

## 5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

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**Primal:**

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$$\min\{[b^T \ -b^T]y \mid [A^T \ -A^T]y \geq c, y \geq 0\}$$

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# Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with **reduced cost**

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to  $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$  is solution to the **dual**  $\min\{b^T y \mid A^T y \geq c\}$ .

Hence, the solution is optimal.

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