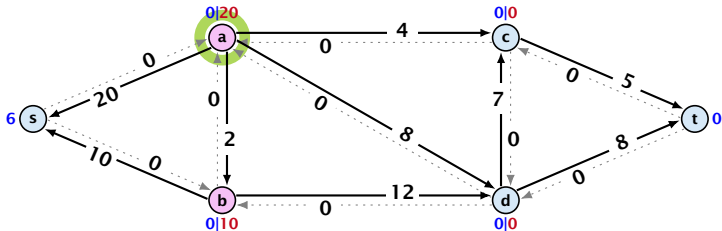


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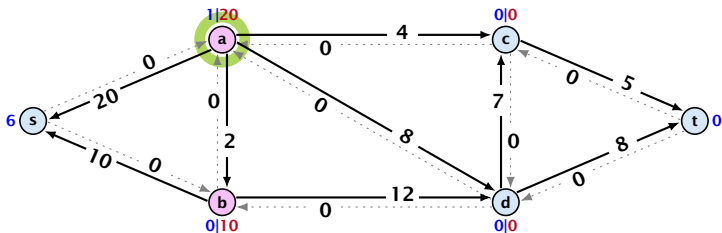
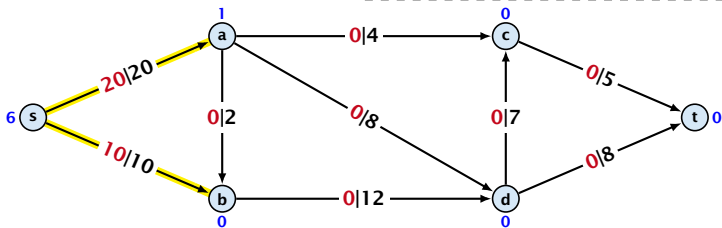


relabel to 1



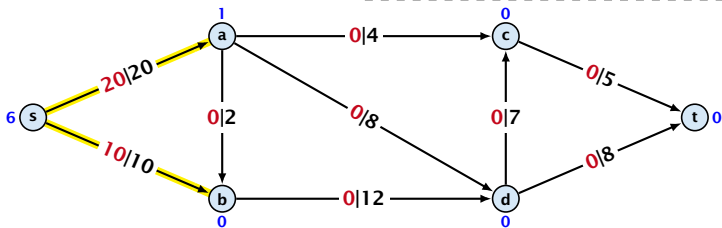
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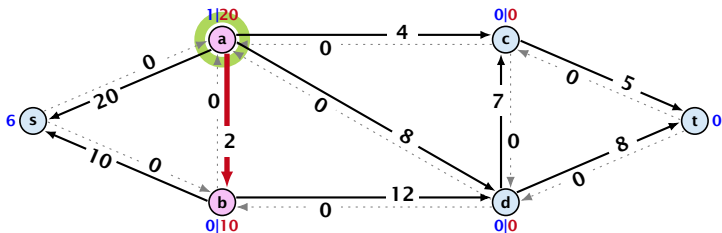


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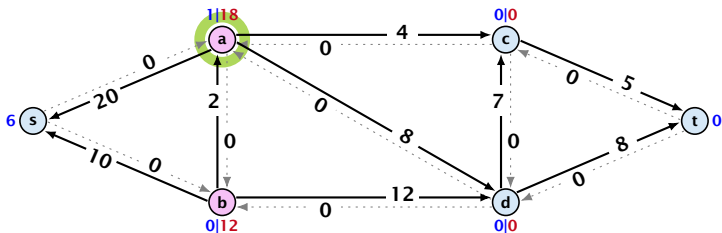
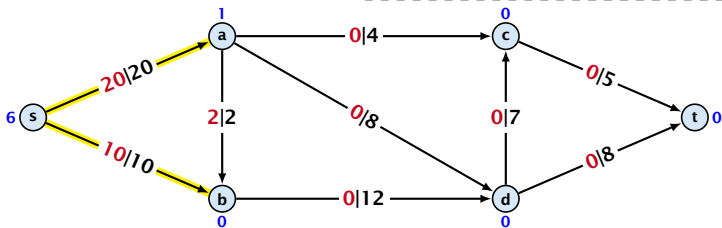


satürating push



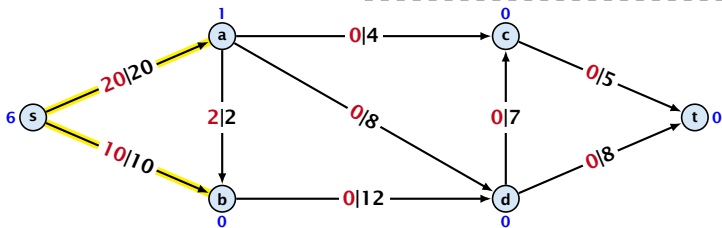
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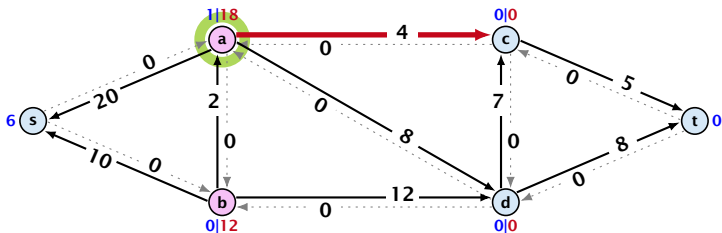


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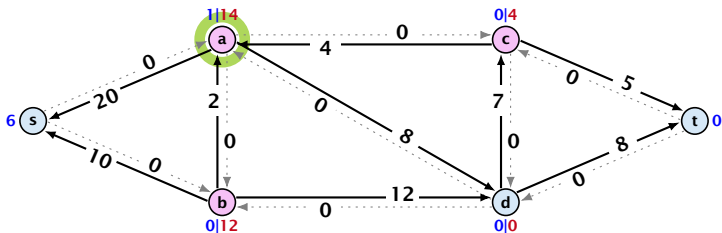
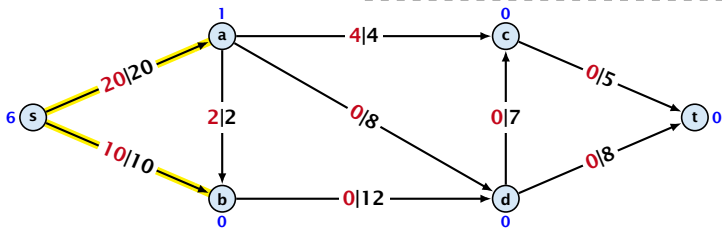


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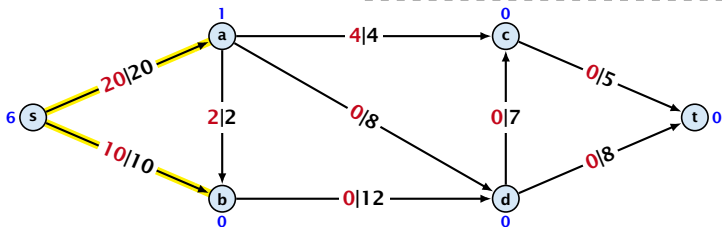
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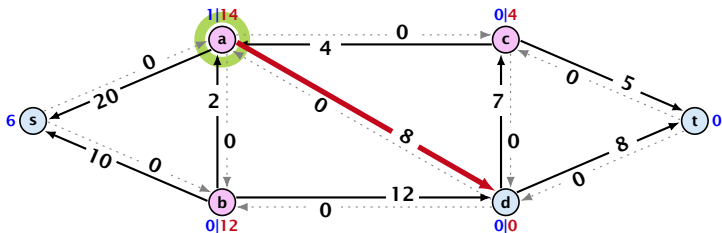


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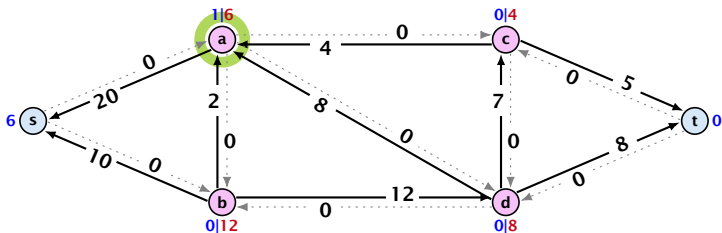
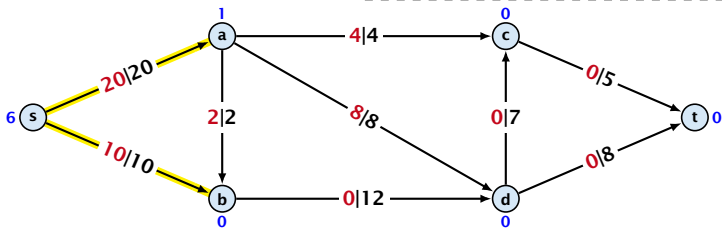


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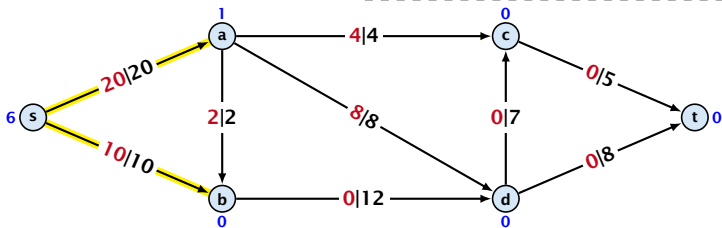
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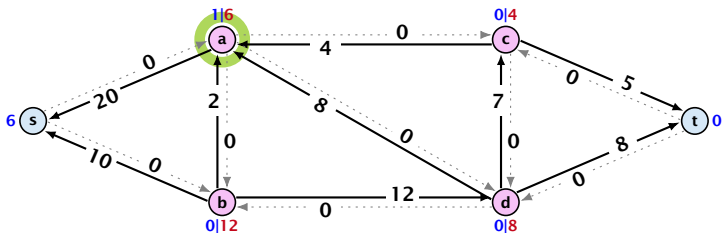


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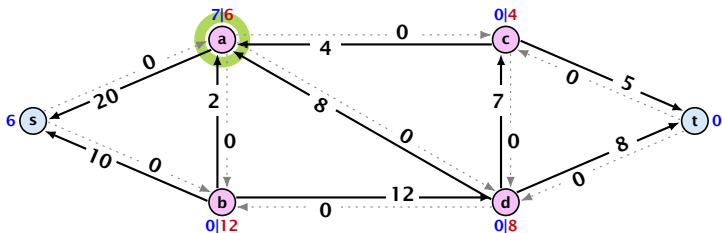
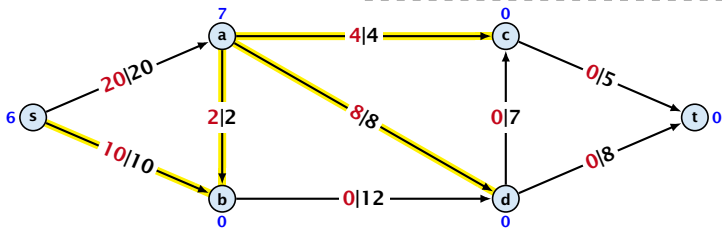


relabel to 7



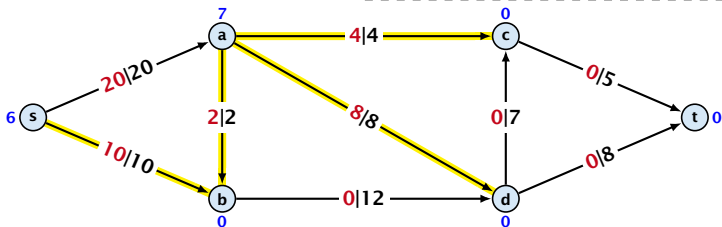
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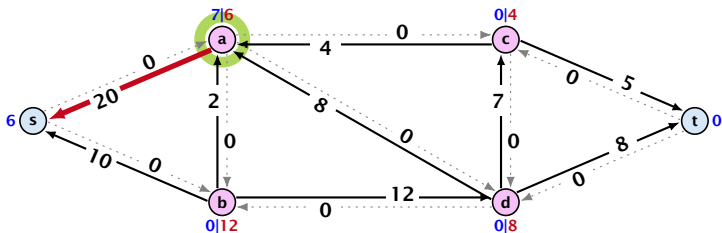


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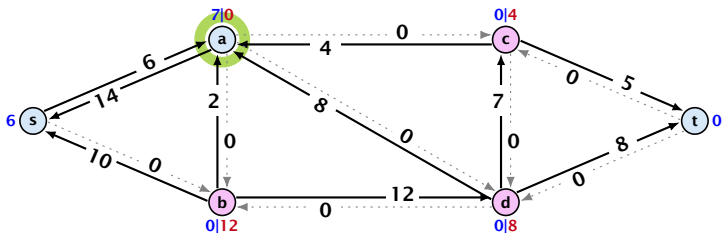
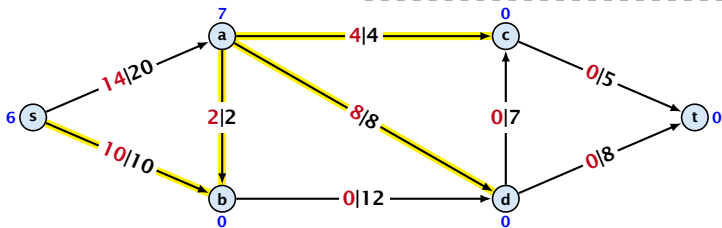


deactivating push



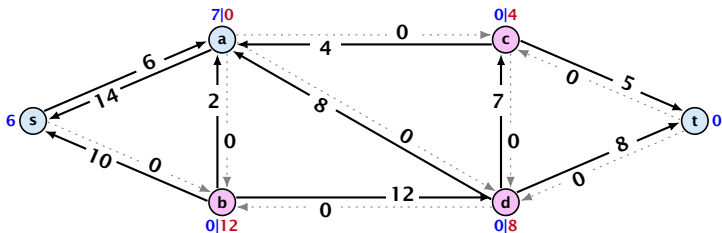
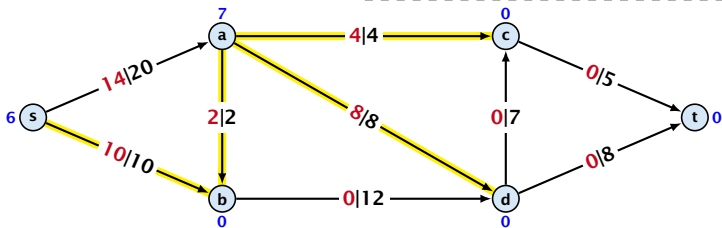
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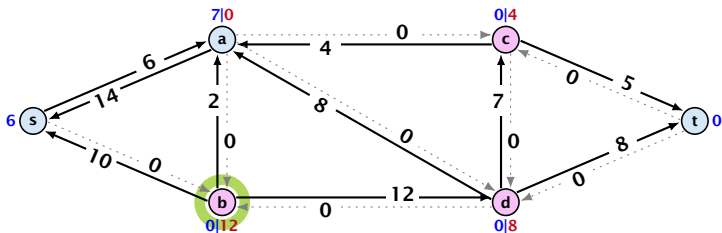
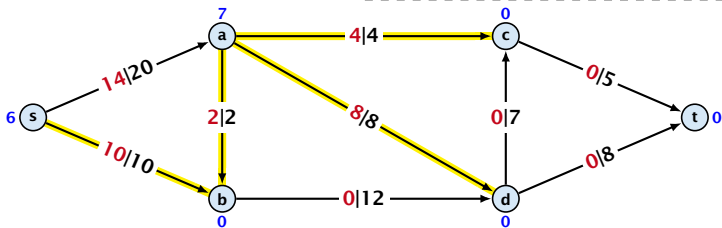
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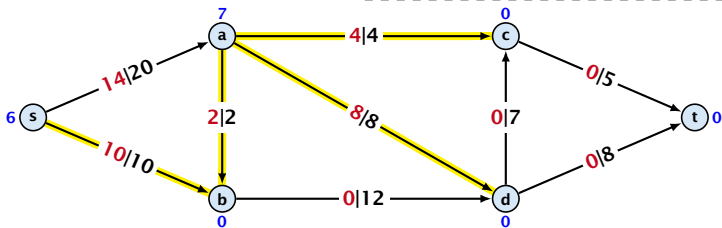
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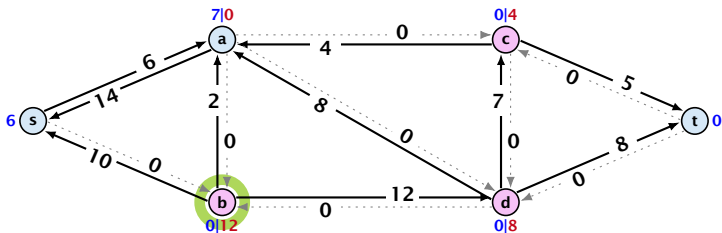


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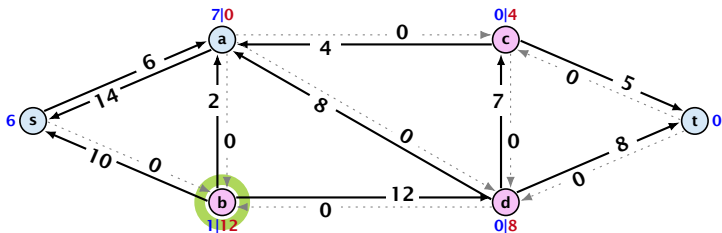
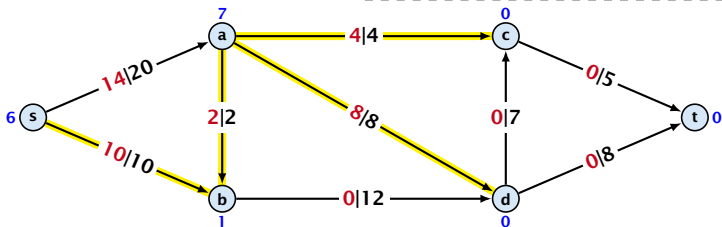


relabel to 1



# Preflow Push

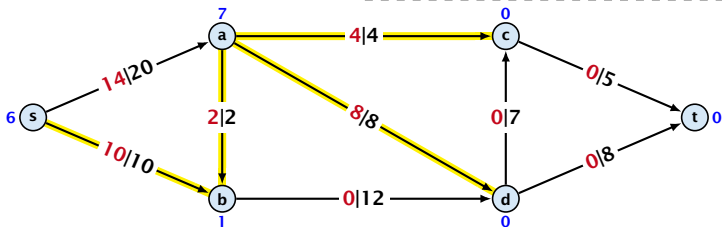
The yellow edges indicate the cut that is introduced by the smallest missing label.



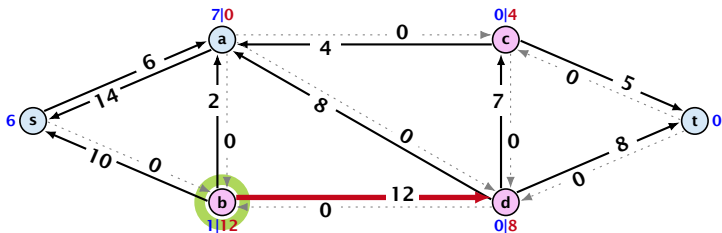


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

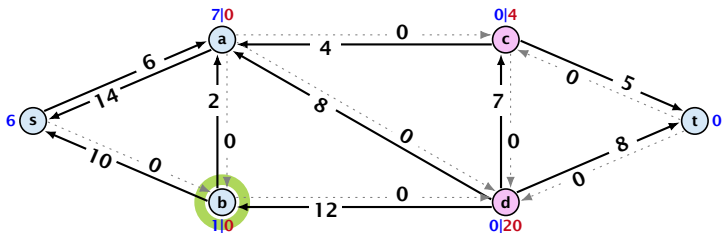
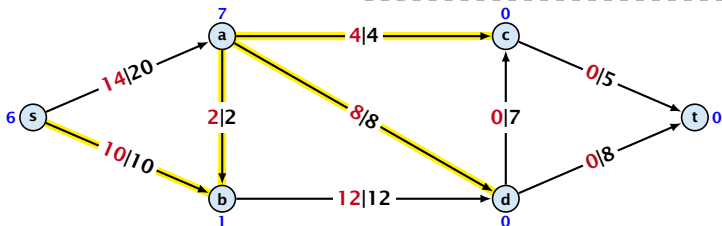


satürating and deactivating push



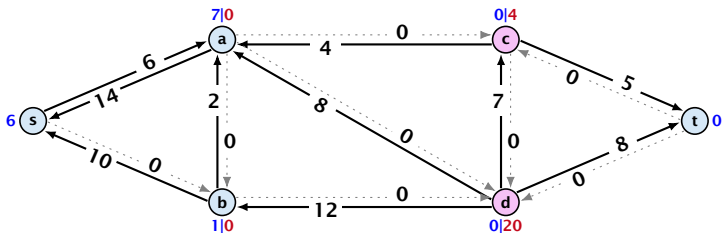
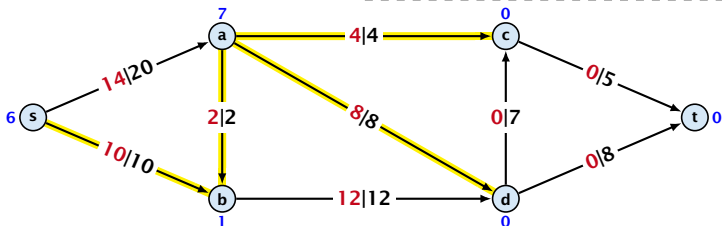
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



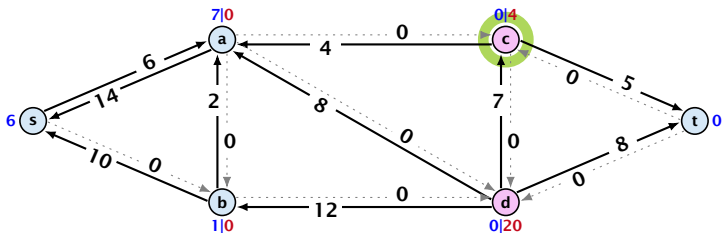
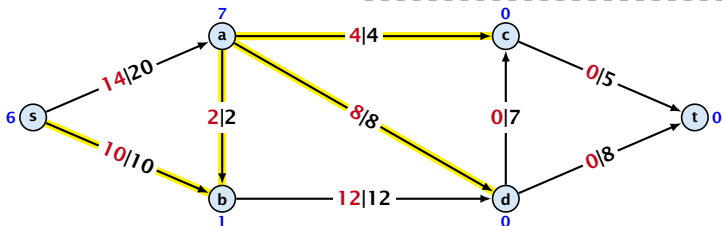
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



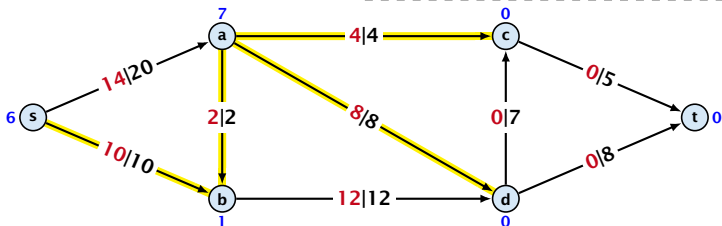
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

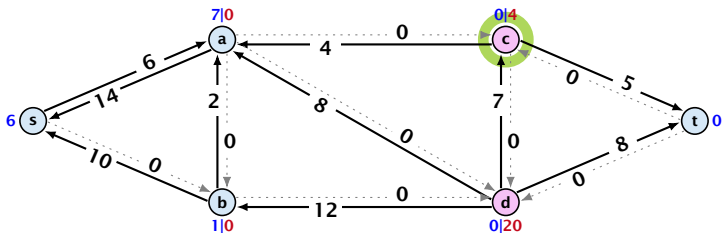


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

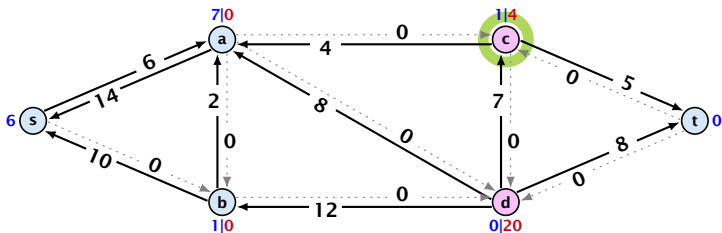
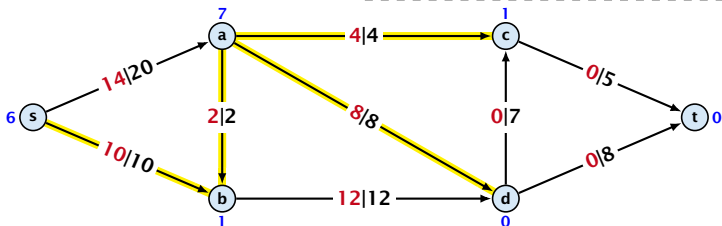


relabel to 1



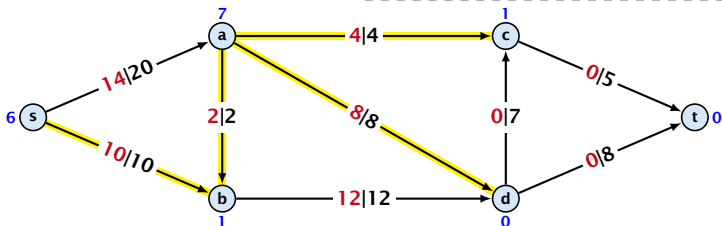
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

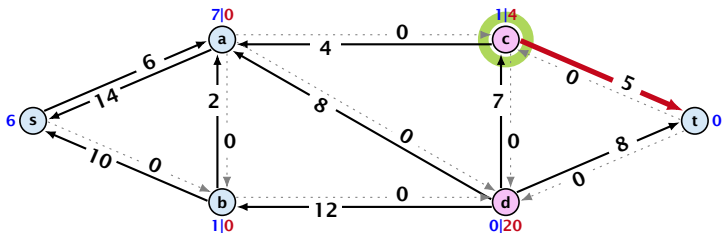


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

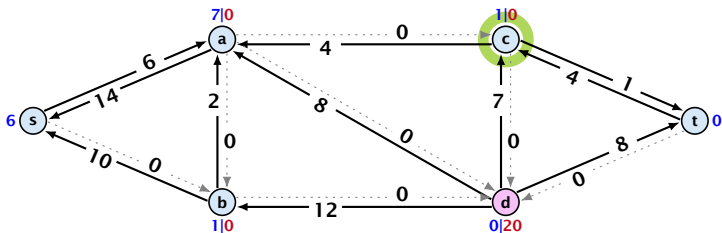
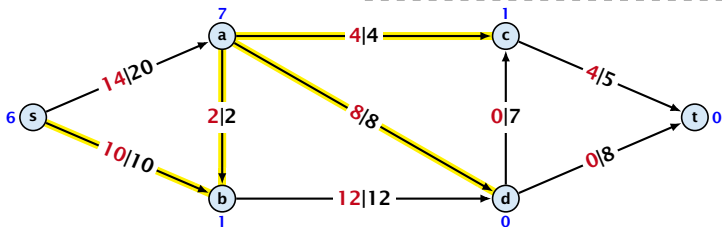


deactivating push



# Preflow Push

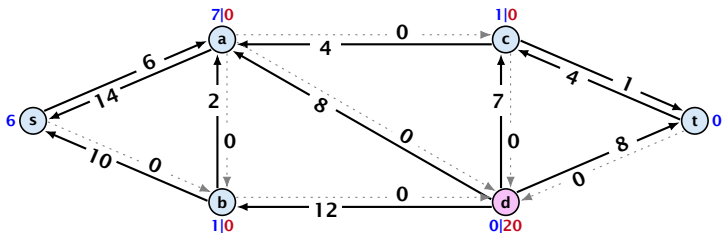
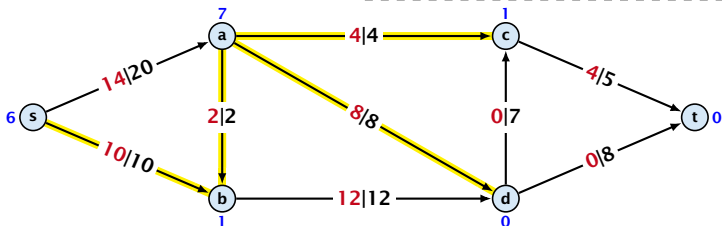
The yellow edges indicate the cut that is introduced by the smallest missing label.





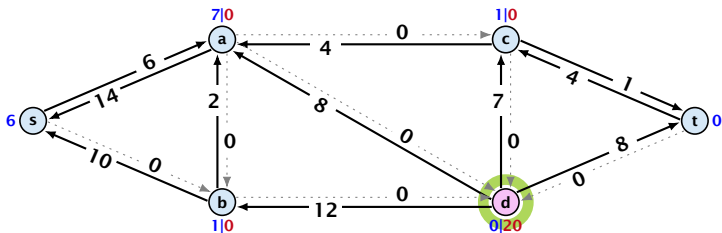
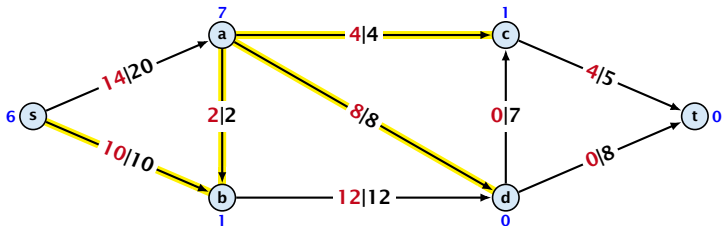
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



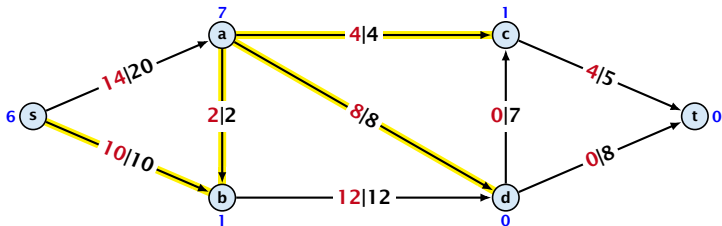
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

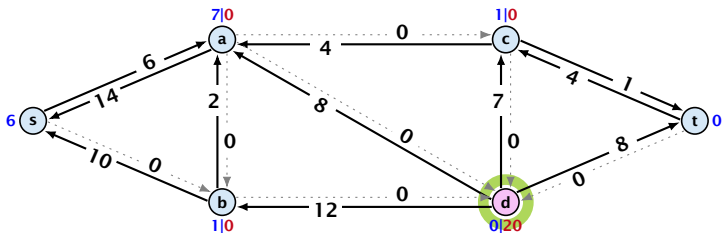


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

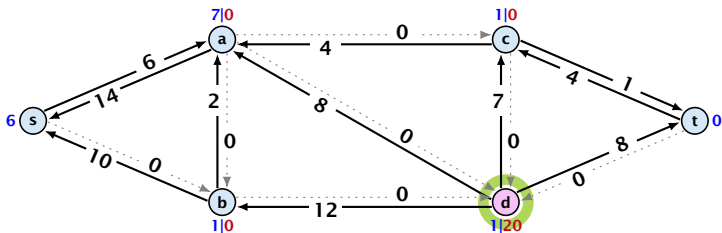
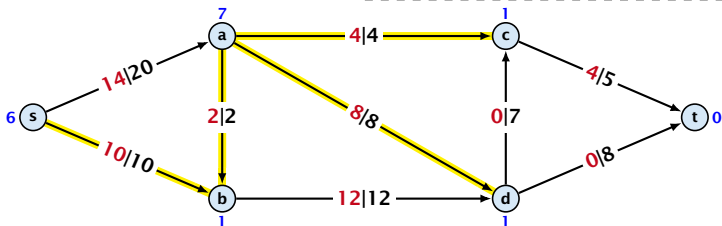


relabel to 1



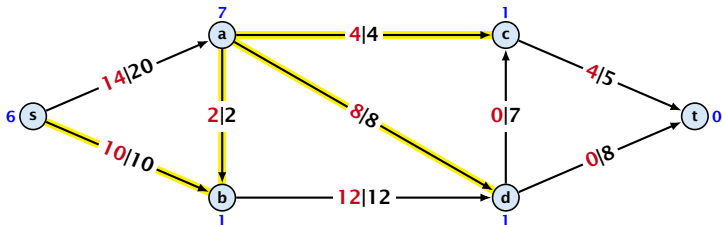
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

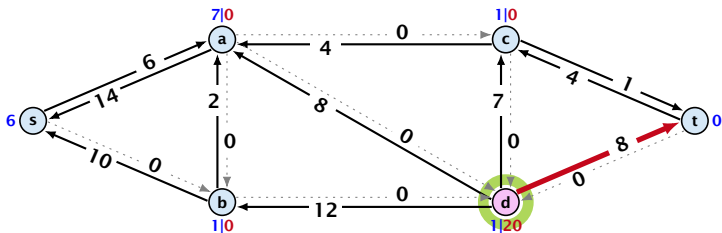


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

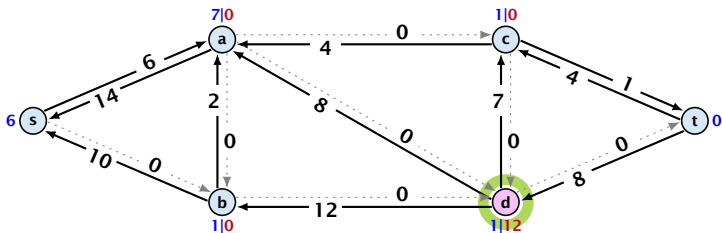
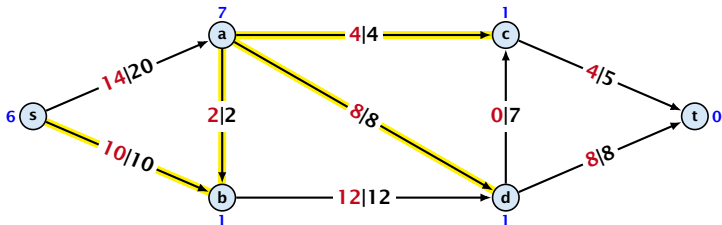


satürating push



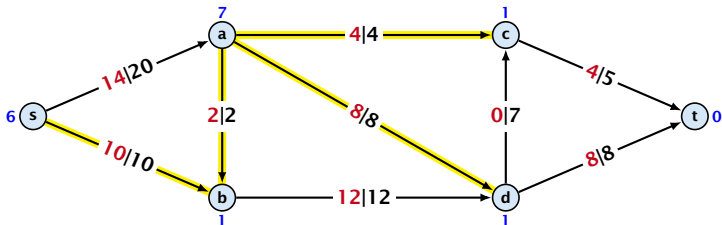
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

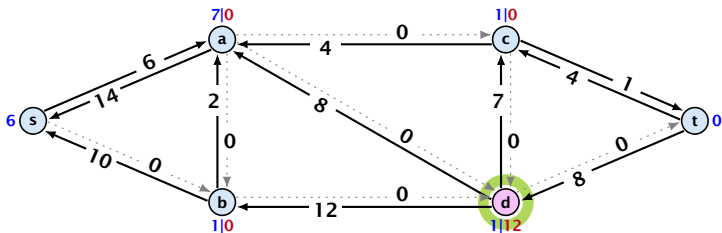


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

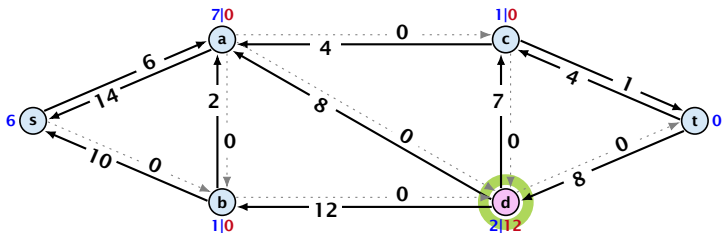
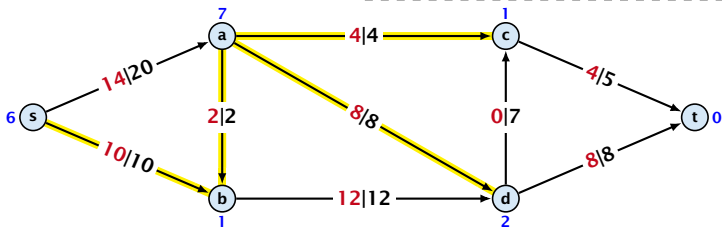


relabel to 2



# Preflow Push

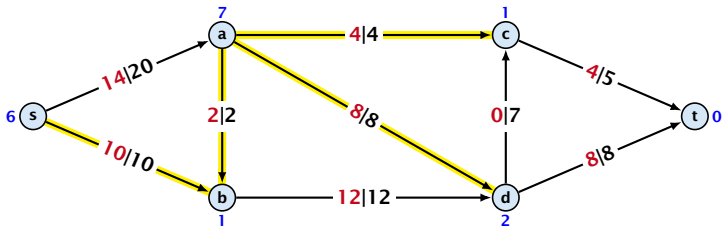
The yellow edges indicate the cut that is introduced by the smallest missing label.



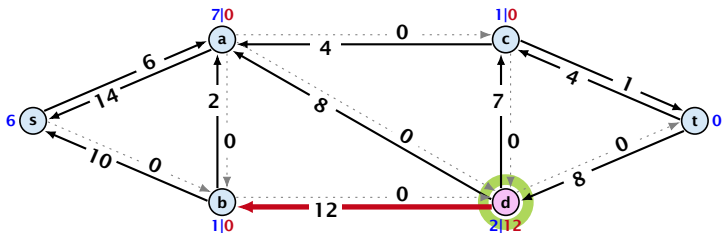


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

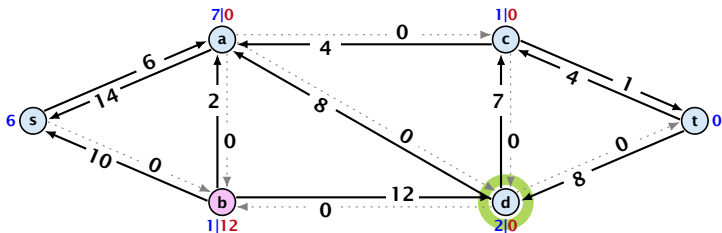
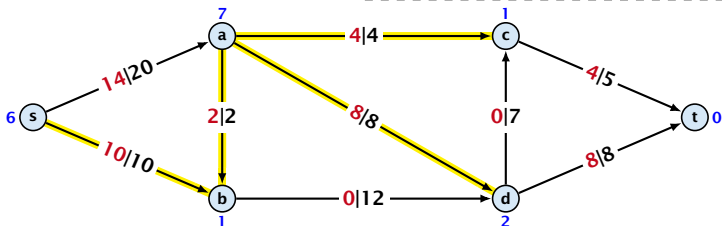


satürating and deactivating push



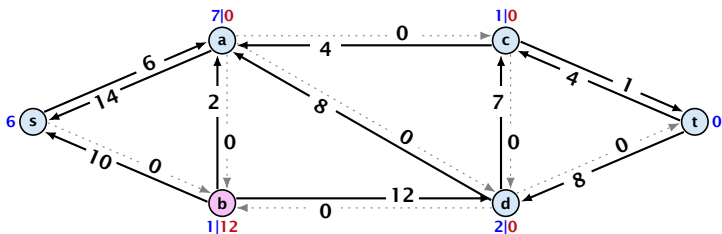
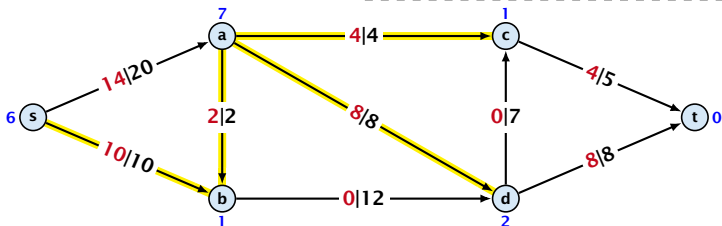
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



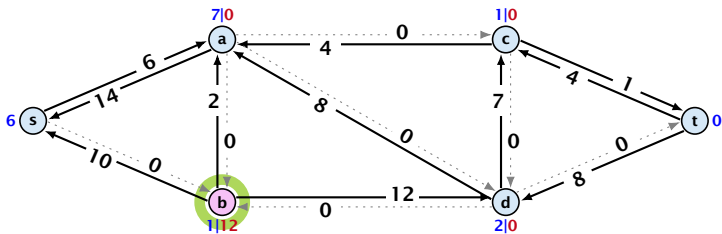
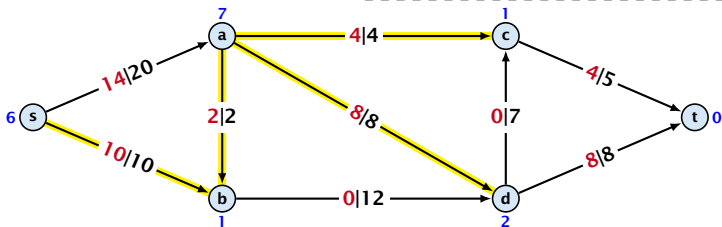
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



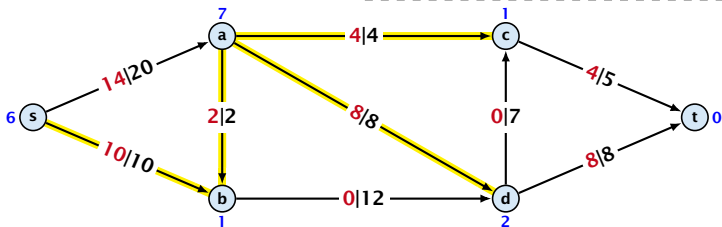
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

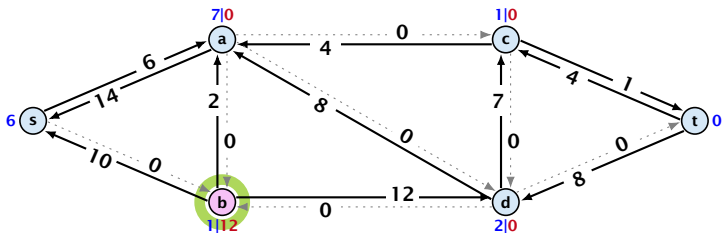


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

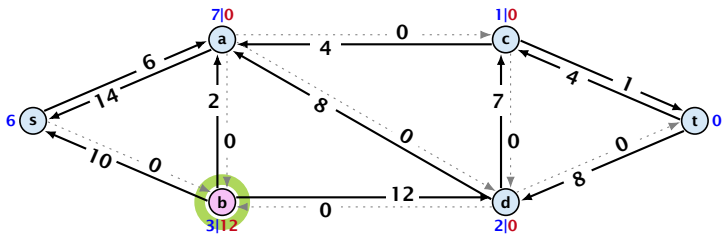
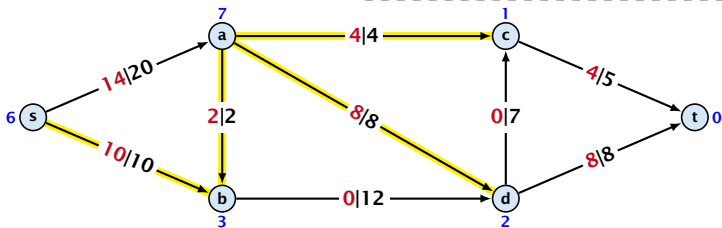


relabel to 3



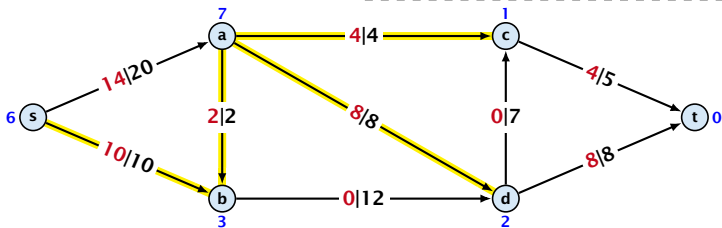
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

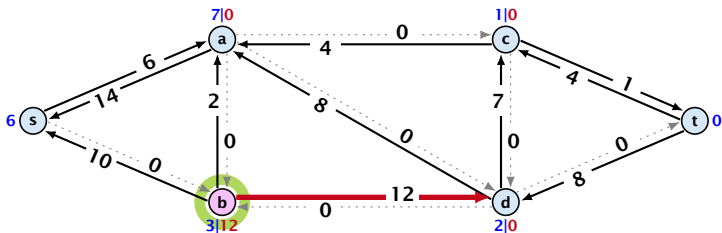


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

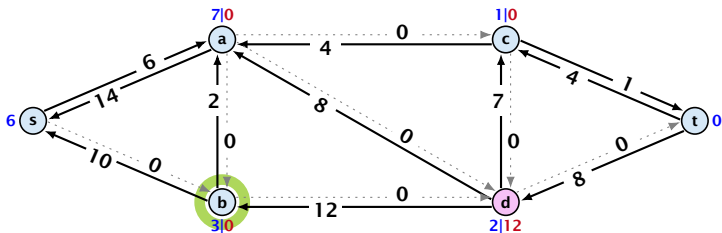
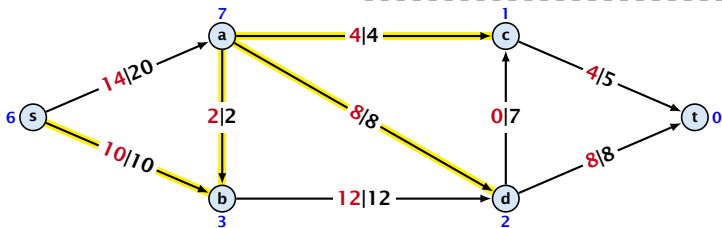


satürating and deactivating push



# Preflow Push

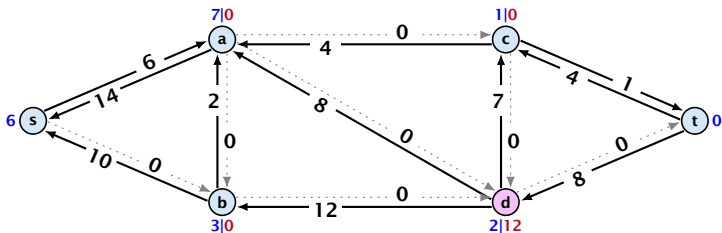
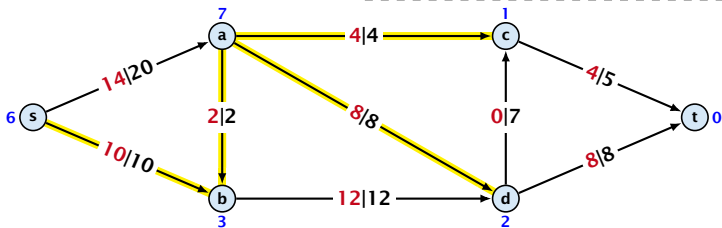
The yellow edges indicate the cut that is introduced by the smallest missing label.





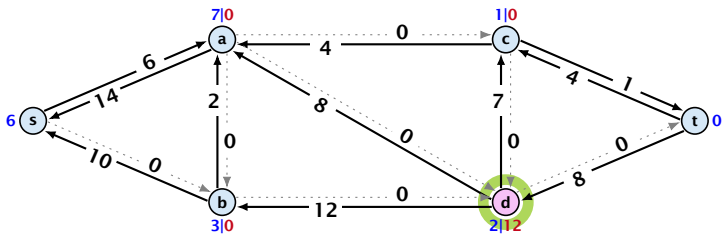
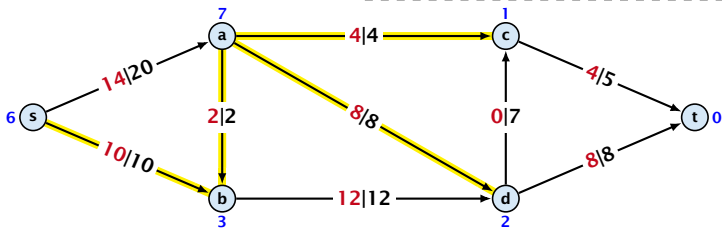
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



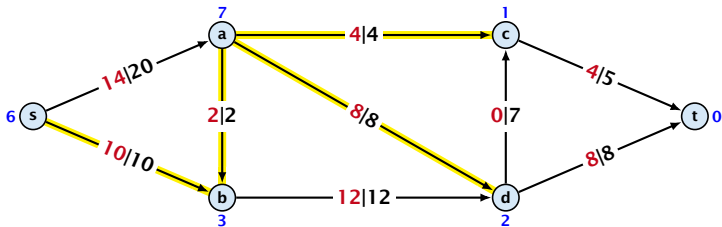
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

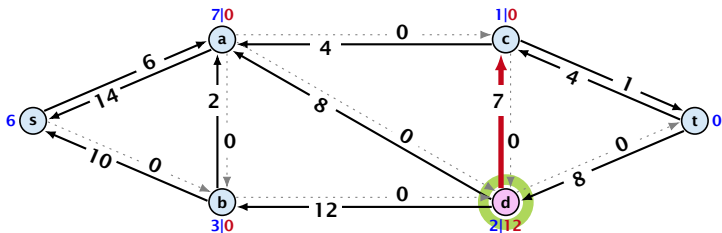


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

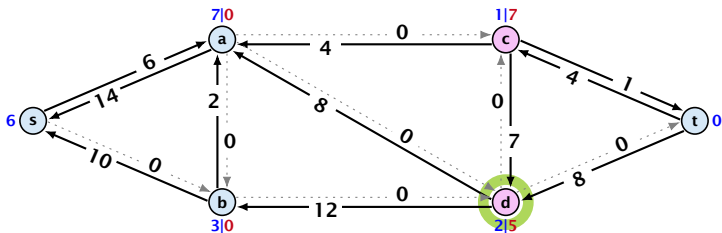
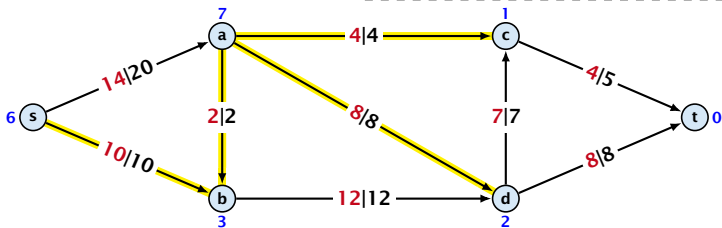


satürating push



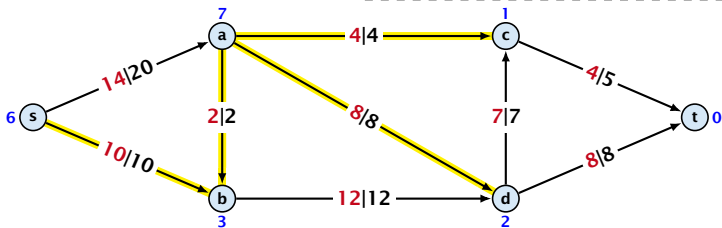
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

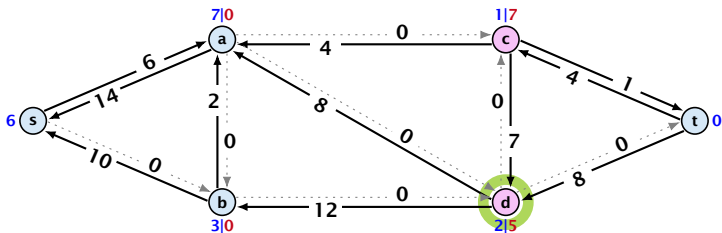


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

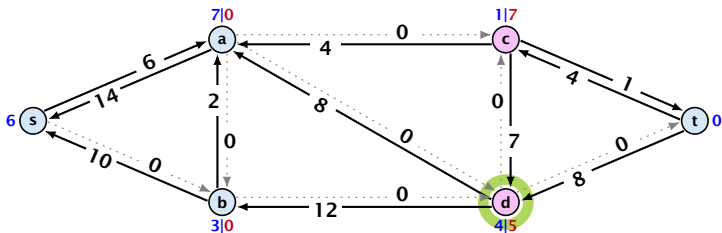
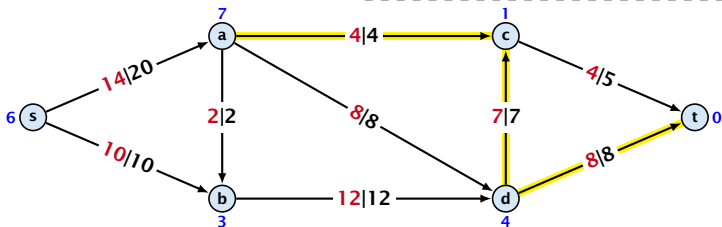


relabel to 4



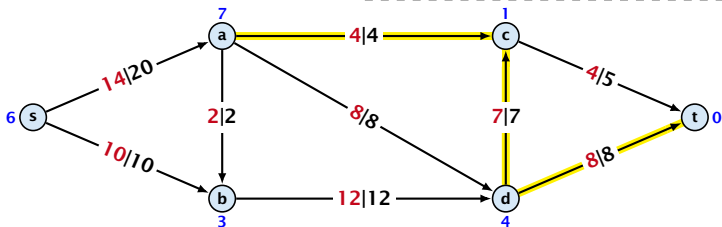
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

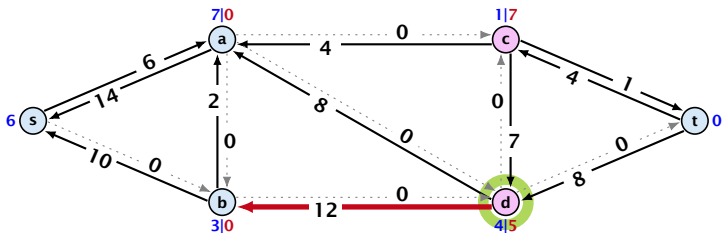


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

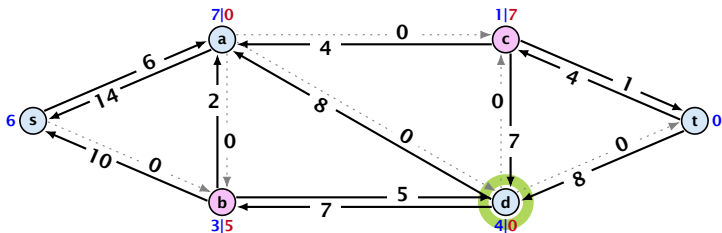
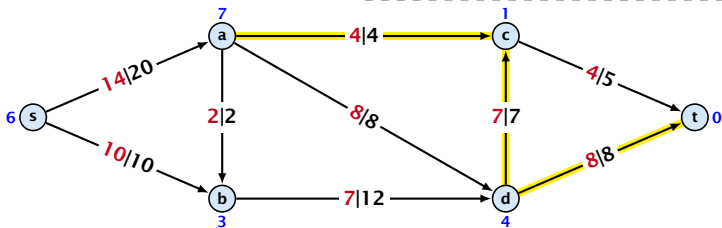


deactivating push



# Preflow Push

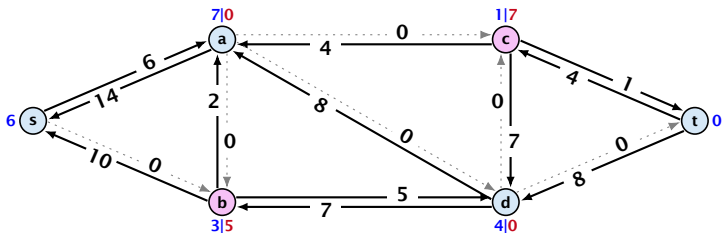
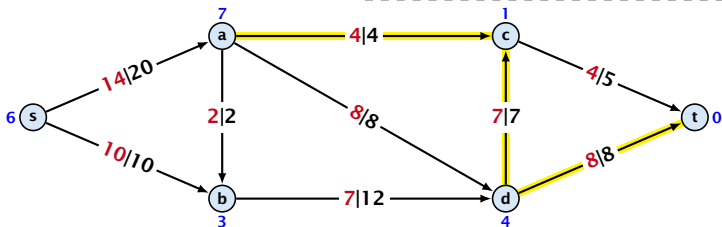
The yellow edges indicate the cut that is introduced by the smallest missing label.





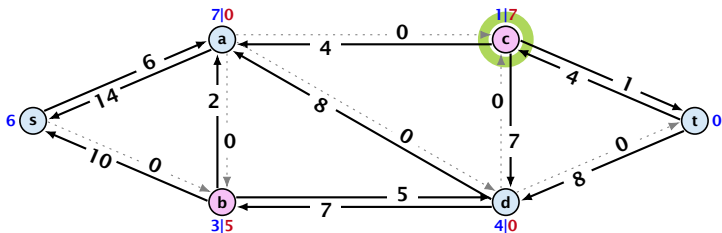
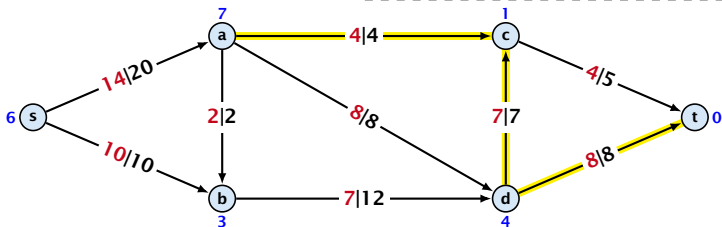
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



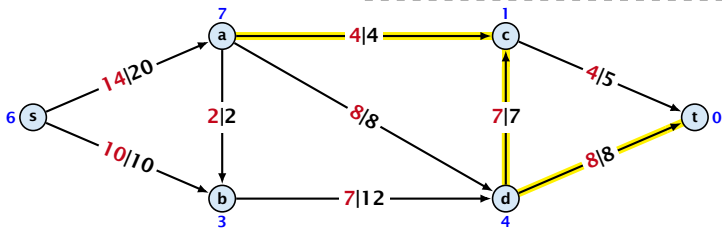
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

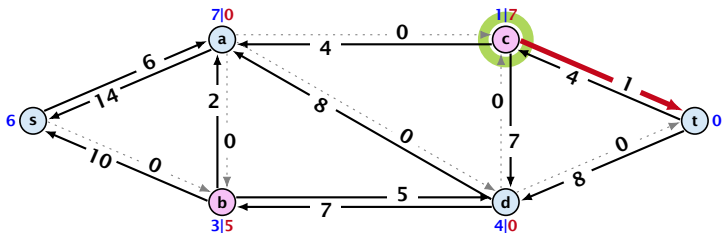


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

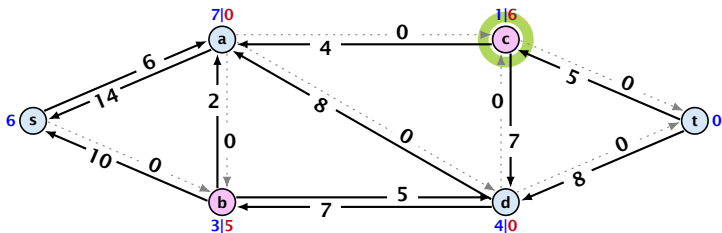
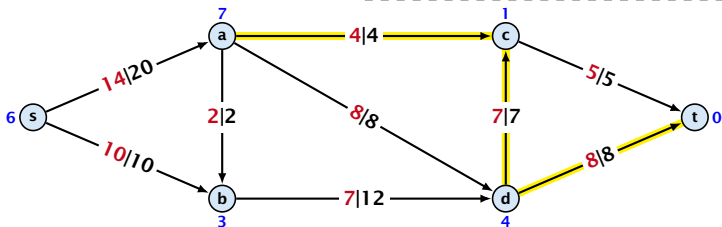


satürating push



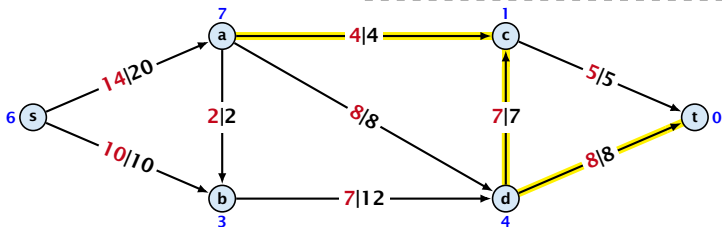
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

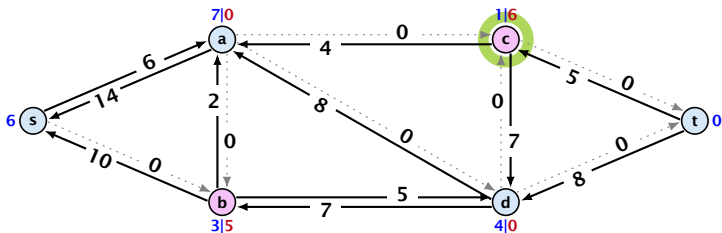


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

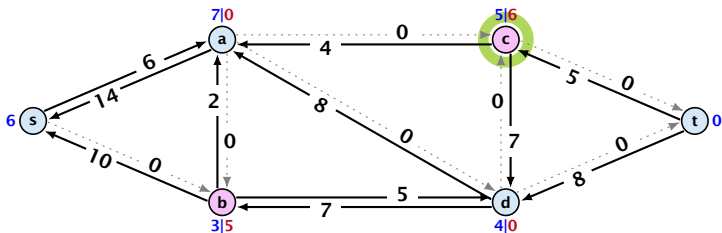
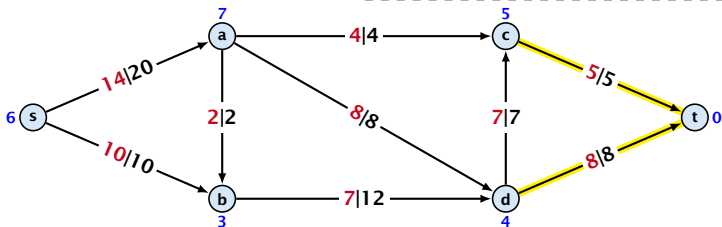


relabel to 5



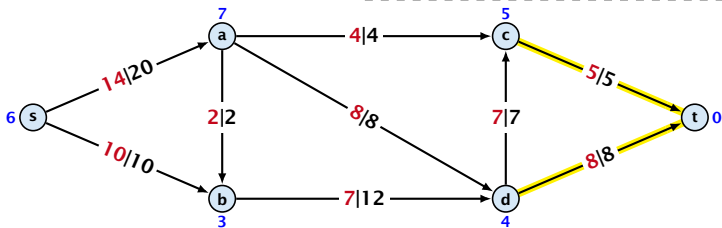
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

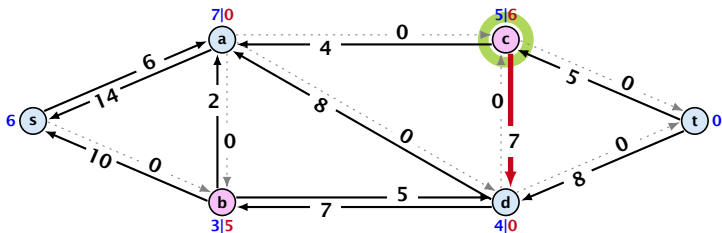


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

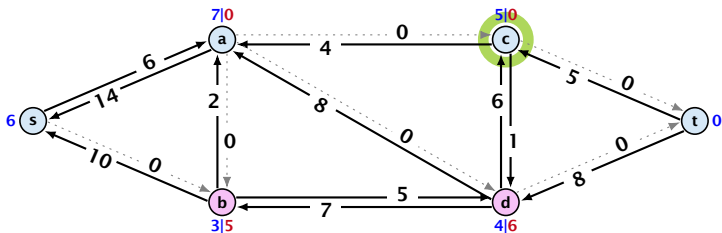
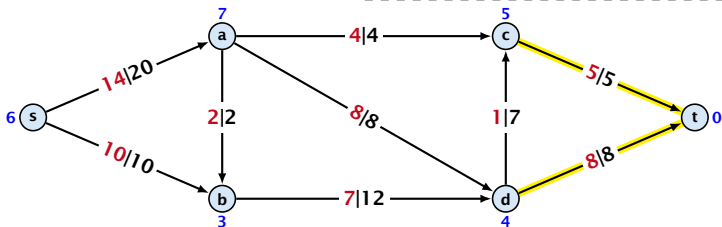


deactivating push



# Preflow Push

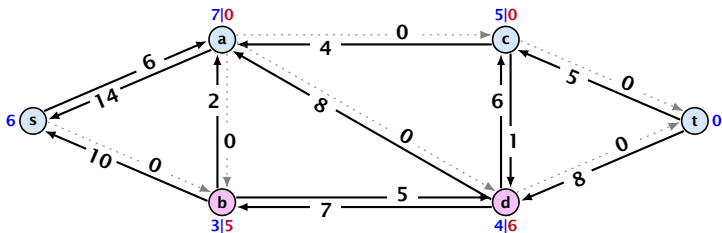
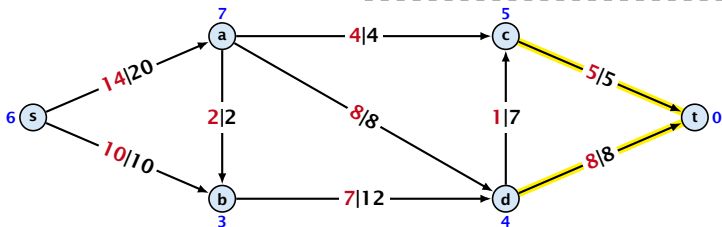
The yellow edges indicate the cut that is introduced by the smallest missing label.





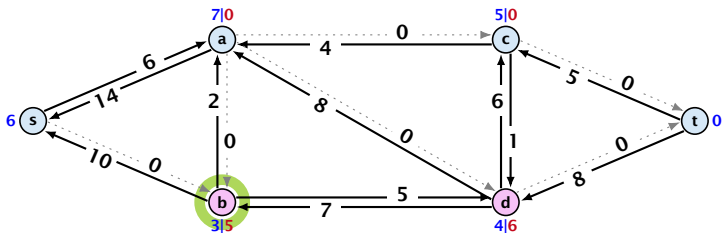
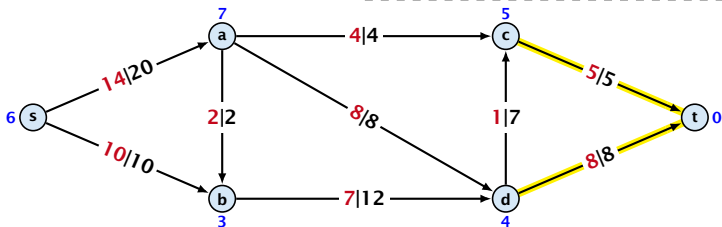
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



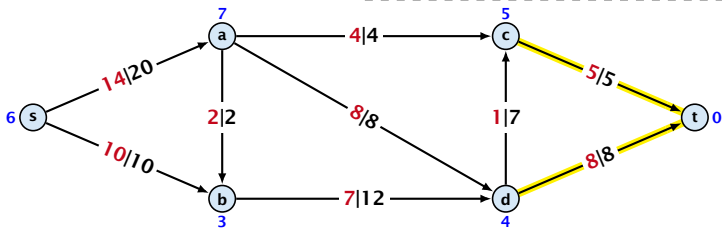
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

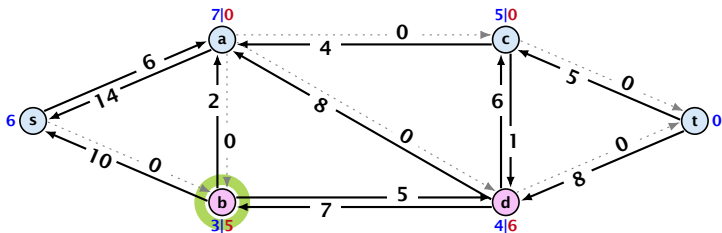


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

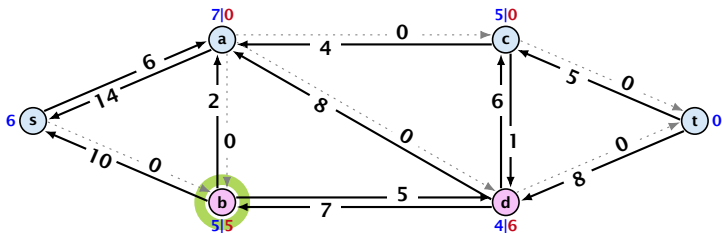
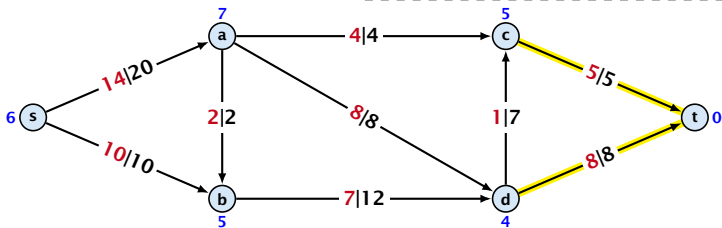


relabel to 5



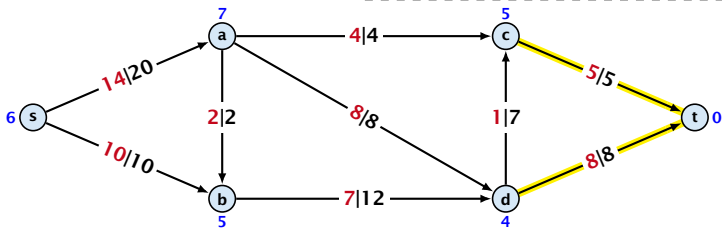
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

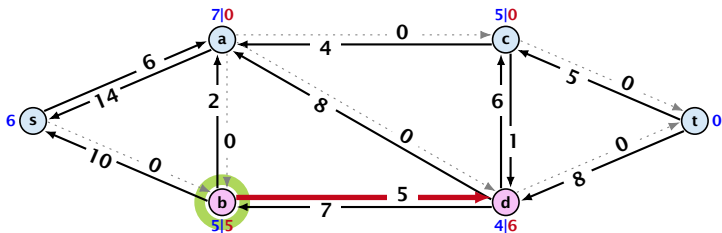


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

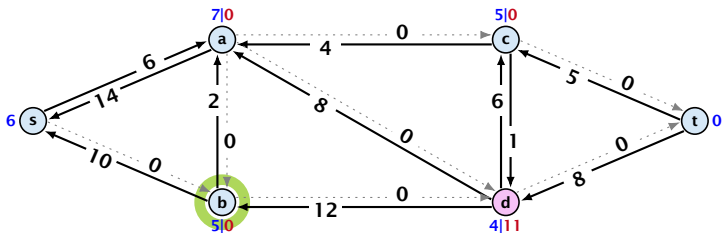
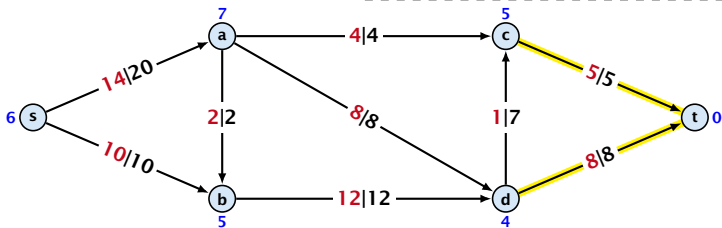


satürating and deactivating push



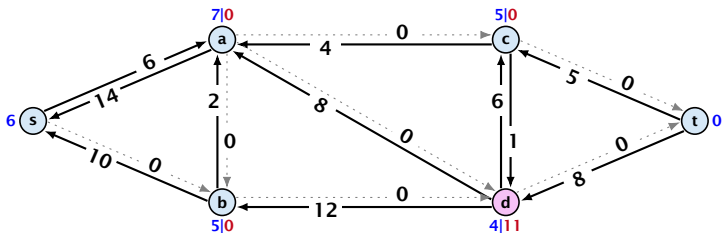
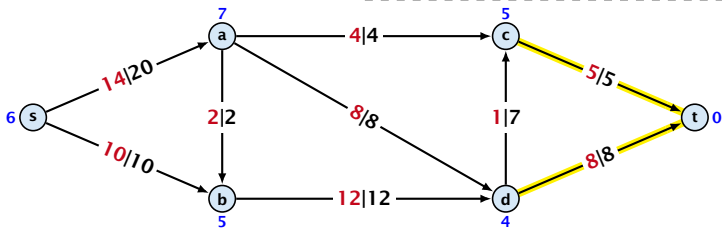
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



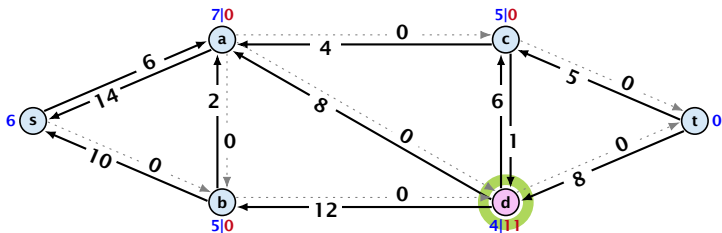
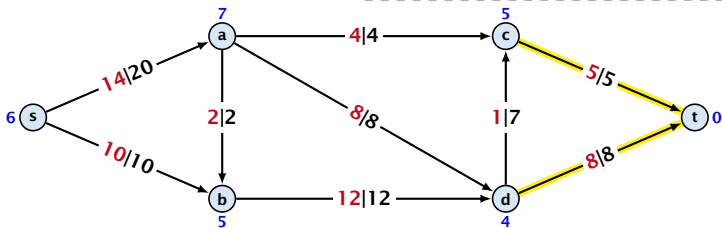
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



# Preflow Push

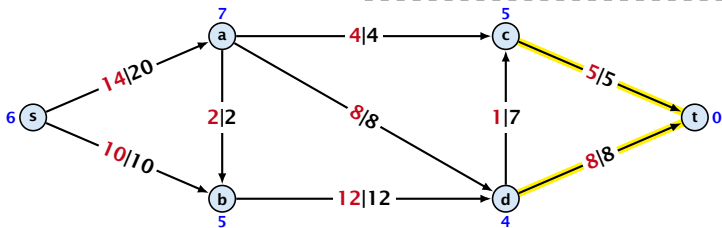
The yellow edges indicate the cut that is introduced by the smallest missing label.



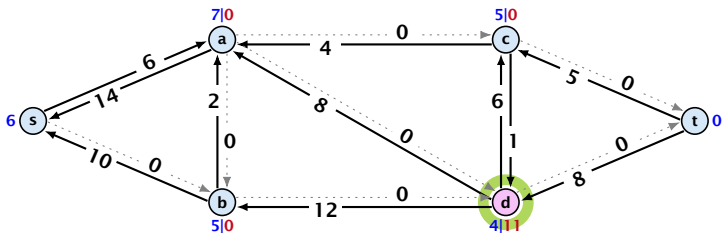


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

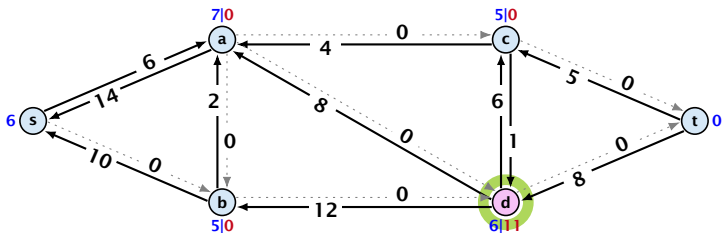
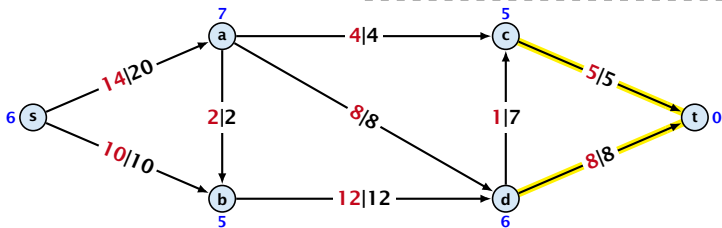


relabel to 6



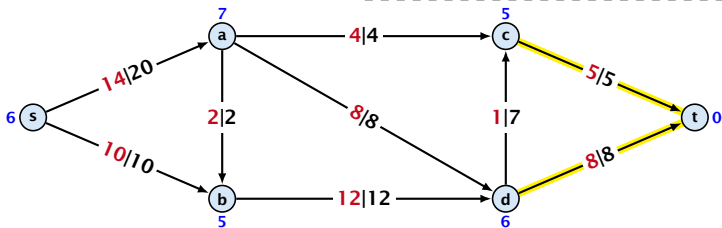
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

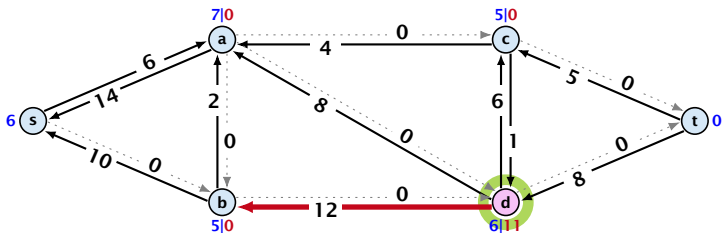


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

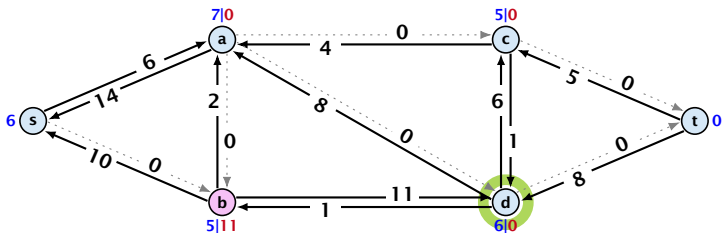
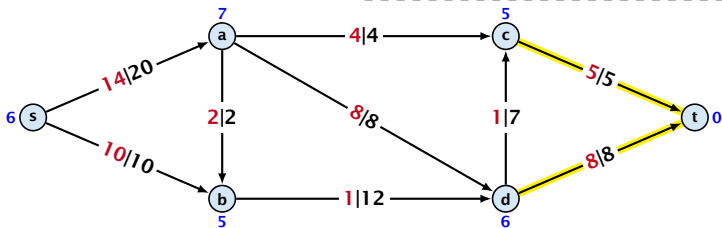


deactivating push



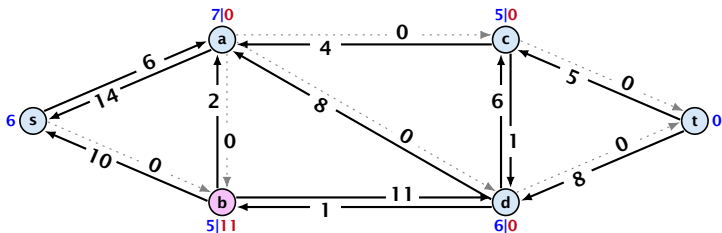
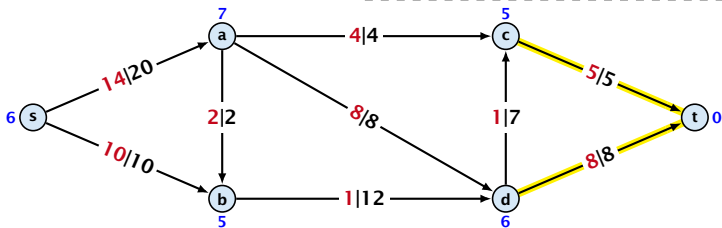
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



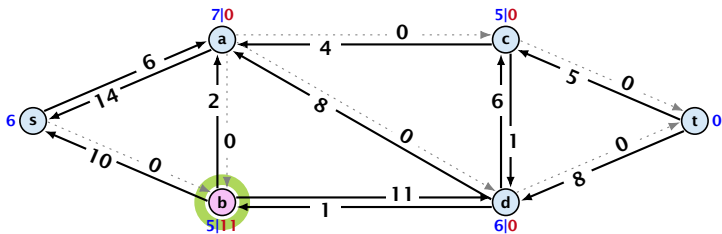
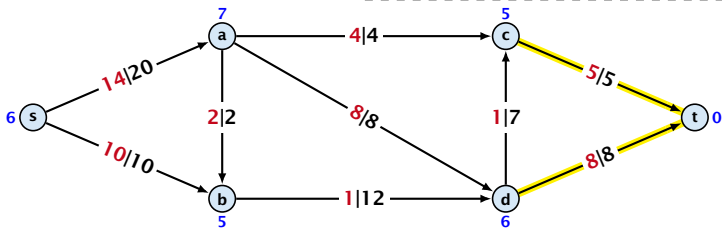
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



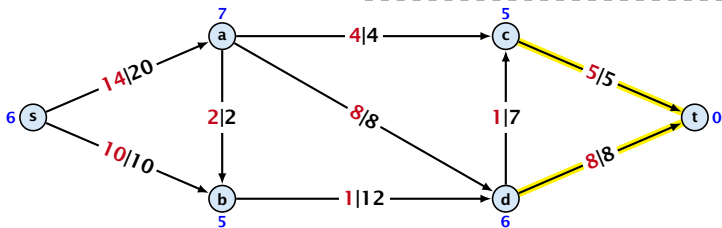
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

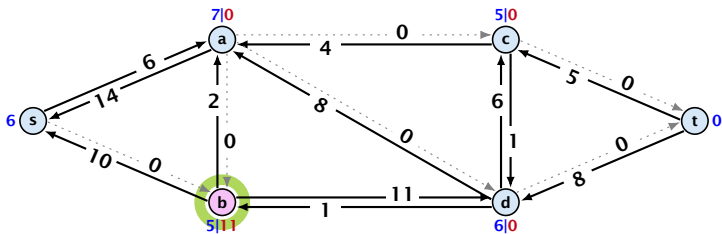


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

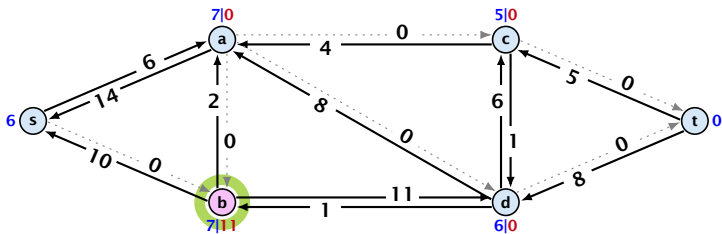
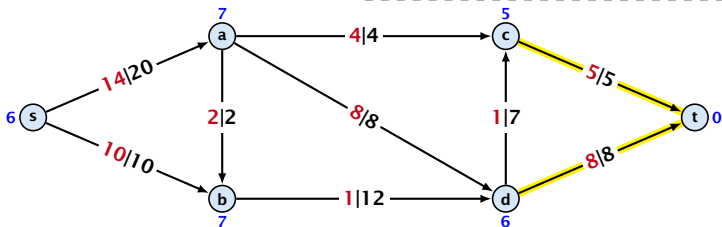


relabel to 7



# Preflow Push

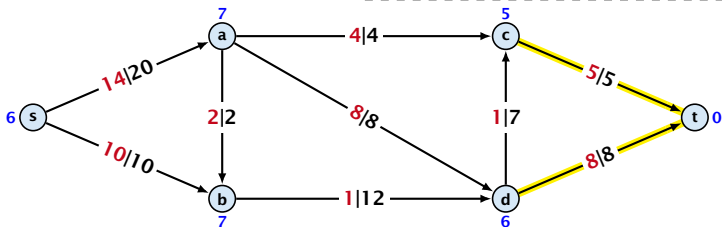
The yellow edges indicate the cut that is introduced by the smallest missing label.



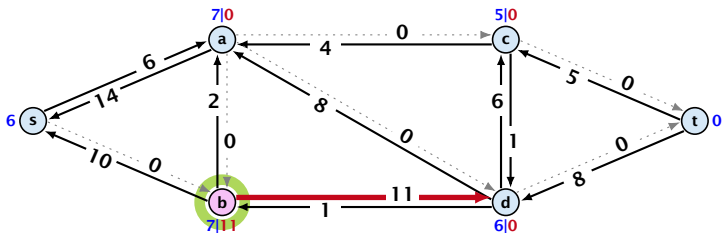


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

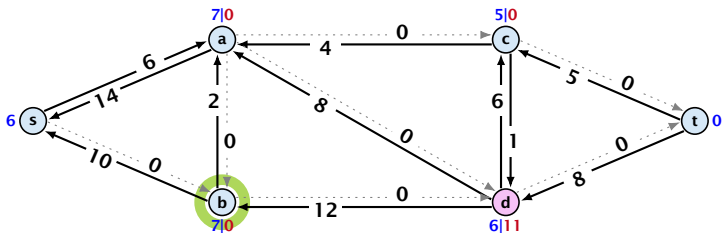
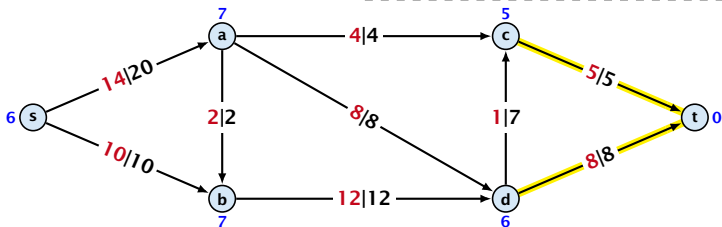


satürating and deactivating push



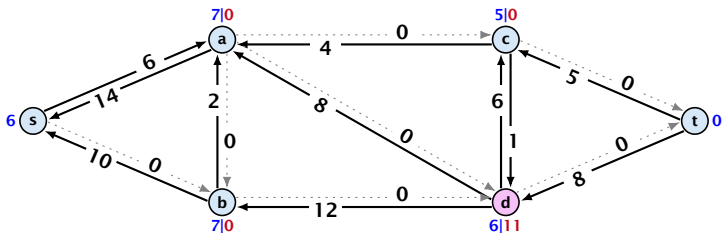
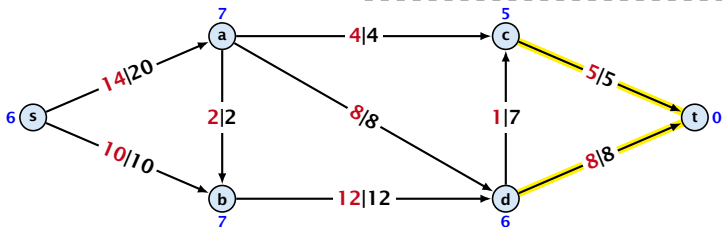
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



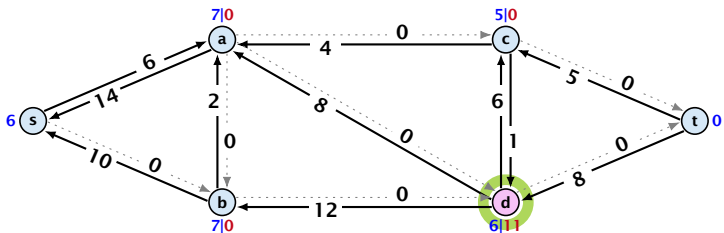
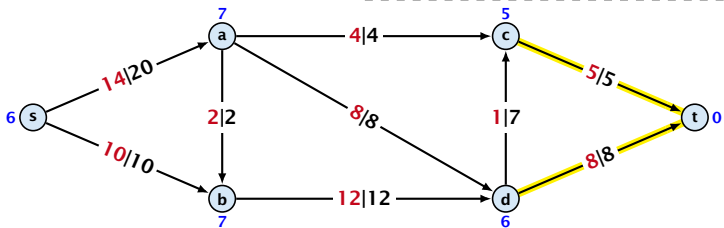
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



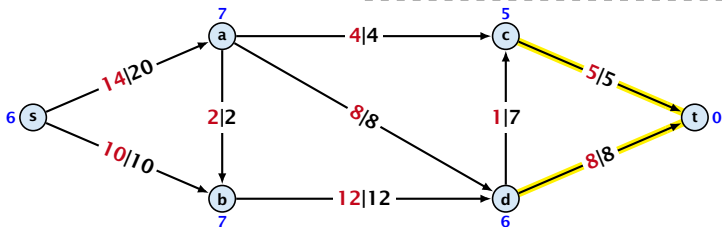
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

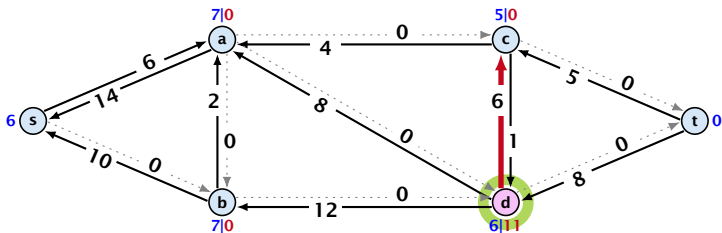


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

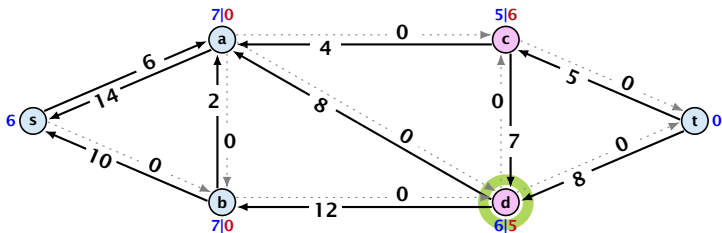
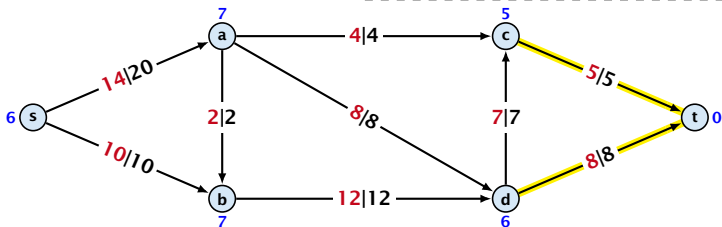


satürating push



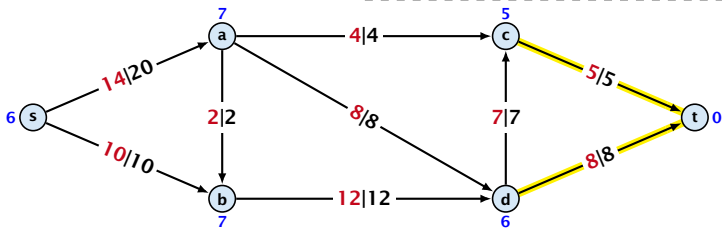
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

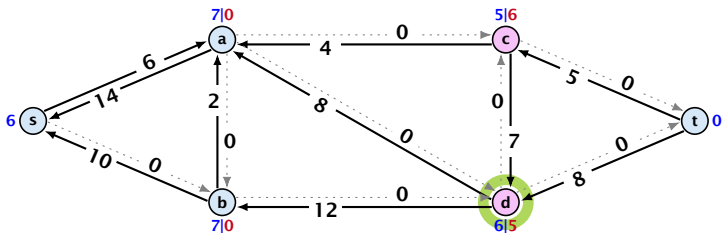


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

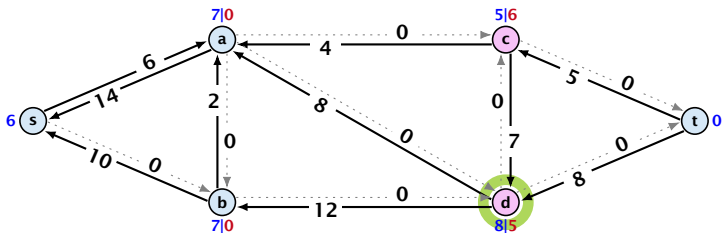
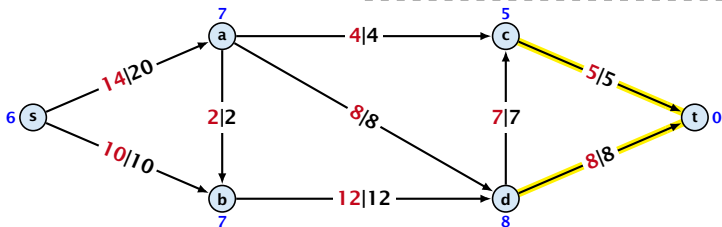


relabel to 8



# Preflow Push

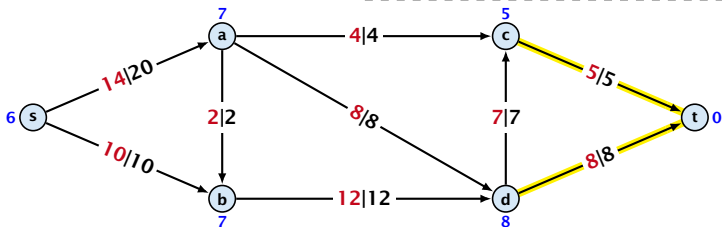
The yellow edges indicate the cut that is introduced by the smallest missing label.



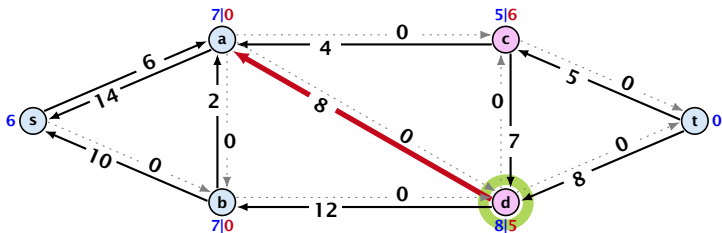


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

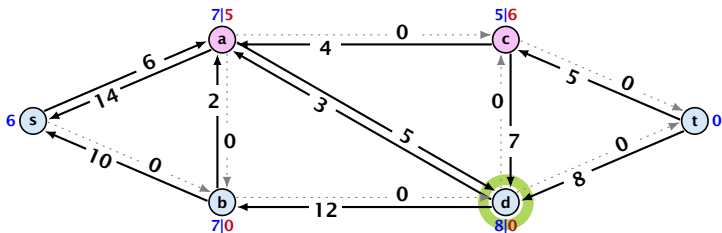
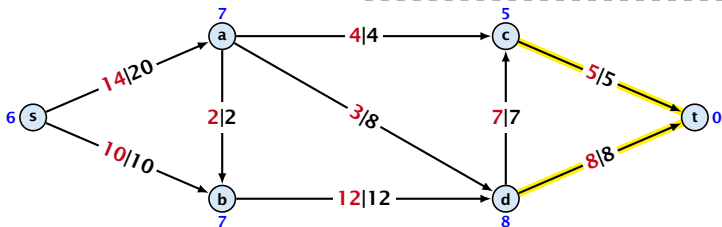


deactivating push



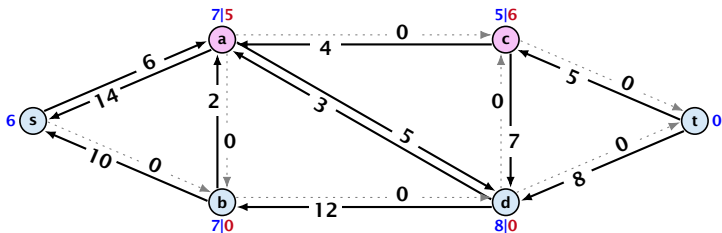
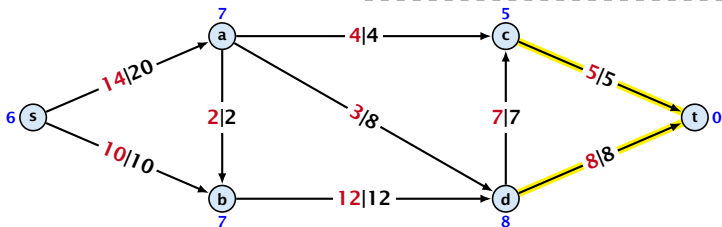
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



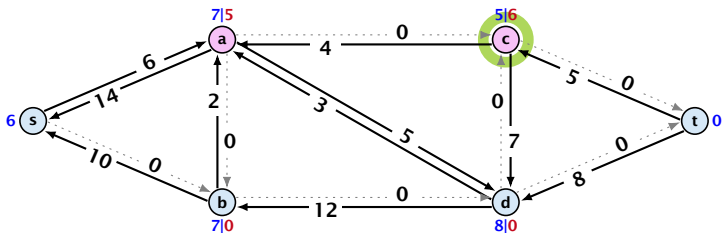
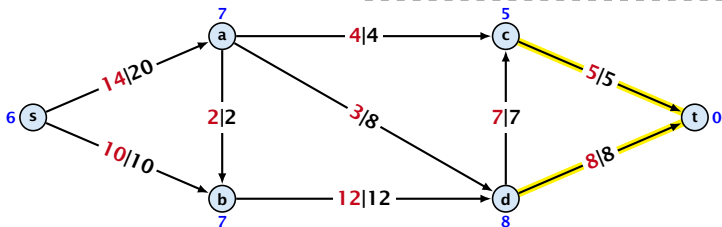
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



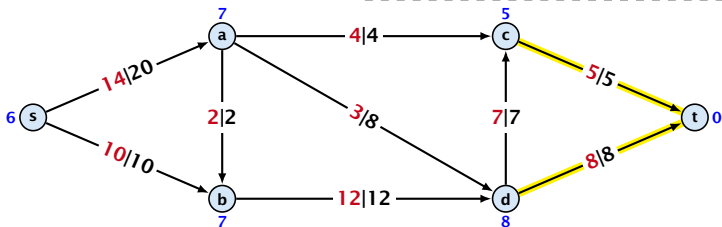
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

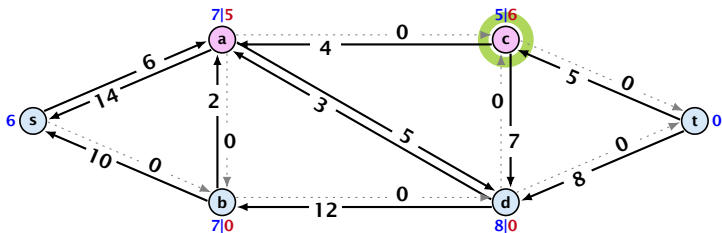


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

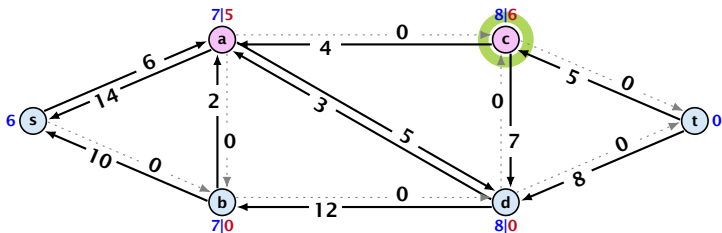
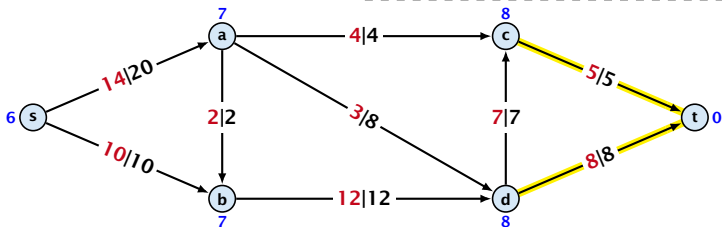


relabel to 8



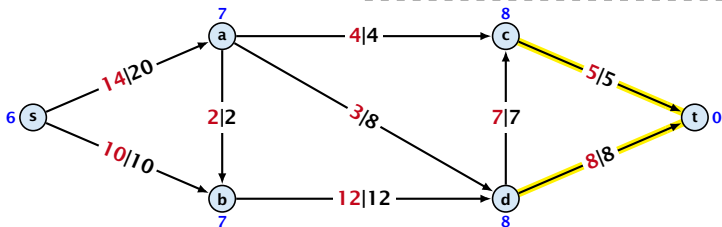
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

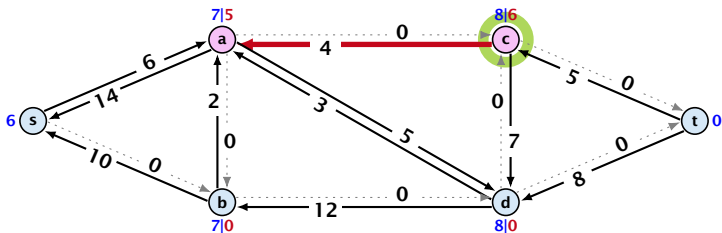


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

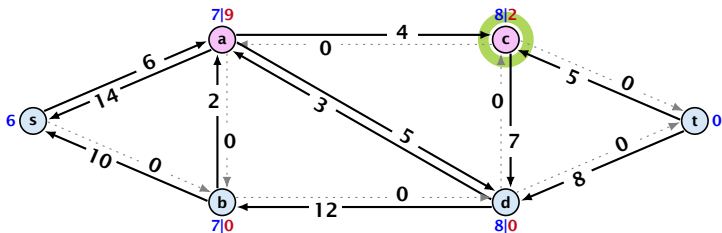
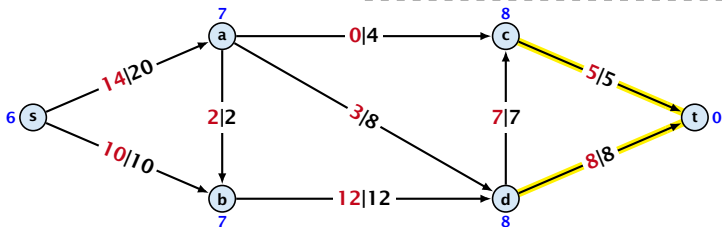


satürating push



# Preflow Push

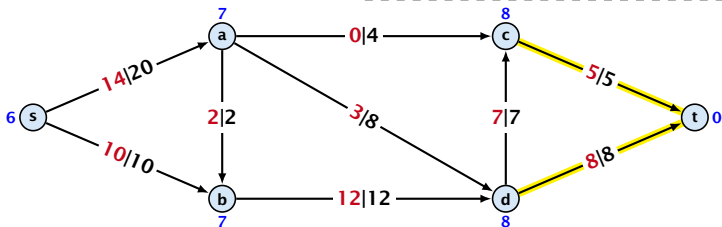
The yellow edges indicate the cut that is introduced by the smallest missing label.



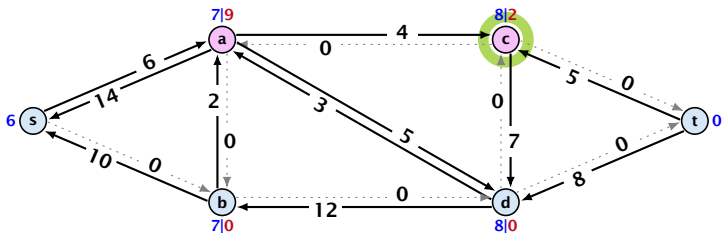


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

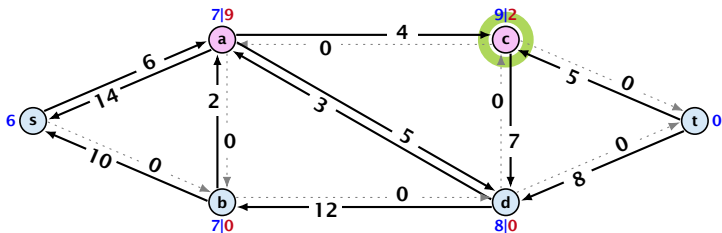
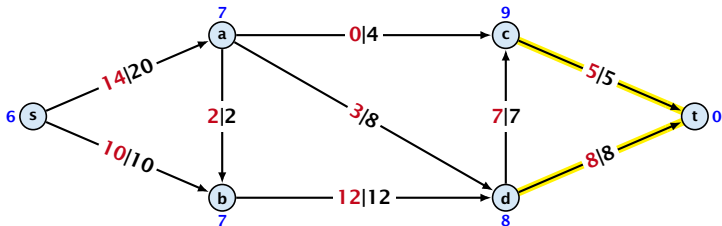


relabel to 9



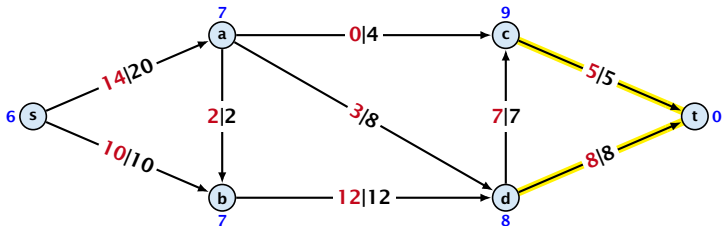
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

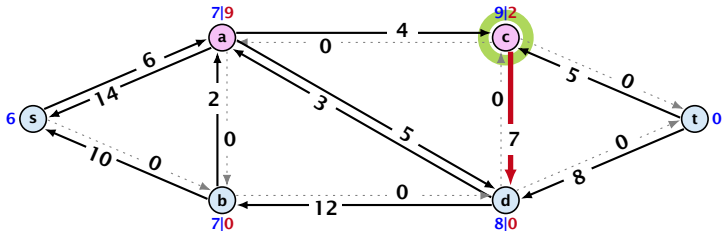


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

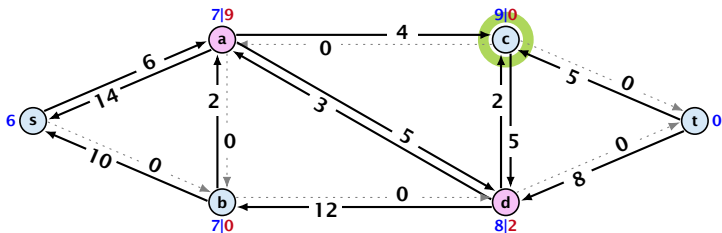
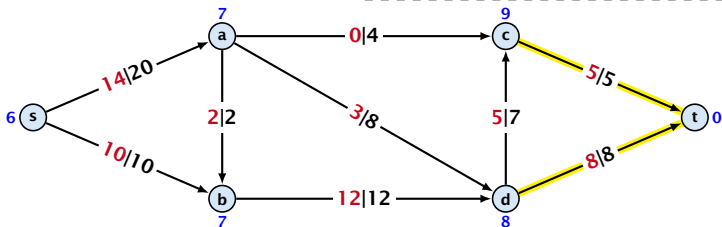


deactivating push



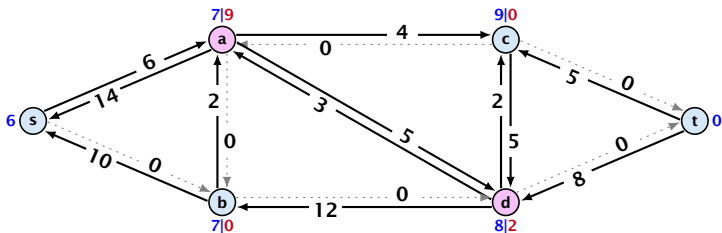
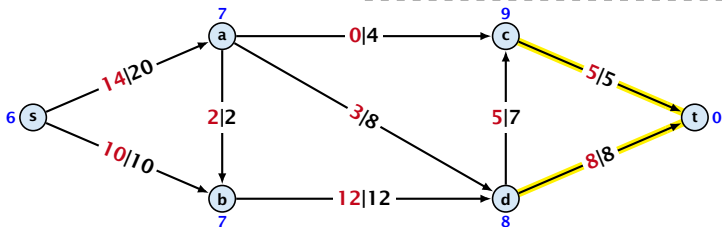
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



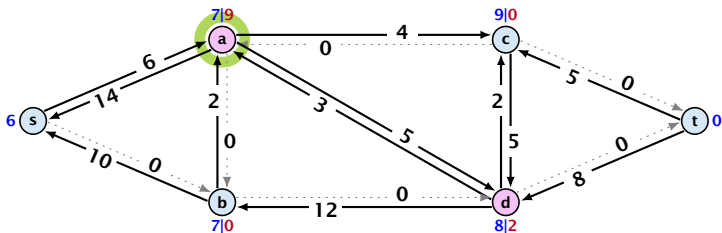
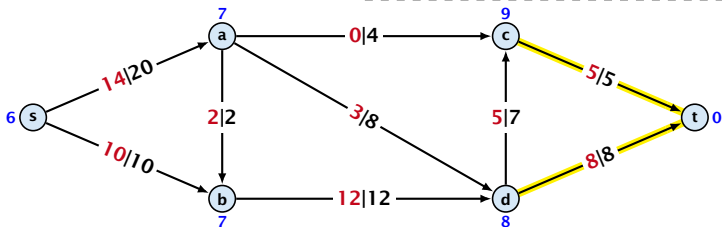
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



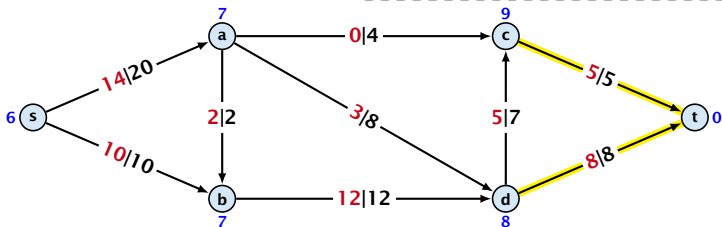
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

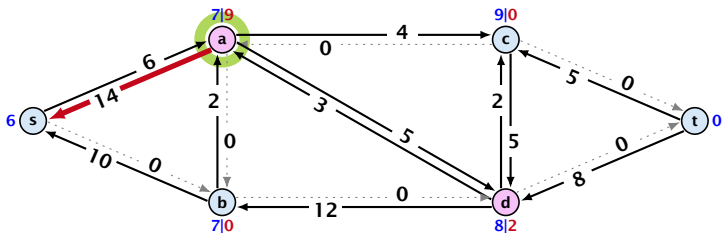


# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.

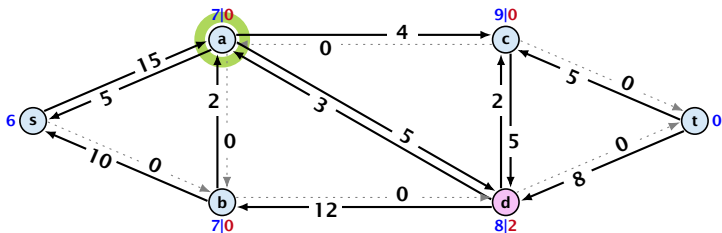
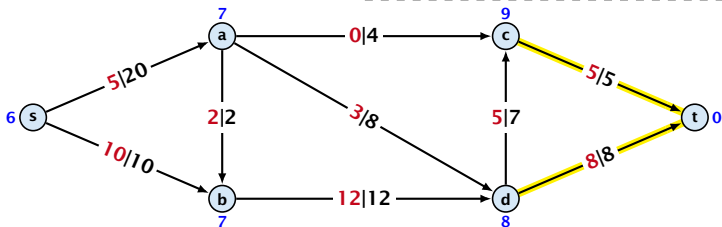


deactivating push



# Preflow Push

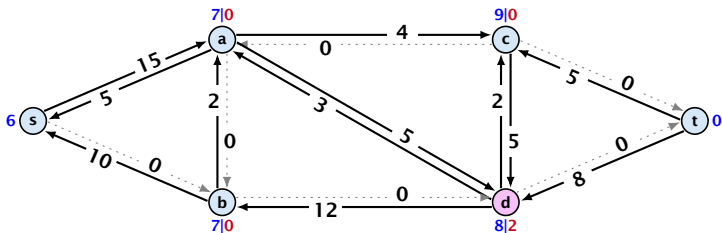
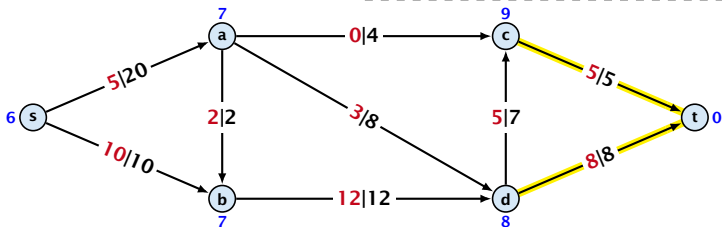
The yellow edges indicate the cut that is introduced by the smallest missing label.





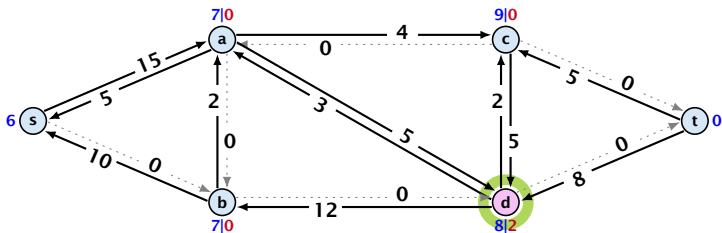
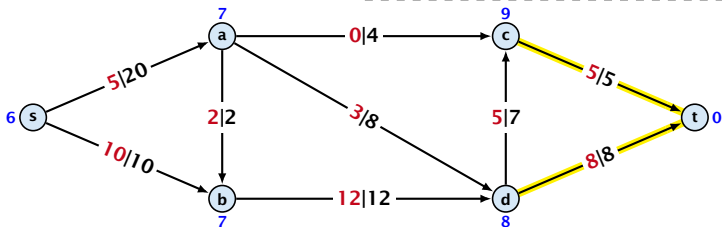
# Preflow Push

The yellow edges indicate the cut that is introduced by the smallest missing label.



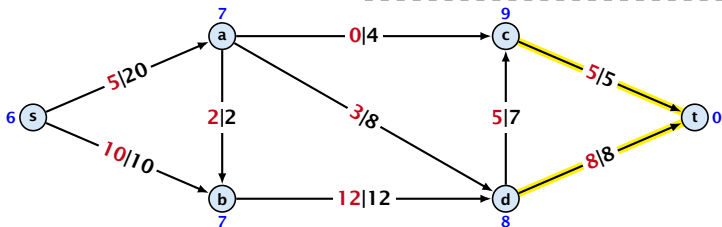
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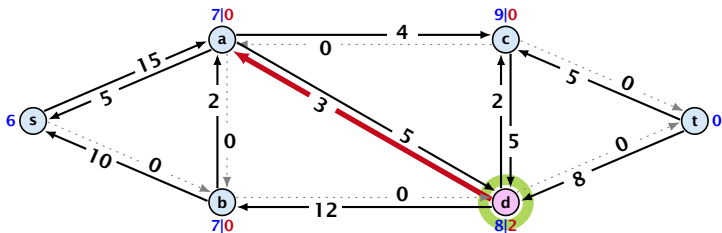


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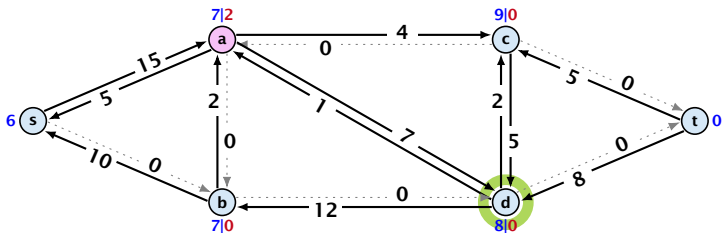
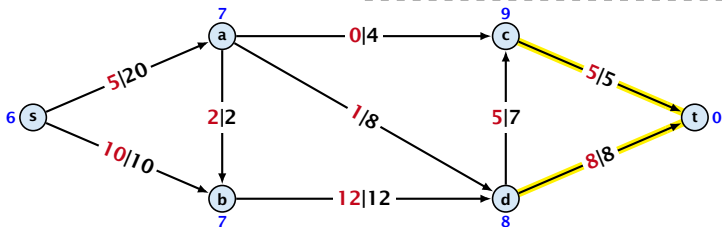


deactivating push



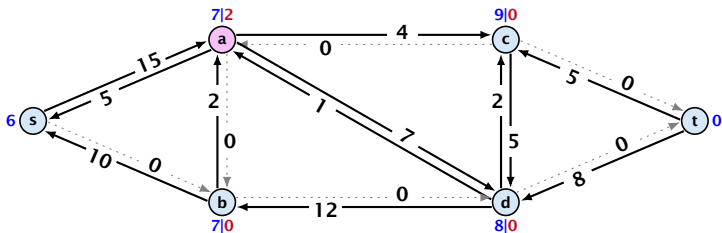
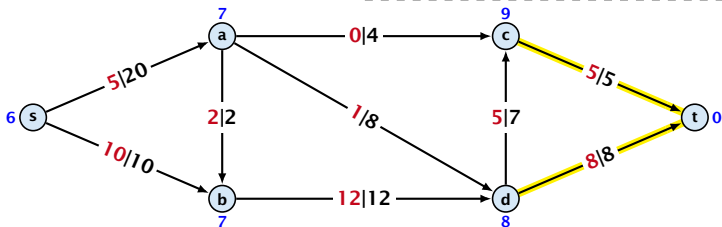
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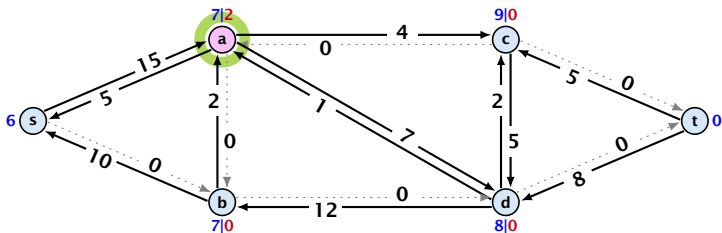
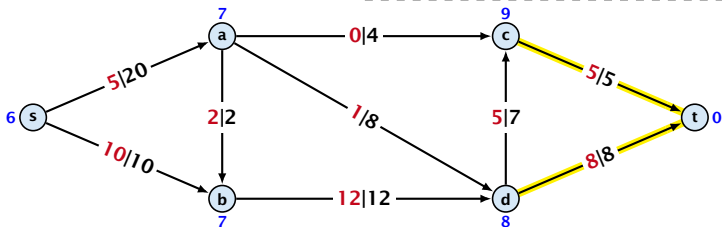
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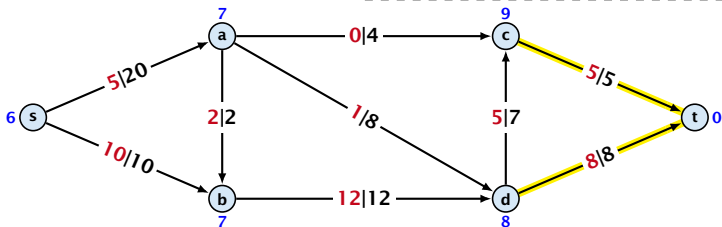
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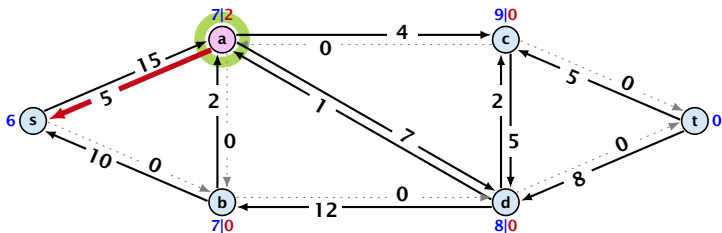


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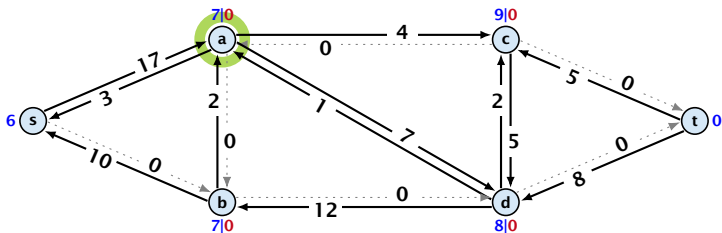
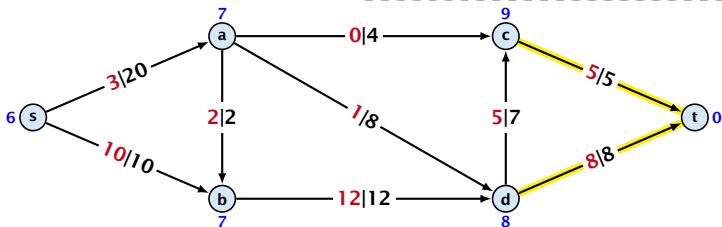


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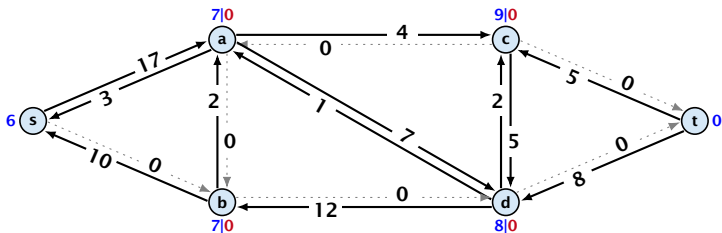
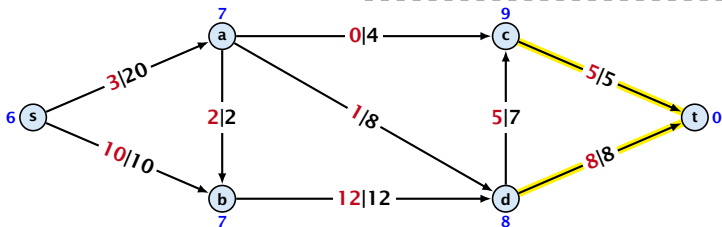
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Note that the lemma is almost trivial. A node  $v$  having excess flow means that the current preflow ships something to  $v$ . The residual graph allows to *undo* flow. Therefore, there must exist a path that can undo the shipment and move it back to  $s$ . However, a formal proof is required.

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*An active node has a path to  $s$  in the residual graph.*

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- ▶ Let  $f(B) = \sum_{v \in B} f(v)$  be the excess flow of all nodes in  $B$ .

Let  $f : E \rightarrow \mathbb{R}_0^+$  be a preflow. We introduce the notation

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Hence, the excess flow  $f(b)$  must be 0 for every node  $b \in B$ .

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- ▶ Hence,

$$\begin{aligned} \#deactivating\_pushes &\leq \#relabels + 2n \cdot \#saturating\_pushes \\ &\leq \mathcal{O}(n^2m) . \end{aligned}$$

## Theorem 14

*There is an implementation of the generic push relabel algorithm with running time  $\mathcal{O}(n^2m)$ .*

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For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

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A relabel at a node  $u$  can be performed in time  $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissible
- ▶ check for all incoming edges if they become non-admissible

## Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph  $G_f$ ). Then we use the discharge-operation:

### Algorithm 2 discharge( $u$ )

```
1: while  $u$  is active do  
2:    $v \leftarrow u.current\text{-neighbour}$   
3:   if  $v = \text{null}$  then  
4:     relabel( $u$ )  
5:      $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list-head}$   
6:   else  
7:     if  $(u, v)$  admissible then push( $u, v$ )  
8:     else  $u.current\text{-neighbour} \leftarrow v.next\text{-in-list}$ 
```

Note that  $u.current\text{-neighbour}$  is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

## Lemma 15

If  $v = \text{null}$  in Line 3, then there is no outgoing admissible edge from  $u$ .

### Proof.

- ▶ While pushing from  $u$  the current-neighbour pointer is only advanced if the current edge is not admissible.
- ▶ The only thing that could make the edge admissible again would be a relabel at  $u$ .
- ▶ If we reach the end of the list ( $v = \text{null}$ ) all edges are not admissible. □

This shows that  $\text{discharge}(u)$  is correct, and that we can perform a relabel in Line 4.

In order for  $e$  to become admissible the other end-point say  $v$  has to push flow to  $u$  (so that the edge  $(u, v)$  re-appears in the residual graph). For this the label of  $v$  needs to be larger than the label of  $u$ . Then in order to make  $(u, v)$  admissible the label of  $u$  has to increase.