

7.5 Skip Lists

Why do we not use a list for implementing the ADT Dynamic Set?

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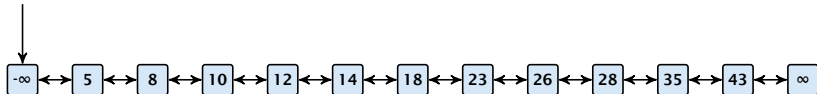
Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search $\Theta(n)$
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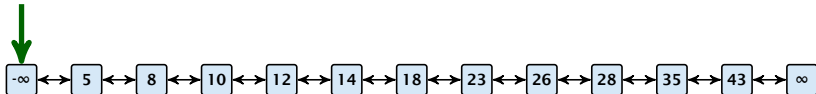
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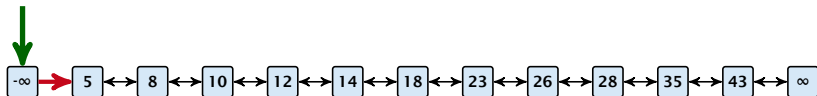
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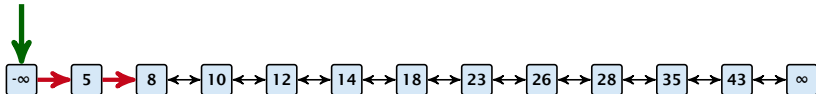
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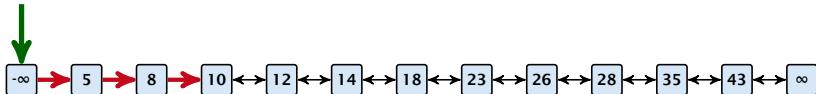
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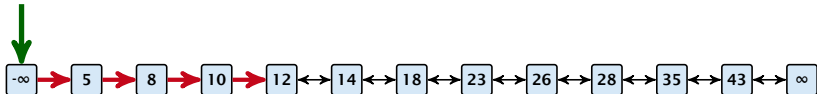
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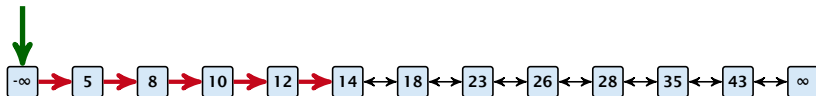
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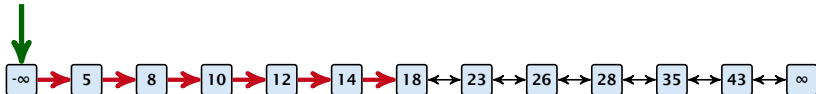
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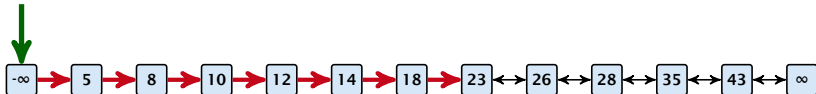
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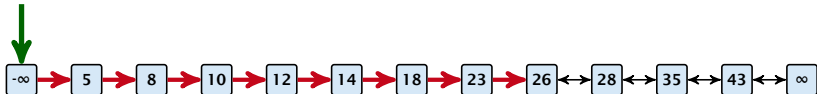
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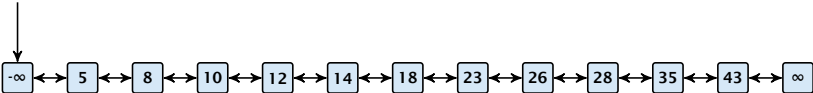
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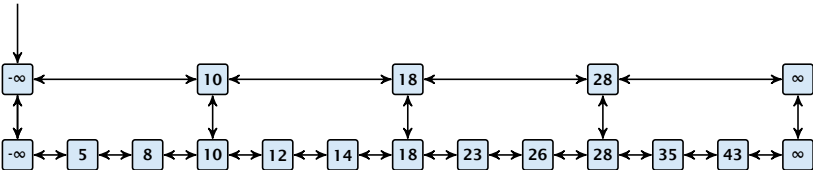
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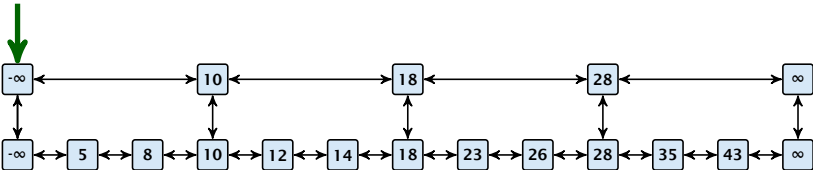
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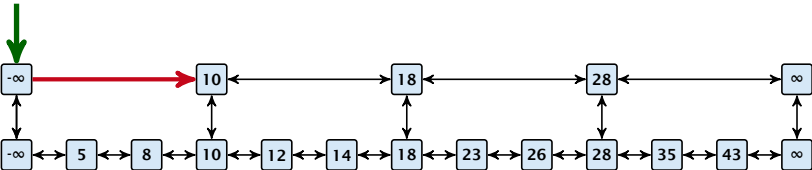
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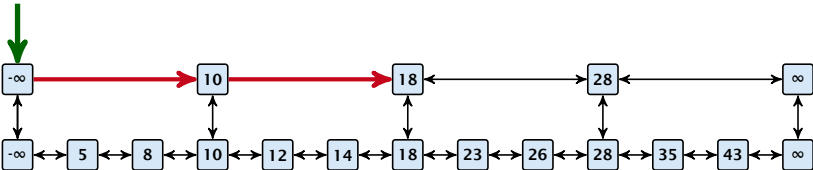
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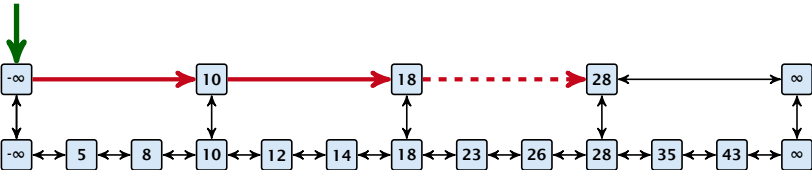
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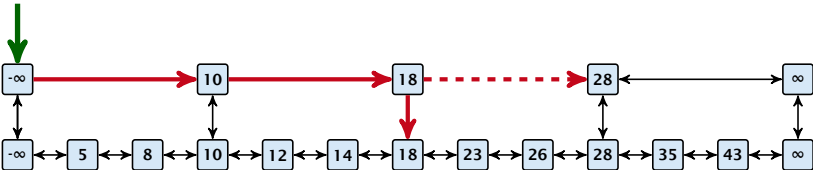
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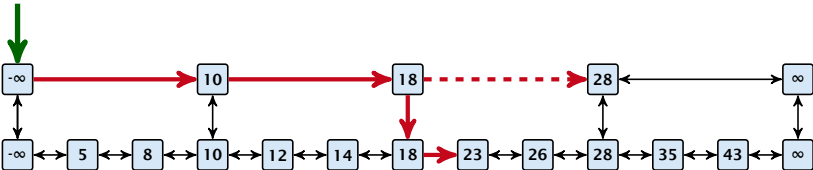
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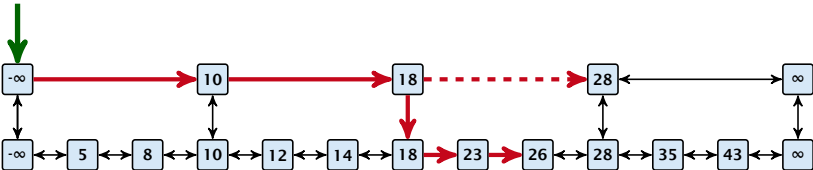
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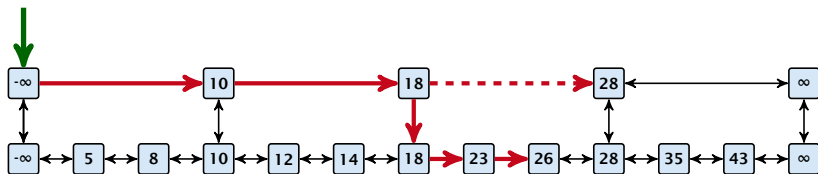
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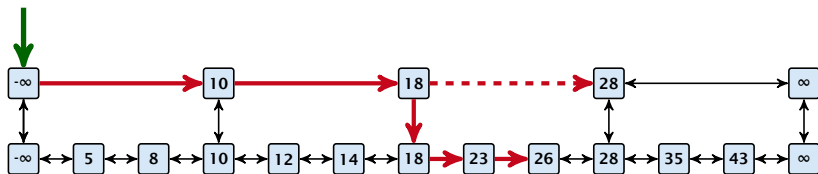


Let $|L_1|$ denote the number of elements in the “express lane”, and $|L_0| = n$ the number of all elements (ignoring dummy elements).

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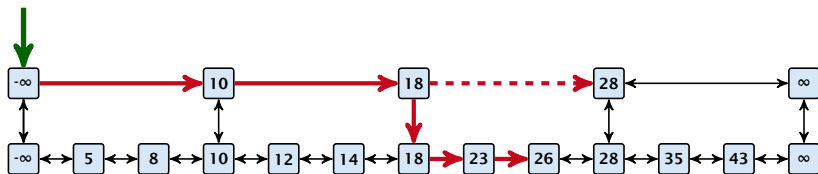
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Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

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- ▶ At most $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$ steps.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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Use randomization instead!

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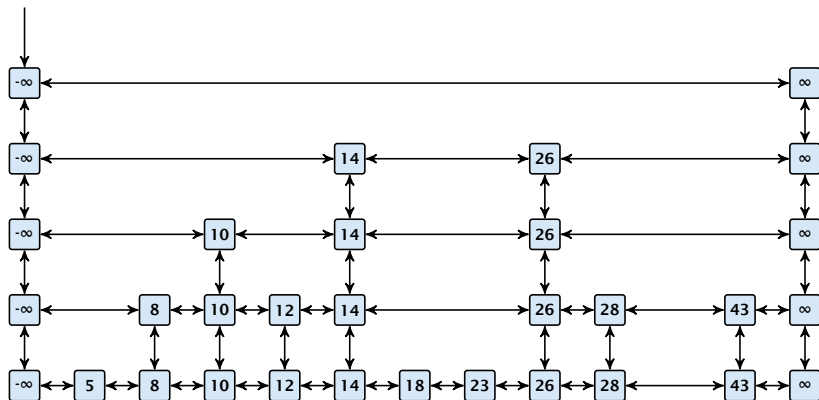
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- ▶ You get all predecessors via backward pointers.
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The time for both operations is dominated by the search time.

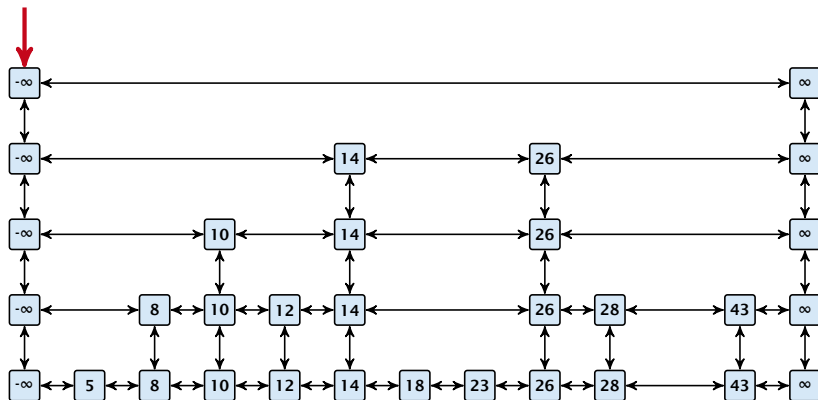
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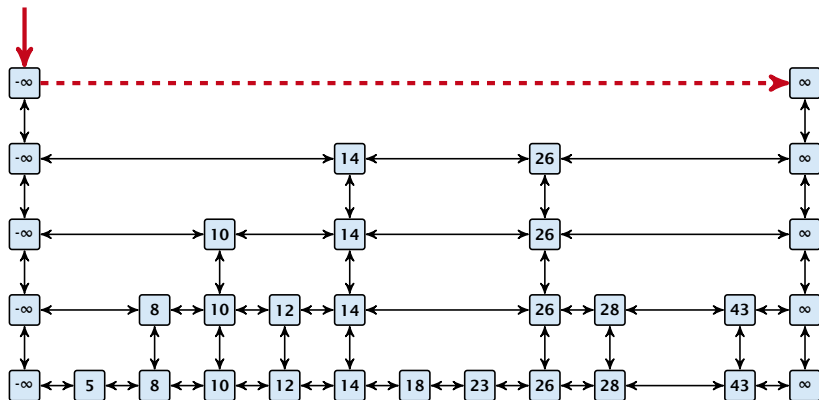
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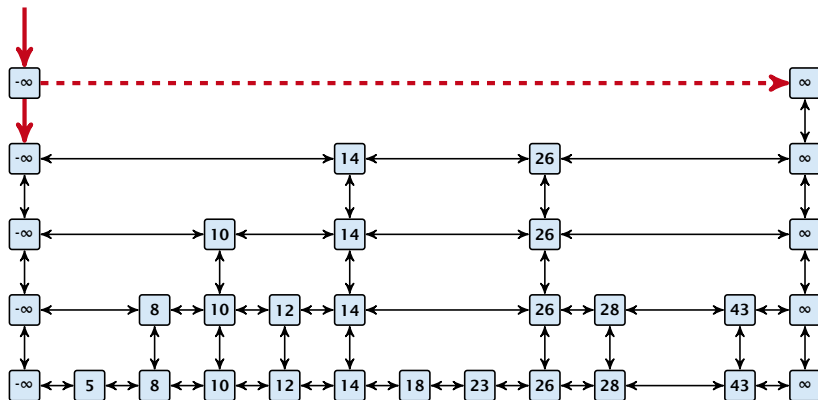
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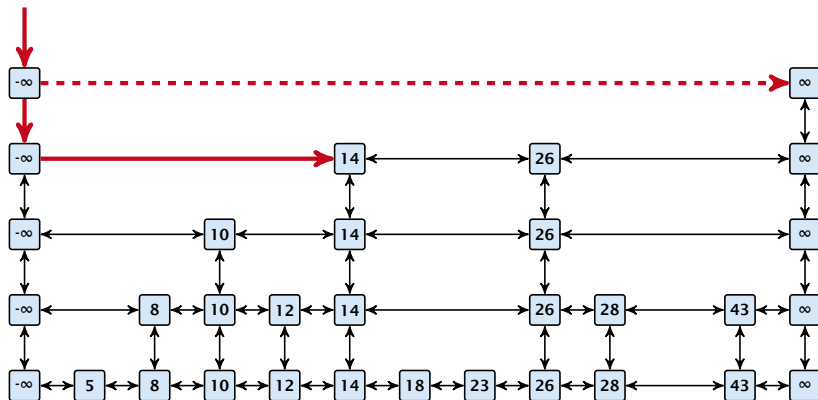
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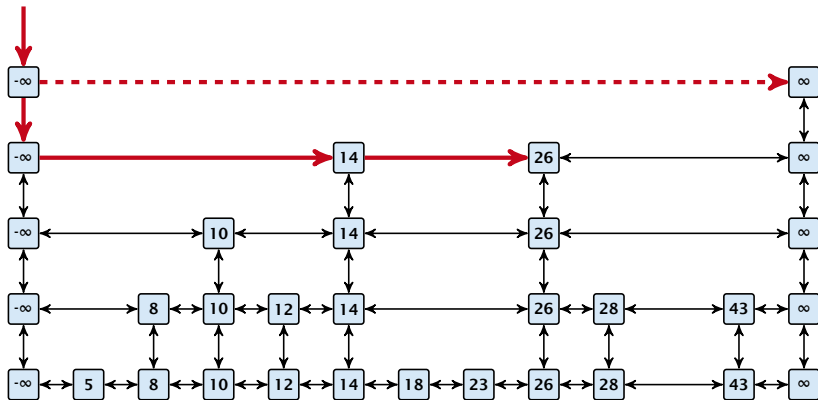
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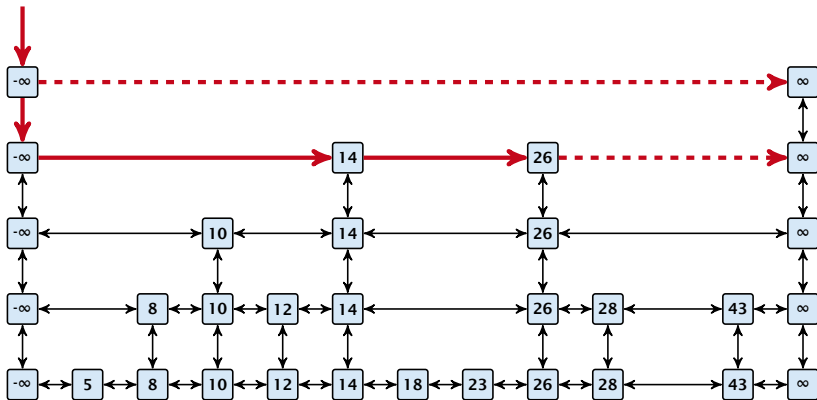
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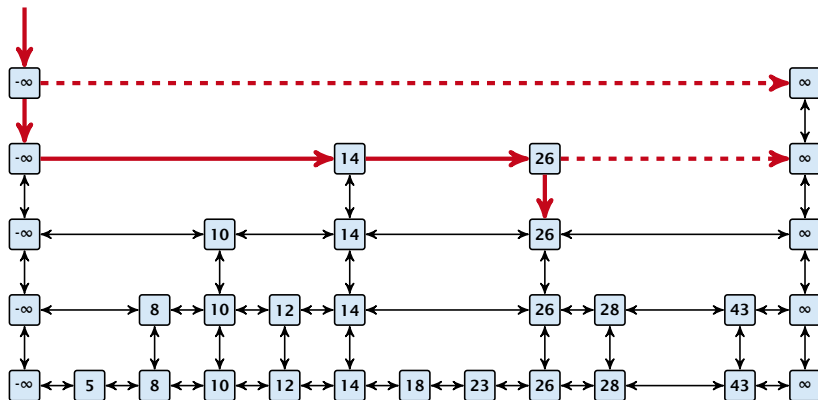
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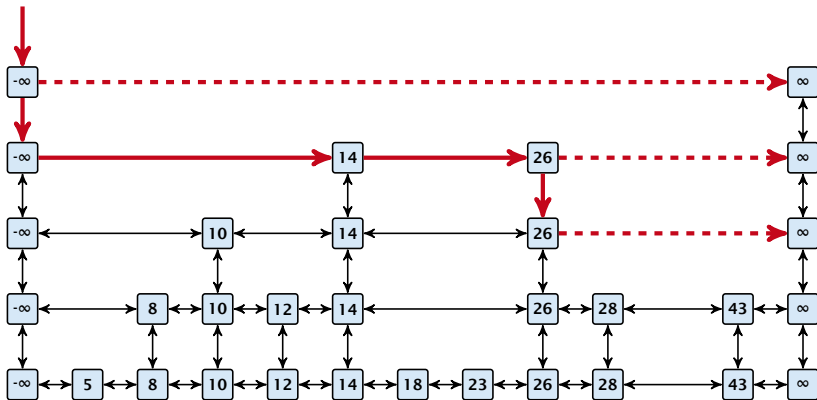
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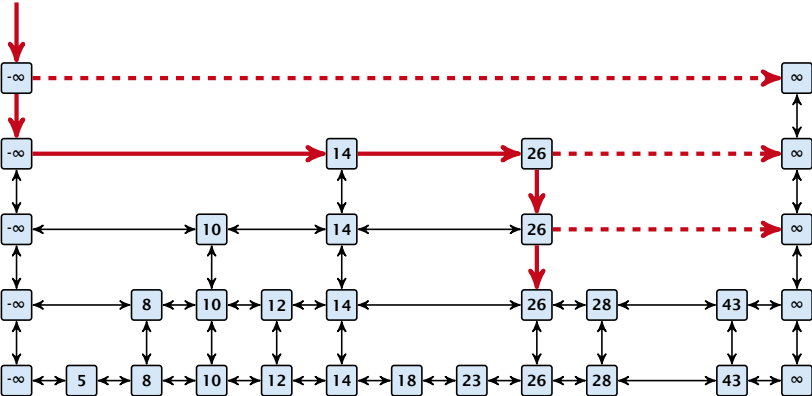
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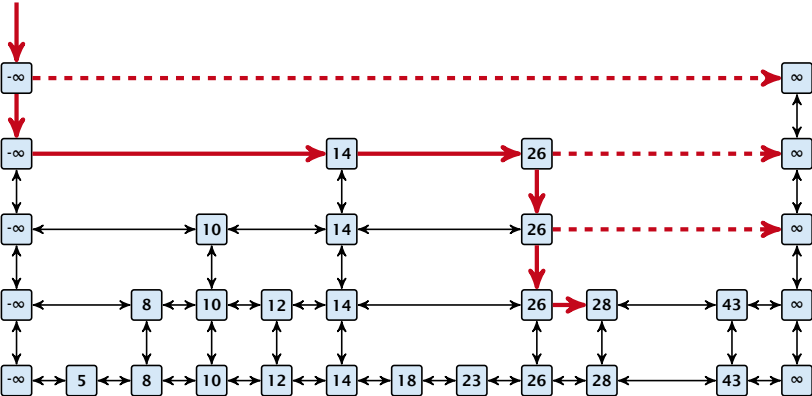
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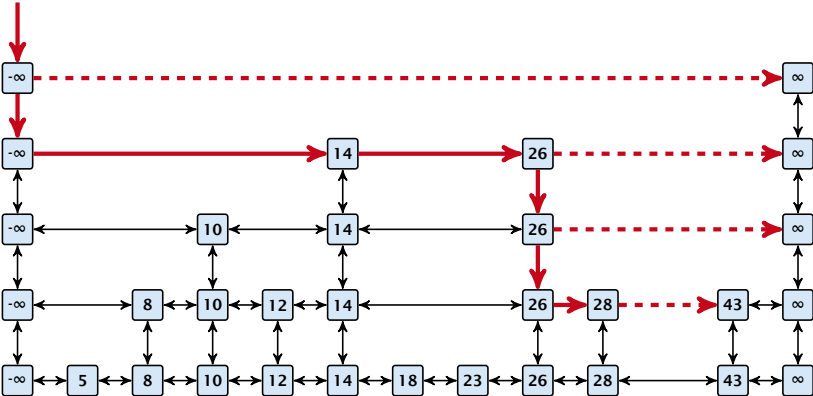
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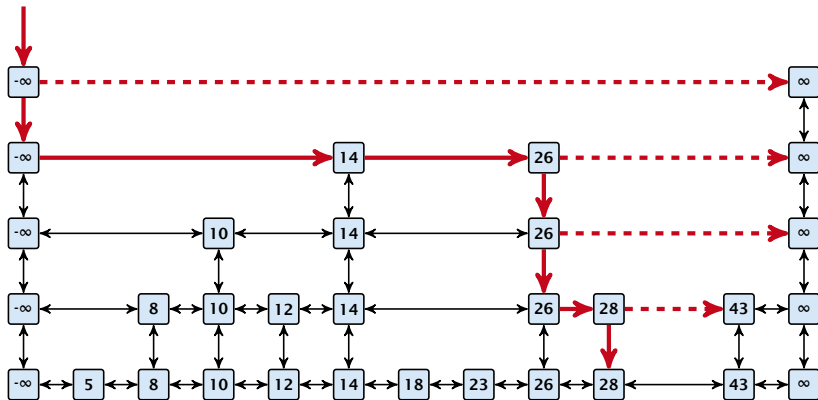
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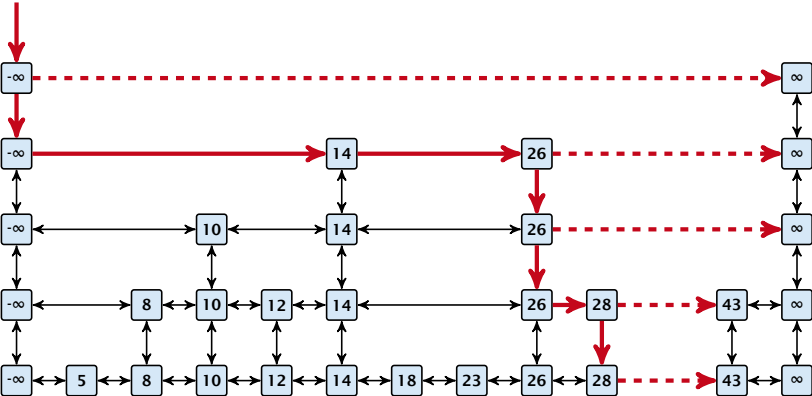
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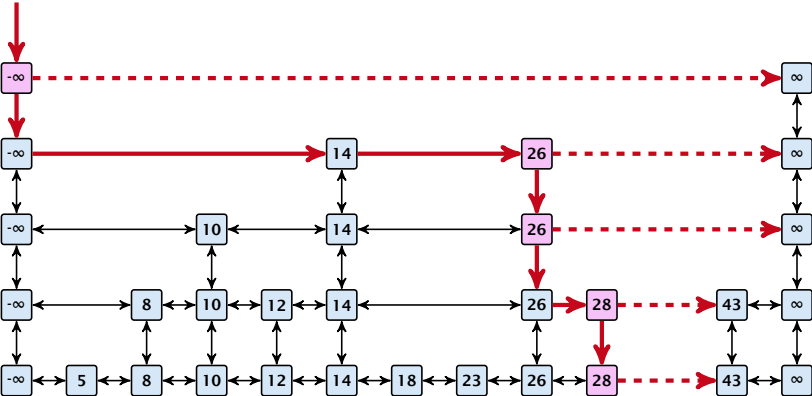
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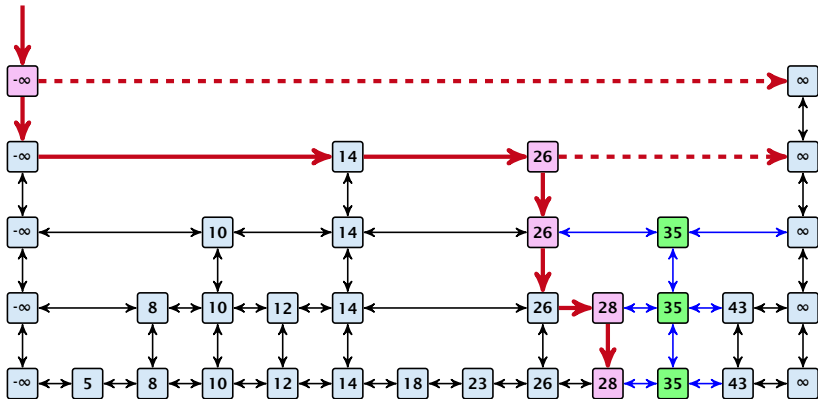
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Here the \mathcal{O} -notation hides a constant that may depend on α .

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Suppose there are **polynomially** many events E_1, E_2, \dots, E_ℓ , $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i -th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \dots \wedge E_\ell]$$

High Probability

Suppose there are **polynomially** many events E_1, E_2, \dots, E_ℓ , $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i -th search in a skip list takes time at most $\mathcal{O}(\log n)$).

Then the probability that all E_i hold is at least

$$\Pr[E_1 \wedge \dots \wedge E_\ell] = 1 - \Pr[\bar{E}_1 \vee \dots \vee \bar{E}_\ell]$$

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This means $\Pr[E_1 \wedge \dots \wedge E_\ell]$ holds with high probability.

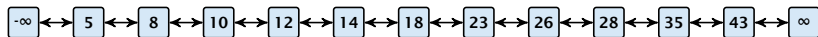
7.5 Skip Lists

Lemma 2

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(\log n)$ with high probability (w. h. p.).

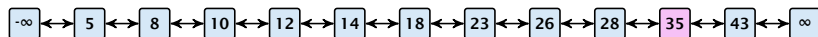
7.5 Skip Lists

Backward analysis:



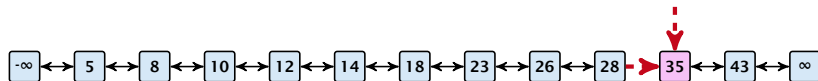
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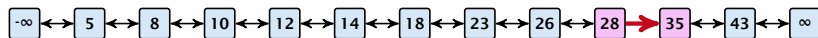
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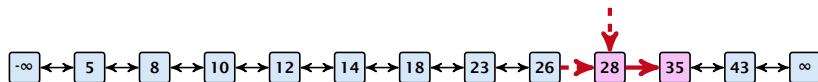
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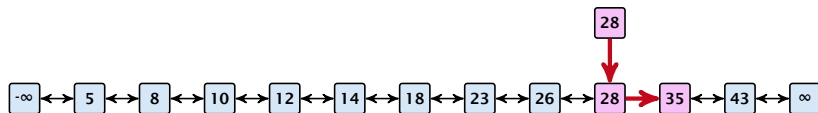
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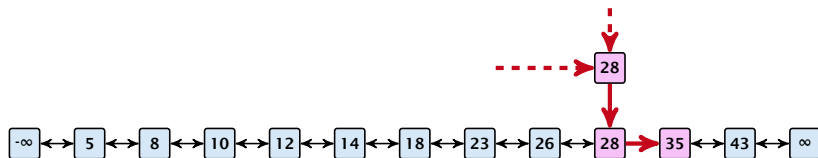
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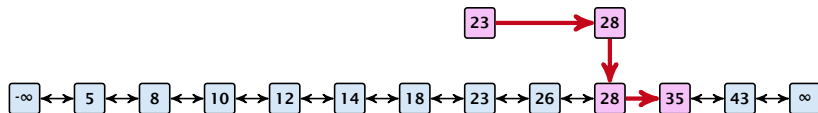
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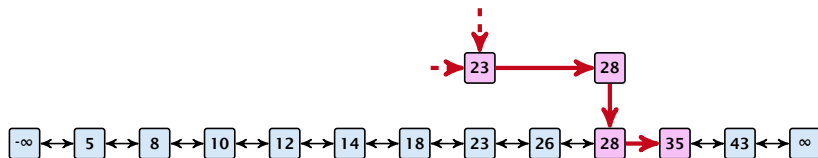
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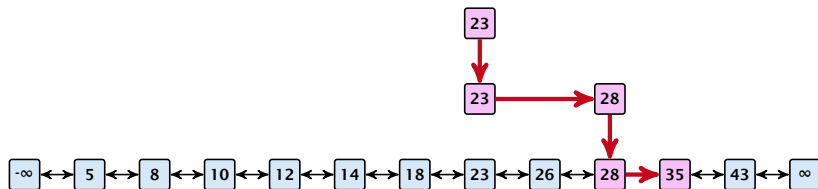
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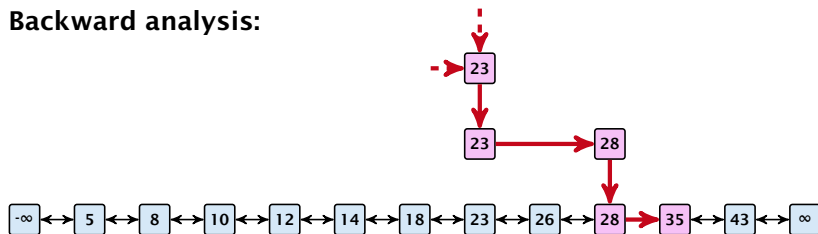
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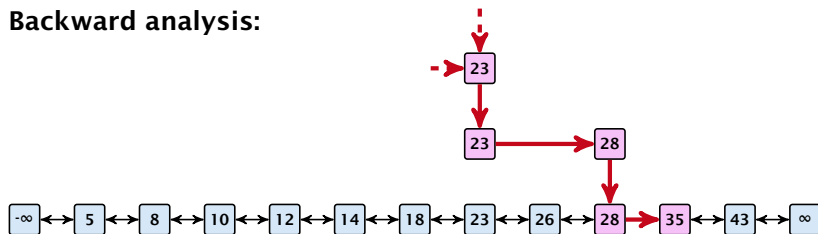
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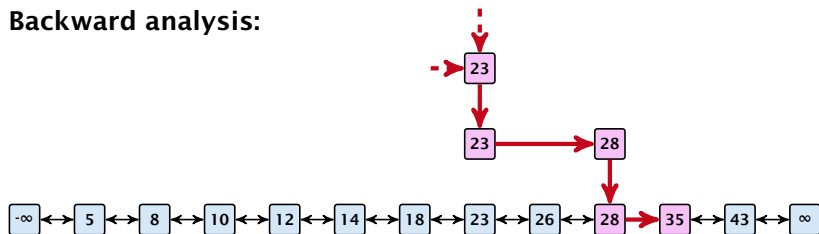
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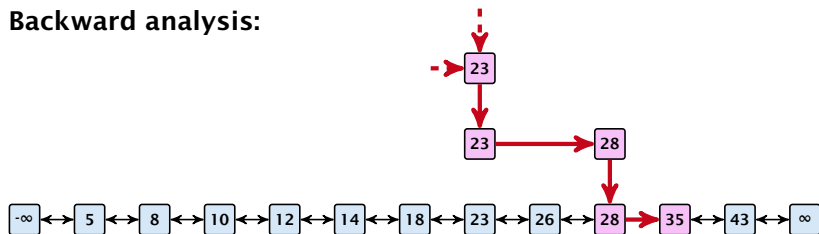
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7.5 Skip Lists

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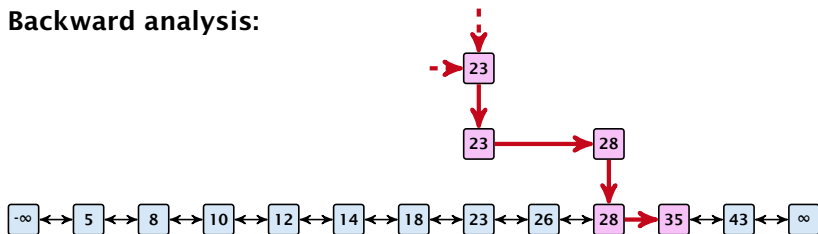
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We show that w.h.p:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

7.5 Skip Lists

Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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This means, the search requires at most z steps, w. h. p.