# 7 Union Find

**Union Find Data Structure $\mathcal{P}$:** Maintains a partition of disjoint sets over elements.

- $\mathcal{P}.\text{makeset}(x)$: Given an element $x$, adds $x$ to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for $x$ in the data-structure.
- $\mathcal{P}.\text{find}(x)$: Given a handle for an element $x$; find the set that contains $x$. Returns a representative/identifier for this set.
- $\mathcal{P}.\text{union}(x, y)$: Given two elements $x$, and $y$ that are currently in sets $S_x$ and $S_y$, respectively, the function replaces $S_x$ and $S_y$ by $S_x \cup S_y$ and returns an identifier for the new set.



# 7 Union Find

**Applications:**

- Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- Kruskals Minimum Spanning Tree Algorithm



# 7 Union Find

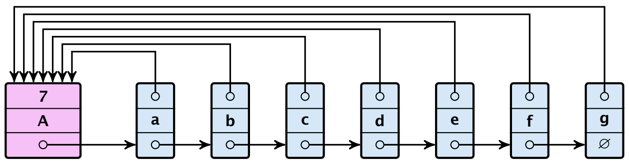
**Algorithm 41** Kruskal-MST($G = (V, E), w$)

$A \leftarrow \emptyset$;
**for all** $v \in V$ **do**
  $v.\text{set} \leftarrow \mathcal{P}.\text{makeset}(v.\text{label})$
sort edges in non-decreasing order of weight $w$
**for all** $(u, v) \in E$ in non-decreasing order **do**
  **if** $\mathcal{P}.\text{find}(u.\text{set}) \neq \mathcal{P}.\text{find}(v.\text{set})$ **then**
    $A \leftarrow A \cup \{(u, v)\}$
    $\mathcal{P}.\text{union}(u.\text{set}, v.\text{set})$



# List Implementation

- The elements of a set are stored in a list; each node has a backward pointer to the head.
- The head of the list contains the identifier for the set and a field that stores the size of the set.




- $\text{makeset}(x)$ can be performed in constant time.
- $\text{find}(x)$ can be performed in constant time.

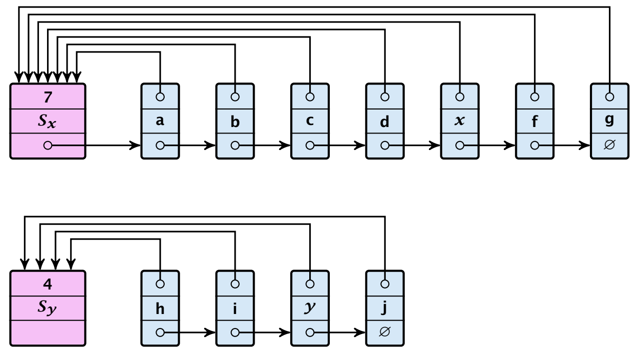
## List Implementation

**union$(x, y)$**

- Determine sets $S_x$ and $S_y$.
- Traverse the smaller list (say $S_y$), and change all backward pointers to the head of list $S_x$.
- Insert list $S_y$ at the head of $S_x$.
- Adjust the size-field of list $S_x$.
- Time: $\min\{|S_x|, |S_y|\}$.

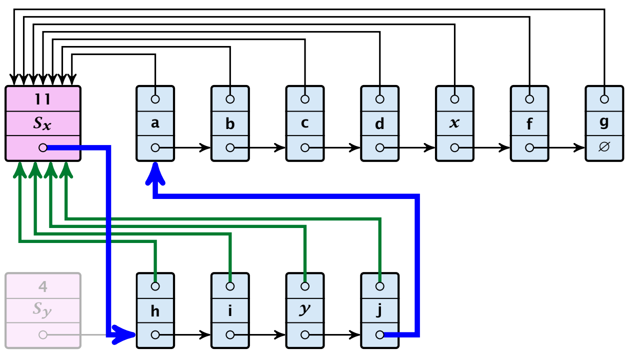


## List Implementation






## List Implementation






## List Implementation

**Running times:**

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x, y)$: $\mathcal{O}(n)$, where $n$ denotes the number of elements contained in the set system.

## List Implementation

**Lemma 2**
*The list implementation for the ADT union find fulfills the following amortized time bounds:*

- $\text{find}(x)$: $\mathcal{O}(1)$.
- $\text{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\text{union}(x, y)$: $\mathcal{O}(1)$.



## The Accounting Method for Amortized Time Bounds

- There is a bank account for every element in the data structure.
- Initially the balance on all accounts is zero.
- Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
- Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.
- If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.



## List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
- Later operations charge the account but the balance never drops below zero.



## List Implementation

**makeset($x$):** The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

**find($x$):** For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

**union($x, y$):**

- If $S_x = S_y$ the cost is constant; no bank accounts change.
- Otw. the actual cost is $\mathcal{O}(\min\{|S_x|, |S_y|\})$.
- Assume wlog. that $S_x$ is the smaller set; let $c$ denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.
- Charge $c$ to every element in set $S_x$.

## List Implementation

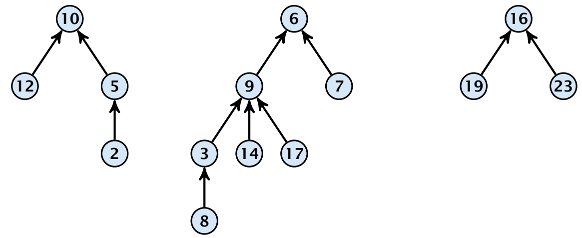
**Lemma 3**
*An element is charged at most $\lfloor \log_2 n \rfloor$ times, where $n$ is the total number of elements in the set system.*

**Proof.**
Whenever an element $x$ is charged the number of elements in $x$'s set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □



## Implementation via Trees

- Maintain nodes of a set in a tree.
- The root of the tree is the label of the set.
- Only pointer to parent exists; we cannot list all elements of a given set.
- Example:




Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.



## Implementation via Trees

**makeset$(x)$**

- Create a singleton tree. Return pointer to the root.
- Time: $\mathcal{O}(1)$.

**find$(x)$**

- Start at element $x$ in the tree. Go upwards until you reach the root.
- Time: $\mathcal{O}(\text{level}(x))$, where $\text{level}(x)$ is the distance of element $x$ to the root in its tree. Not constant.

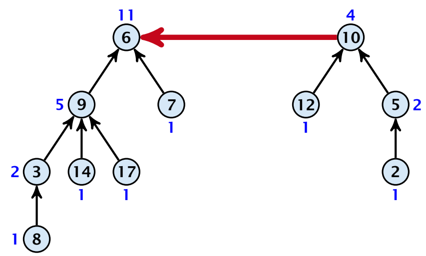


## Implementation via Trees

To support union we store the size of a tree in its root.

**union$(x, y)$**

- Perform $a \leftarrow \text{find}(x)$; $b \leftarrow \text{find}(y)$. Then: $\text{link}(a, b)$.
- $\text{link}(a, b)$ attaches the smaller tree as the child of the larger.
- In addition it updates the size-field of the new root.




- Time: constant for $\text{link}(a, b)$ plus two find-operations.

## Implementation via Trees

**Lemma 4**

*The running time (non-amortized!!!) for* $\text{find}(x)$ *is* $\mathcal{O}(\log n)$.

**Proof.**

- When we attach a tree with root $c$ to become a child of a tree with root $p$, then $\text{size}(p) \geq 2\,\text{size}(c)$, where size denotes the value of the size-field right after the operation.
- After that the value of $\text{size}(c)$ stays fixed, while the value of $\text{size}(p)$ may still increase.
- Hence, at any point in time a tree fulfills $\text{size}(p) \geq 2\,\text{size}(c)$, for any pair of nodes $(p, c)$, where $p$ is a parent of $c$.

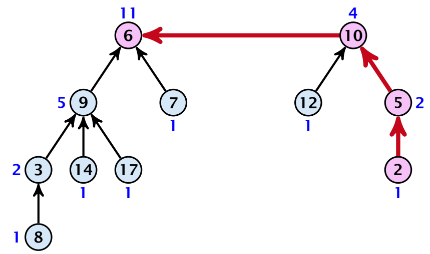
□



## Path Compression

**find($x$):**

- Go upward until you find the root.
- Re-attach all visited nodes as children of the root.
- Speeds up successive find-operations.



- Note that the size-fields now only give an upper bound on the size of a sub-tree.



## Path Compression

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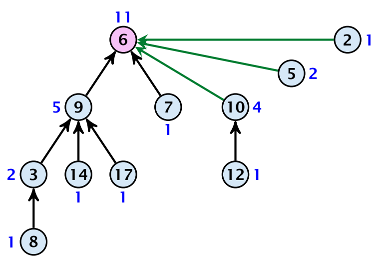

One could change the algorithm to update the size-fields. This could be done without asymptotically affecting the running time.

However, the only size-field that is actually required is the field at the root, which is always correct.

We will only use the other size-fields for the proof of Theorem 7.

- Note that the size-fields now only give an upper bound on the size of a sub-tree.



## Path Compression

Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

## Amortized Analysis

**Definitions:**

- $\text{size}(v) :=$ the number of nodes that were in the sub-tree rooted at $v$ when $v$ became the child of another node (or the number of nodes if $v$ is the root).

  Note that this is the same as the size of $v$'s subtree in the case that there are no find-operations.
- $\text{rank}(v) := \lfloor \log(\text{size}(v)) \rfloor$.
- $\Longrightarrow \text{size}(v) \geq 2^{\text{rank}(v)}$.

**Lemma 5**

*The rank of a parent must be strictly larger than the rank of a child.*



## Amortized Analysis

**Lemma 6**

*There are at most $n/2^s$ nodes of rank $s$.*

**Proof.**

- Let's say a node $v$ sees node $x$ if $v$ is in $x$'s sub-tree at the time that $x$ becomes a child.
- A node $v$ sees at most one node of rank $s$ during the running time of the algorithm.
- This holds because the rank-sequence of the roots of the different trees that contain $v$ during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node *sees* at most one rank $s$ node, but every rank $s$ node is seen by at least $2^s$ different nodes. $\square$



## Amortized Analysis

We define

$$\text{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\text{tow}(i-1)} & \text{otw.} \end{cases} \qquad \text{tow}(i) = \left.2^{2^{2^{2^{2}}}}\right\}i \text{ times}$$

and

$$\log^*(n) := \min\{i \mid \text{tow}(i) \geq n\} \ .$$

**Theorem 7**

*Union find with path compression fulfills the following amortized running times:*

- $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\text{find}(x) : \mathcal{O}(\log^*(n))$
- $\text{union}(x, y) : \mathcal{O}(\log^*(n))$



## Amortized Analysis

In the following we assume $n \geq 2$.

**rank-group:**

- A node with rank $\text{rank}(v)$ is in rank group $\log^*(\text{rank}(v))$.
- The rank-group $g = 0$ contains only nodes with rank $0$ or rank $1$.
- A rank group $g \geq 1$ contains ranks $\text{tow}(g-1) + 1, \ldots, \text{tow}(g)$.
- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- Hence, the total number of rank-groups is at most $\log^* n$.

## Amortized Analysis

**Accounting Scheme:**

- create an account for every find-operation
- create an account for every node $v$

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from $v$ to $\text{parent}[v]$ as follows:

- If $\text{parent}[v]$ is the root we charge the cost to the find-account.
- If the group-number of $\text{rank}(v)$ is the same as that of $\text{rank}(\text{parent}[v])$ (before starting path compression) we charge the cost to the node-account of $v$.
- Otherwise we charge the cost to the find-account.



## Amortized Analysis

**Observations:**

- A find-account is charged at most $\log^*(n)$ times (once for the root and at most $\log^*(n) - 1$ times when increasing the rank-group).
- After a node $v$ is charged its parent-edge is re-assigned. The rank of the parent strictly increases.
- After some charges to $v$ the parent will be in a larger rank-group. $\Rightarrow$ $v$ will never be charged again.
- The total charge made to a node in rank-group $g$ is at most $\text{tow}(g) - \text{tow}(g-1) - 1 \le \text{tow}(g)$.



## Amortized Analysis

**What is the total charge made to nodes?**

- The total charge is at most

$$\sum_g n(g) \cdot \text{tow}(g) \ ,$$

where $n(g)$ is the number of nodes in group $g$.



## Amortized Analysis

For $g \ge 1$ we have

$$\begin{aligned} n(g) &\le \sum_{s=\text{tow}(g-1)+1}^{\text{tow}(g)} \frac{n}{2^s} \le \sum_{s=\text{tow}(g-1)+1}^{\infty} \frac{n}{2^s} \\ &= \frac{n}{2^{\text{tow}(g-1)+1}} \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{n}{2^{\text{tow}(g-1)+1}} \cdot 2 \\ &= \frac{n}{2^{\text{tow}(g-1)}} = \frac{n}{\text{tow}(g)} \ . \end{aligned}$$

Hence,

$$\sum_g n(g)\,\text{tow}(g) \le n(0)\,\text{tow}(0) + \sum_{g\ge 1} n(g)\,\text{tow}(g) \le n\log^*(n)$$

## Amortized Analysis

Without loss of generality we can assume that all makeset-operations occur at the start.

This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of $v$ then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).



## Amortized Analysis

The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of $m$ operations on at most $n$ elements).

There is also a lower bound of $\Omega(\alpha(m,n))$.



## Amortized Analysis

$$A(x,y) = \begin{cases} y+1 & \text{if } x = 0 \\ A(x-1,1) & \text{if } y = 0 \\ A(x-1, A(x,y-1)) & \text{otw.} \end{cases}$$

$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

- $A(0,y) = y+1$
- $A(1,y) = y+2$
- $A(2,y) = 2y+3$
- $A(3,y) = 2^{y+3} - 3$
- $A(4,y) = \underbrace{2^{2^{2^2}}}_{y+3 \text{ times}} - 3$



## Union Find

**Bibliography**


[CLRS90a] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest: *Introduction to Algorithms (1st ed.)*, MIT Press and McGraw-Hill, 1990

[CLRS90b] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein: *Introduction to Algorithms (2nd ed.)*, MIT Press and McGraw-Hill, 2001

[CLRS90c] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein: *Introduction to Algorithms (3rd ed.)*, MIT Press and McGraw-Hill, 2009

[AHU74] Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman: *The Design and Analysis of Computer Algorithms*, Addison-Wesley, 1974


Union find data structures are discussed in Chapter 21 of [CLRS90b] and [CLRS90c] and in Chapter 22 of [CLRS90a]. The analysis of union by rank with path compression can be found in [CLRS90a] but neither in [CLRS90b] in nor in [CLRS90c]. The latter books contains a more involved analysis that gives a better bound than $\mathcal{O}(\log^* n)$.

A description of the $\mathcal{O}(\log^*)$-bound can also be found in Chapter 4.8 of [AHU74].