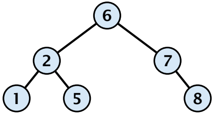
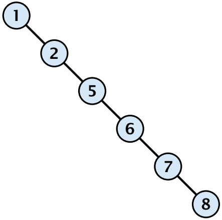
5.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:



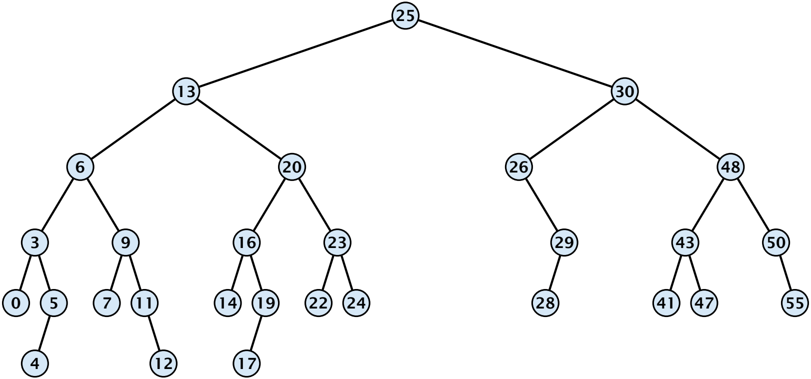


5.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\text{insert}(x)$
- $T.\text{delete}(x)$
- $T.\text{search}(k)$
- $T.\text{successor}(x)$
- $T.\text{predecessor}(x)$
- $T.\text{minimum}()$
- $T.\text{maximum}()$

Binary Search Trees: Searching



Algorithm 1 TreeSearch(x, k)

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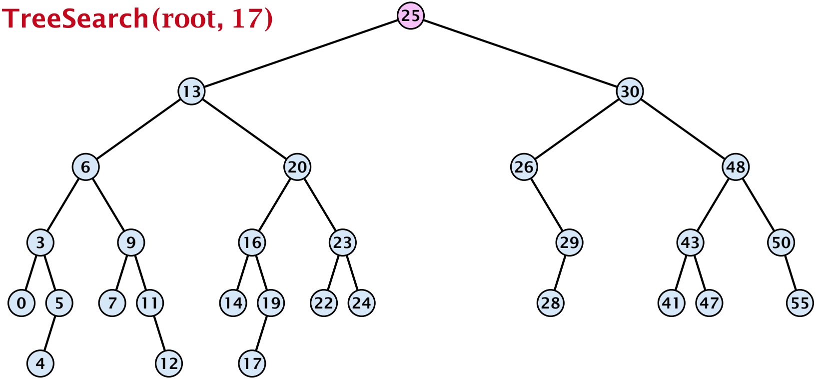
Binary Search Trees: Searching

TreeSearch(root, 17)



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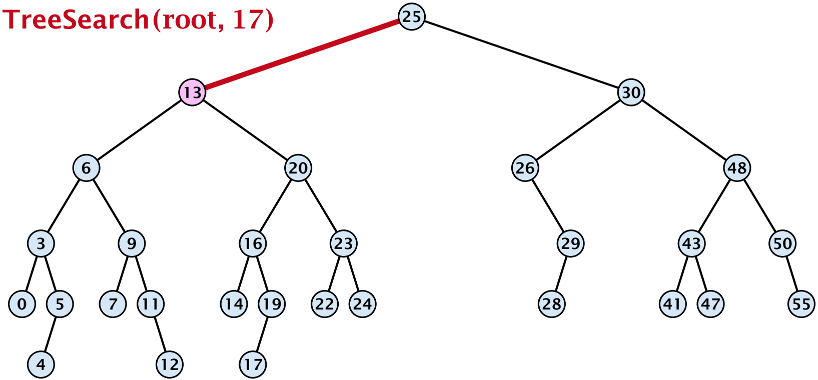

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Binary Search Trees: Searching

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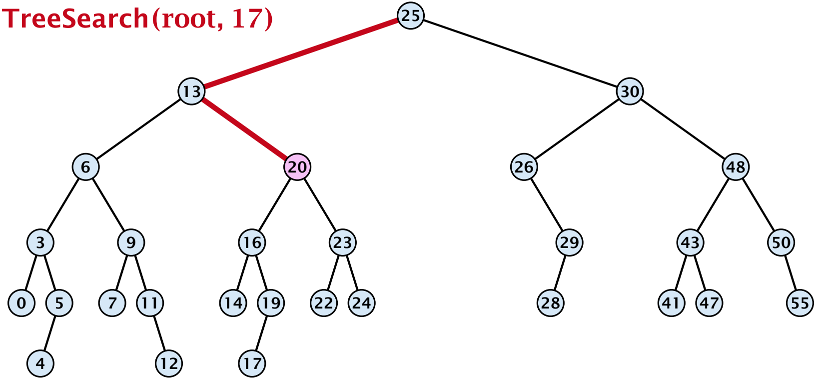

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Binary Search Trees: Searching

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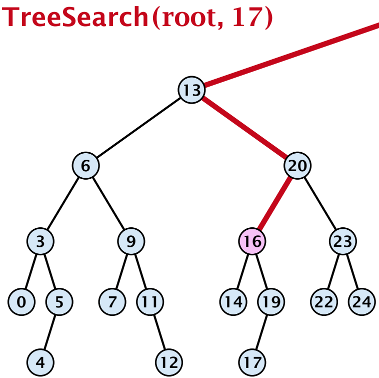

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Binary Search Trees: Searching

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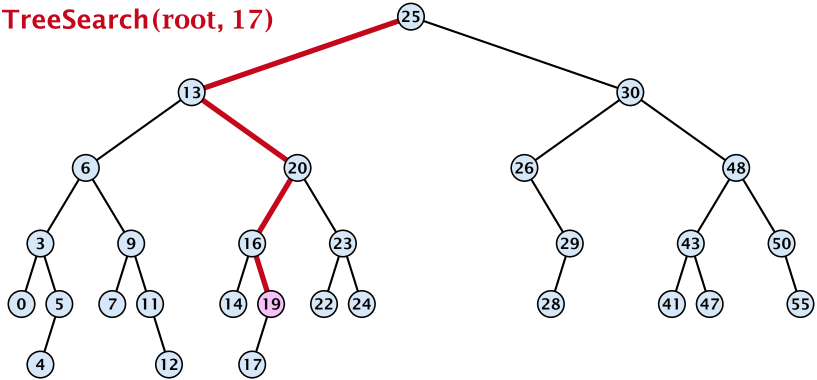

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Binary Search Trees: Searching

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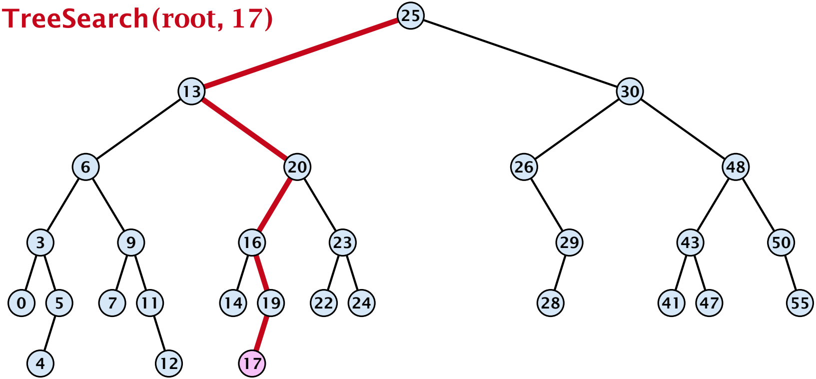

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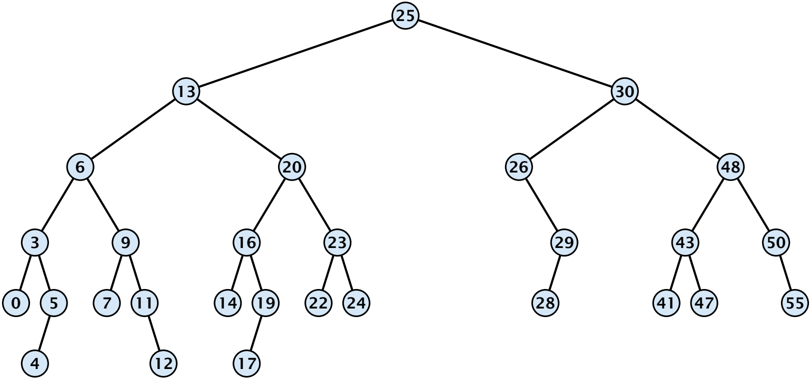
Binary Search Trees: Searching



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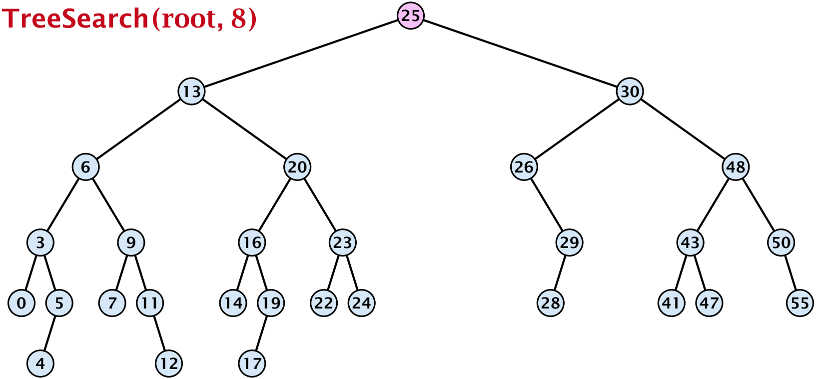
Binary Search Trees: Searching

TreeSearch(root, 8)



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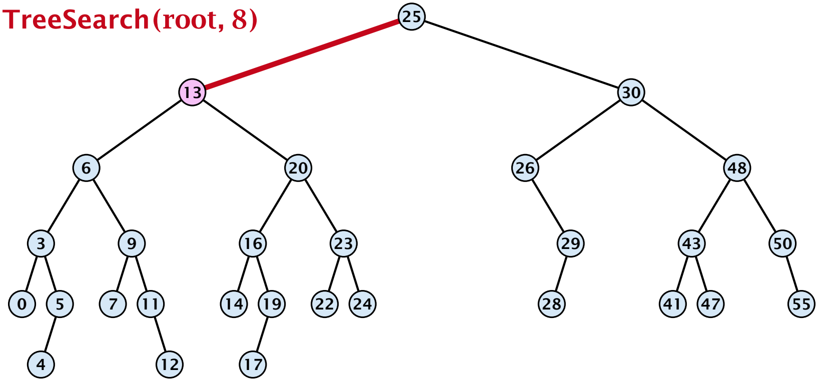

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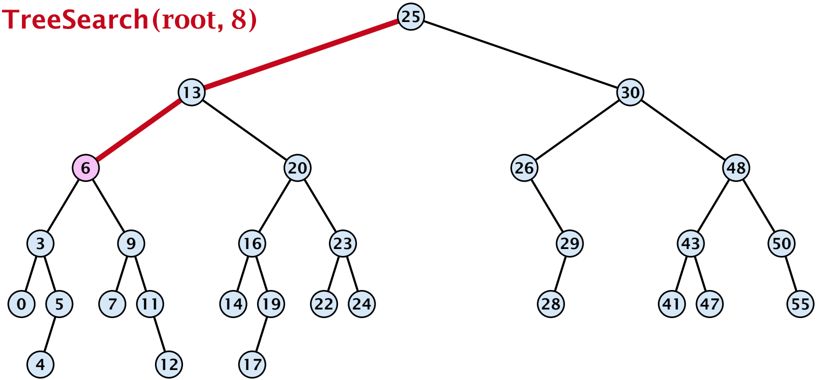

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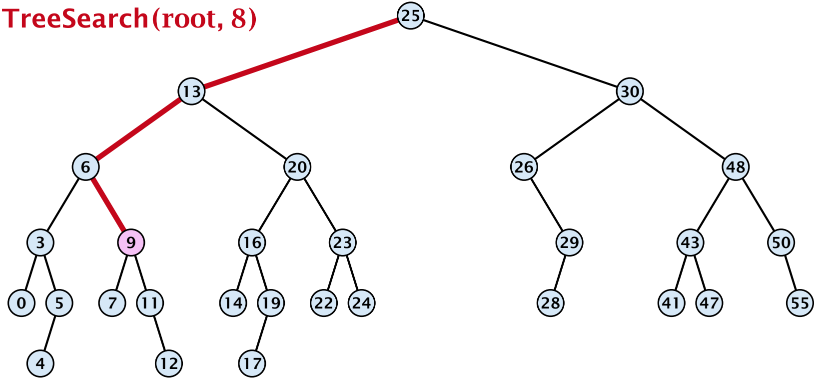

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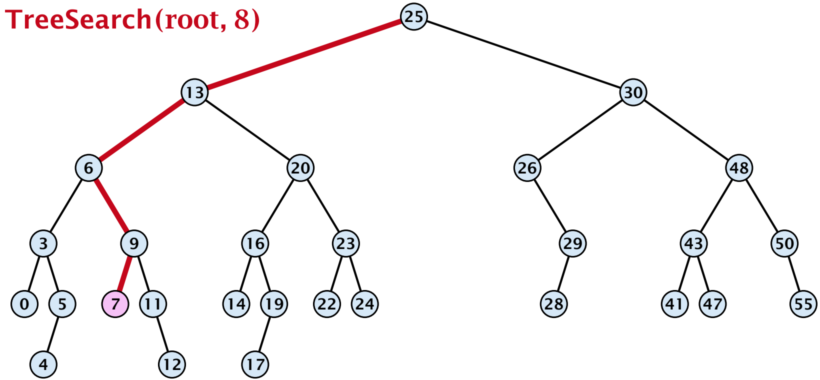

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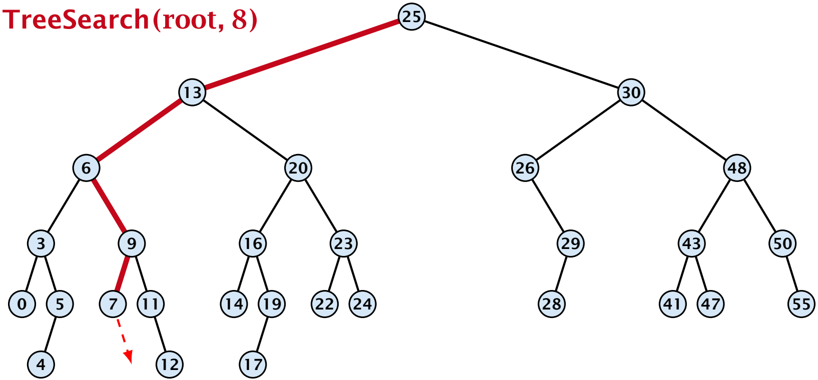

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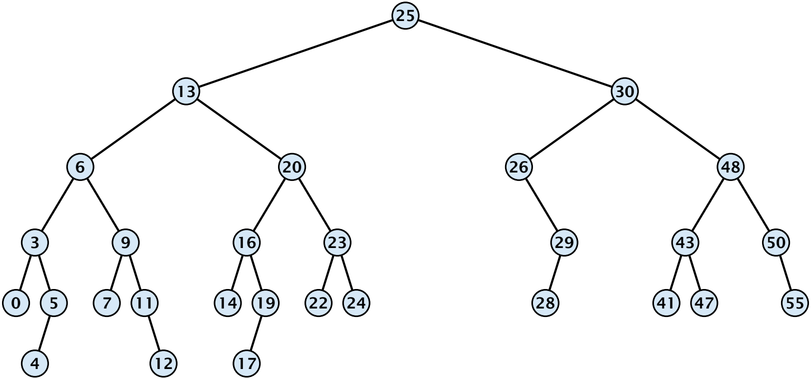
Binary Search Trees: Minimum

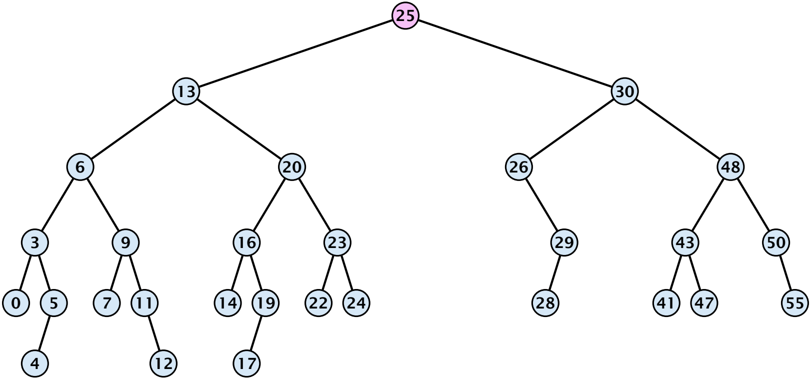


Algorithm 2 TreeMin(x)

1: **if** x = null **or** left[x] = null **return** x

2: **return** TreeMin(left[x])

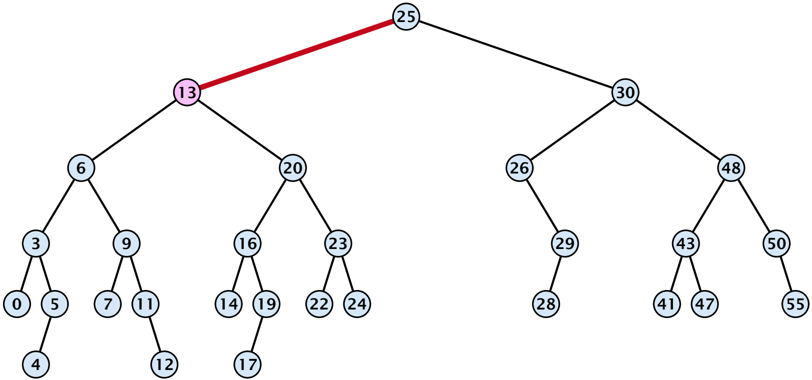
Binary Search Trees: Minimum



Algorithm 2 TreeMin(x)

1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x
2: **return** TreeMin($\text{left}[x]$)

Binary Search Trees: Minimum



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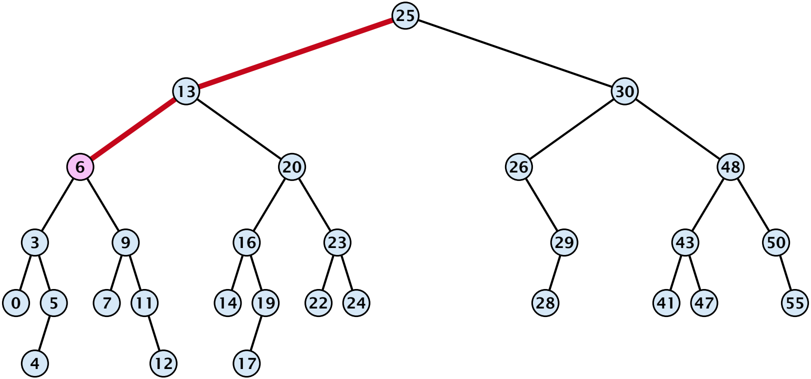

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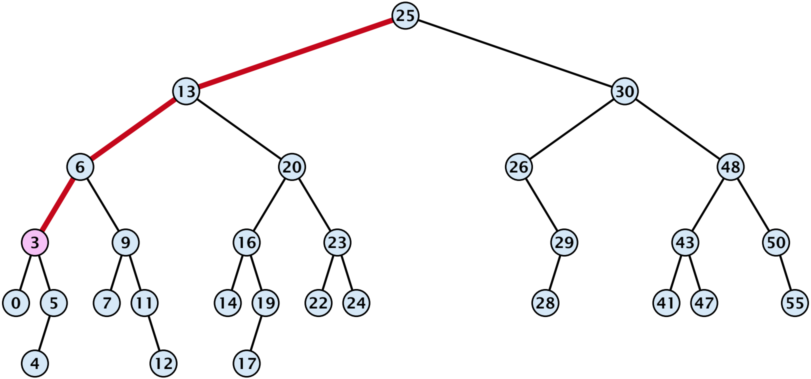

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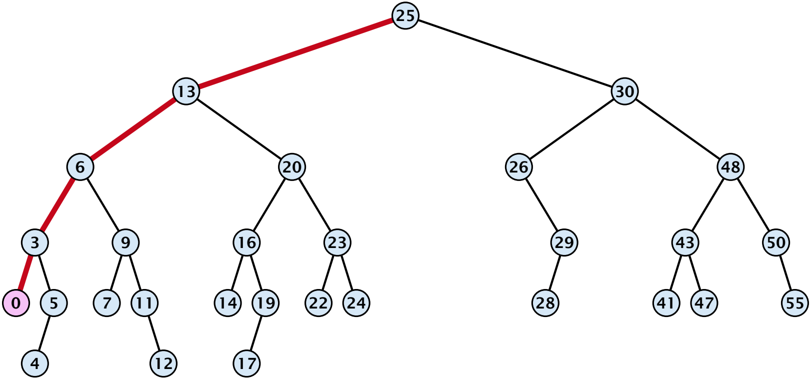

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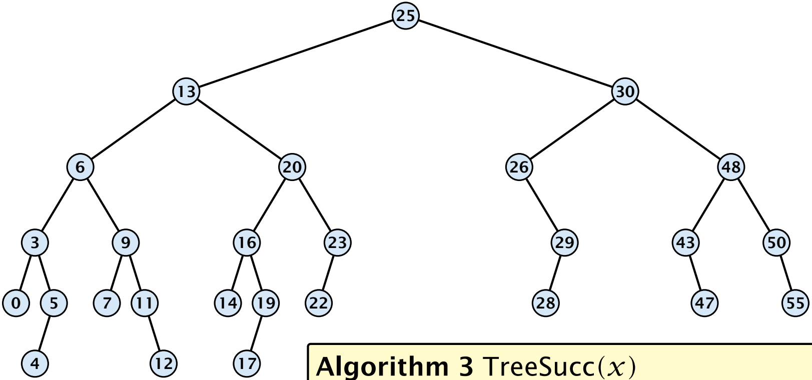
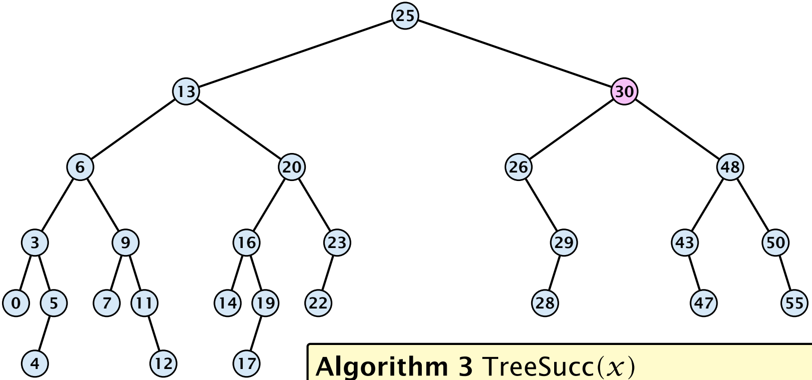
Binary Search Trees: Successor



Algorithm 3 TreeSucc(x)

1: **if** right[x] $\neq$ null **return** TreeMin(right[x])
2: $y \leftarrow$ parent[x]
3: **while** $y \neq$ null **and** $x =$ right[y] **do**
4: $\quad x \leftarrow y; y \leftarrow$ parent[x]
5: **return** y;

Binary Search Trees: Successor



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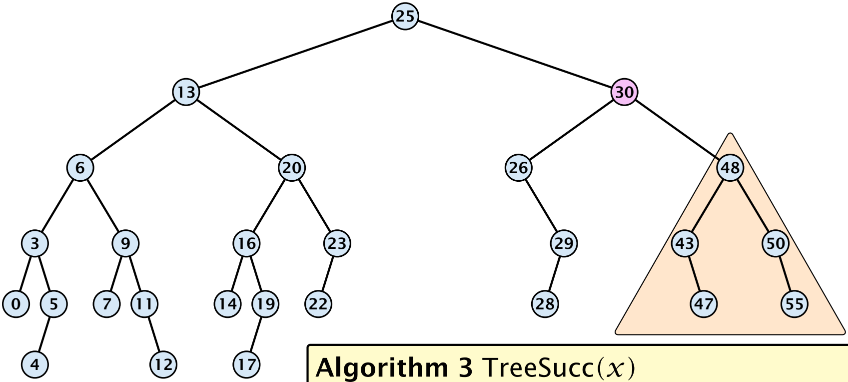

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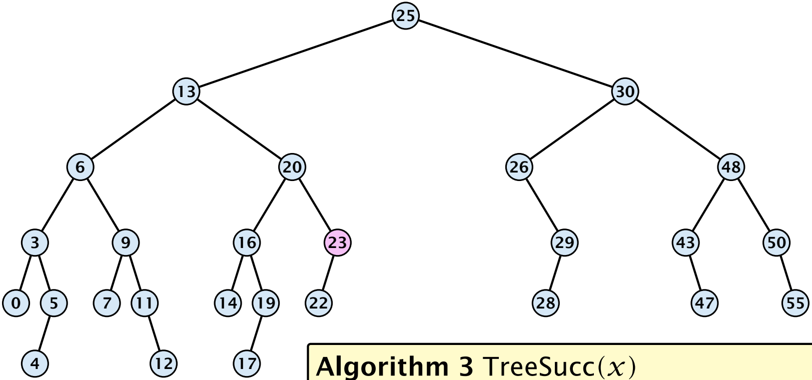

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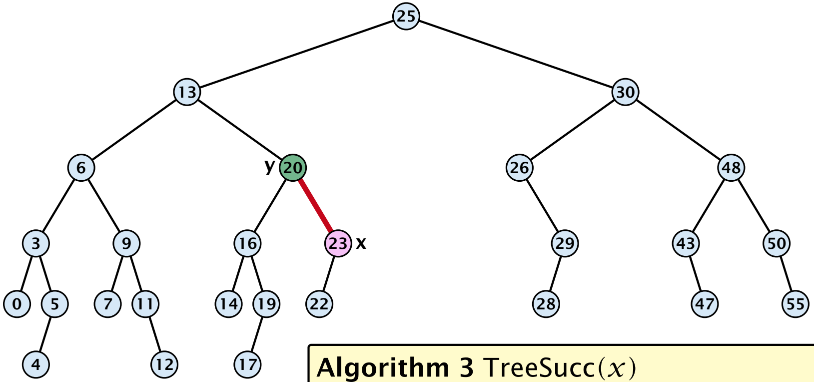

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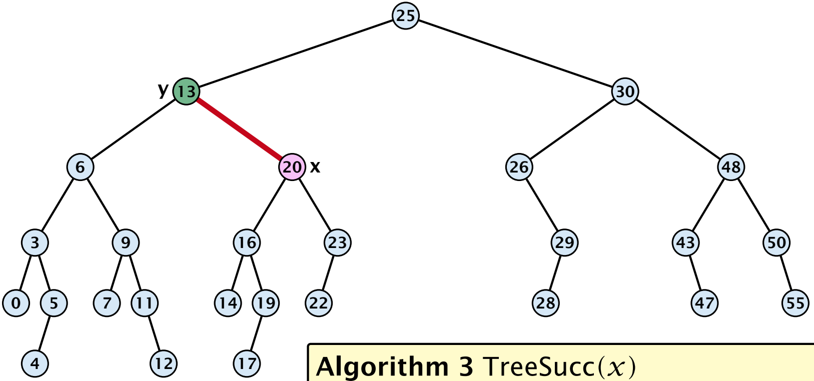

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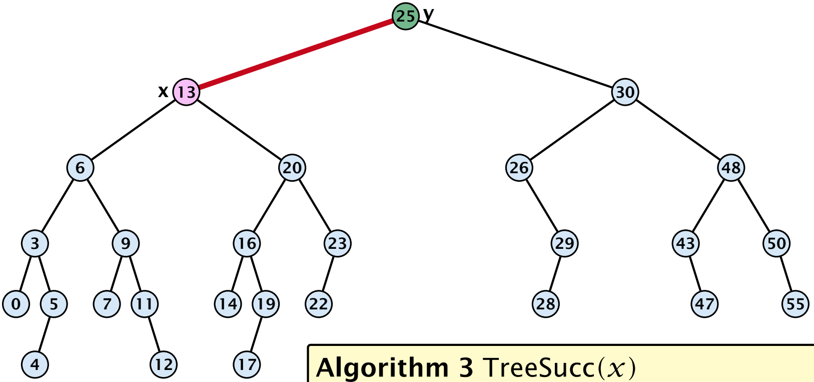

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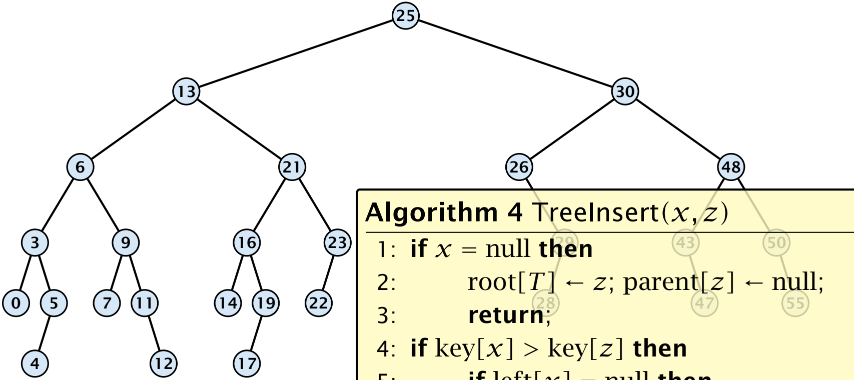
Binary Search Trees: Insert



Algorithm 4 TreeInsert(x, z)

if x = null **then**
    root$[T]$ ← z; parent$[z]$ ← null;
    return;
if key$[x]$ > key$[z]$ **then**
    if left$[x]$ = null **then**
        left$[x]$ ← z; parent$[z]$ ← x;
    else TreeInsert(left$[x]$, z);
else
    if right$[x]$ = null **then**
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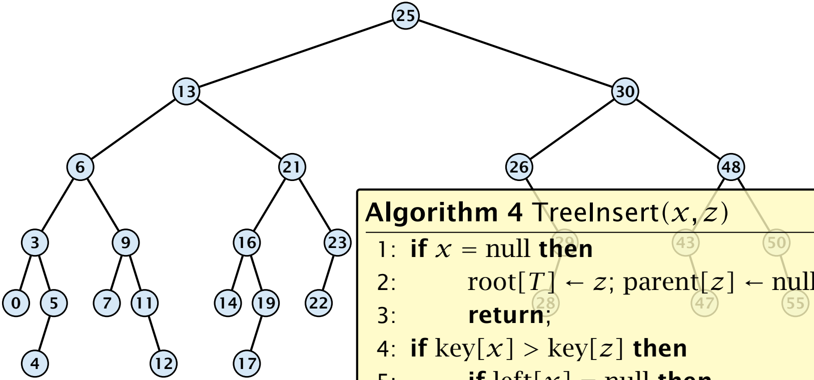
Binary Search Trees: Insert

Insert element **not** in the tree.



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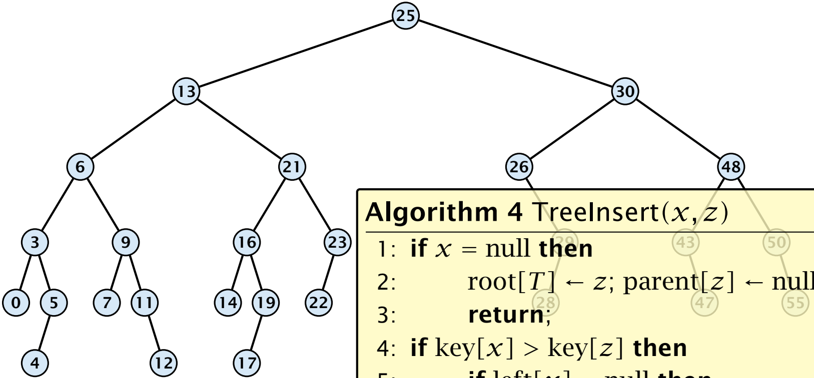

Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

Algorithm 4 TreeInsert(x, z)

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Binary Search Trees: Insert

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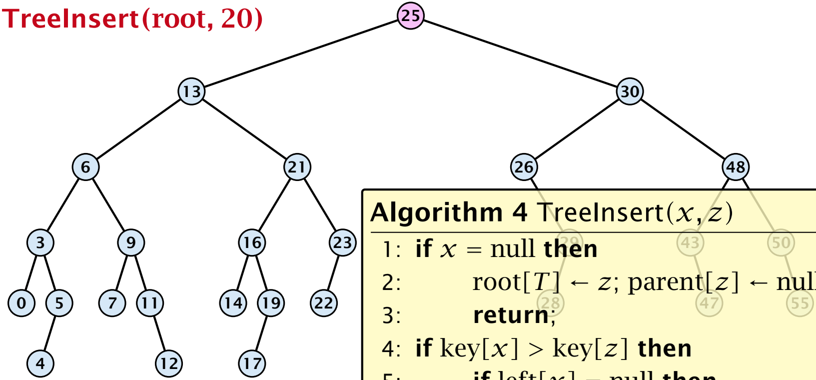
TreeInsert(root, 20)



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Binary Search Trees: Insert

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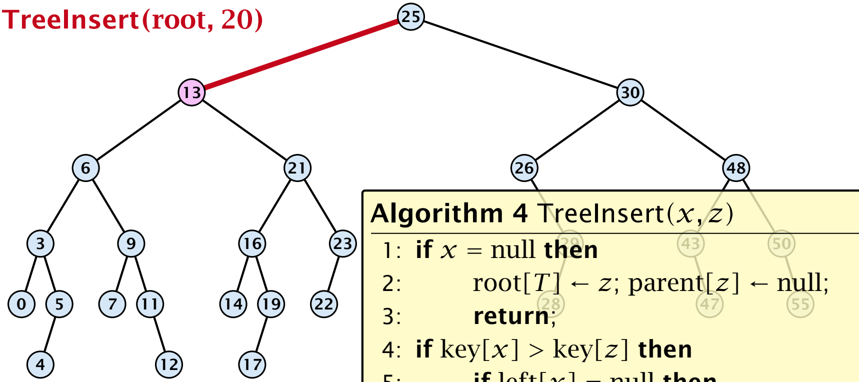

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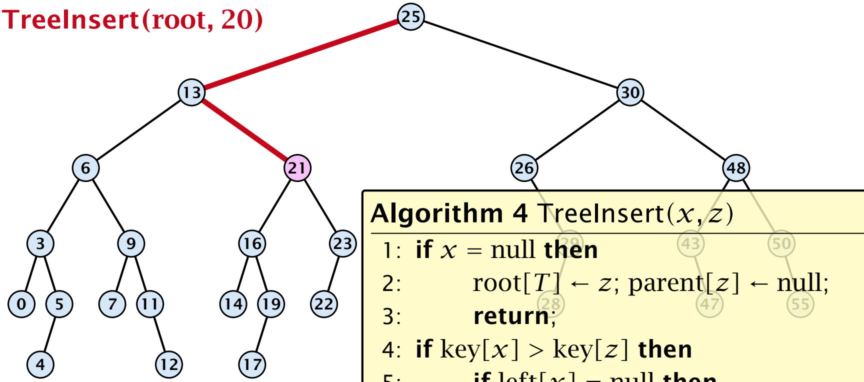

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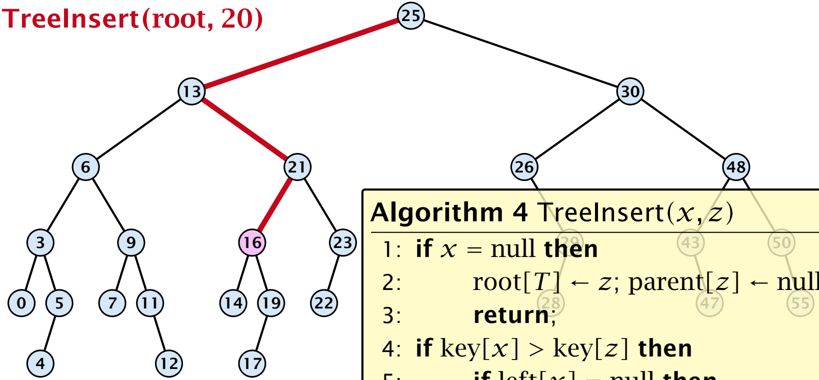

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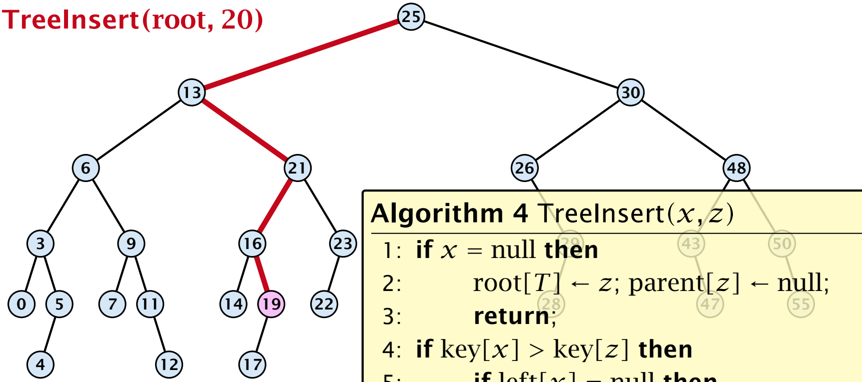

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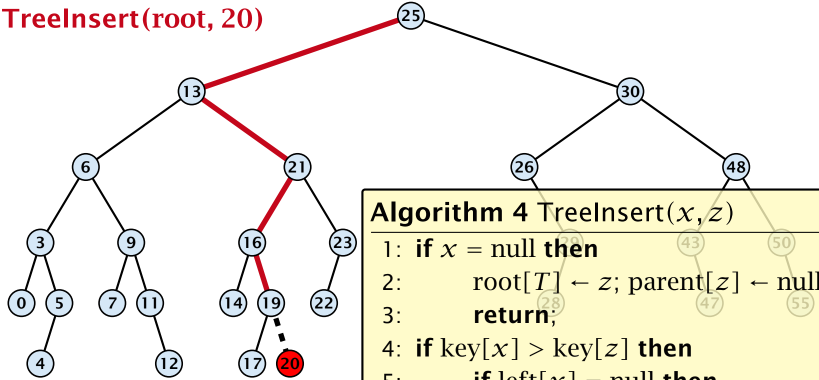
if key$[x] >$ key$[z]$ **then**
    if left$[x]$ = null **then**
        left$[x] \leftarrow z$; parent$[z] \leftarrow x$;
    else TreeInsert(left$[x], z$);
else
    if right$[x]$ = null **then**
        right$[x] \leftarrow z$; parent$[z] \leftarrow x$;
    else TreeInsert(right$[x], z$);

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

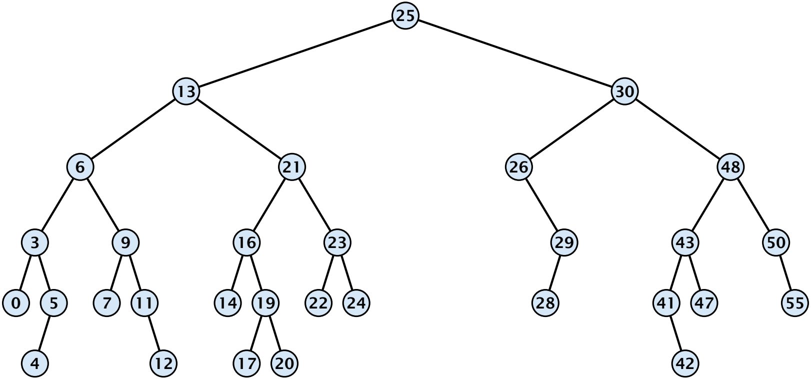


Search for z. At some point the search stops at a null-pointer. This is the place to insert z.

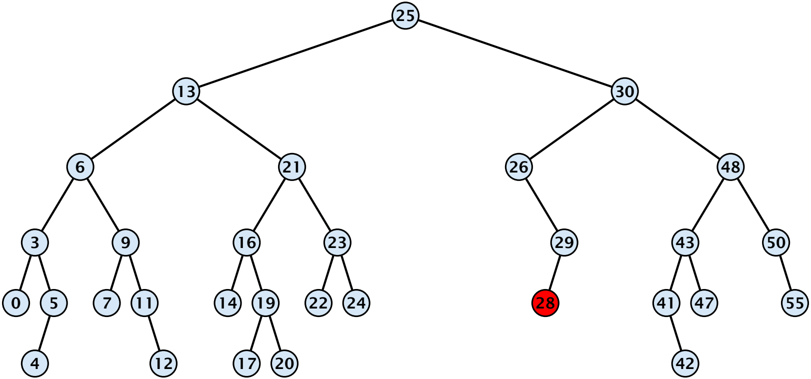
Algorithm 4 TreeInsert(x, z)

```
if x = null then
    root[T] ← z; parent[z] ← null;
    return;
if key[x] > key[z] then
    if left[x] = null then
        left[x] ← z; parent[z] ← x;
    else TreeInsert(left[x], z);
else
    if right[x] = null then
        right[x] ← z; parent[z] ← x;
    else TreeInsert(right[x], z);
```

Binary Search Trees: Delete



Binary Search Trees: Delete

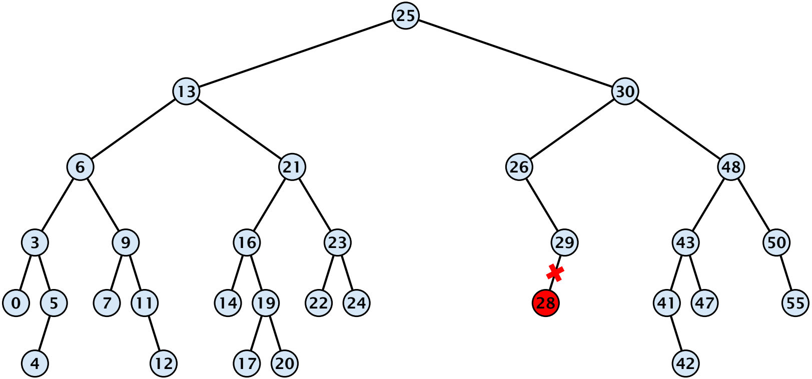


Case 1:

Element does not have any children

- Simply go to the parent and set the corresponding pointer to null.

Binary Search Trees: Delete



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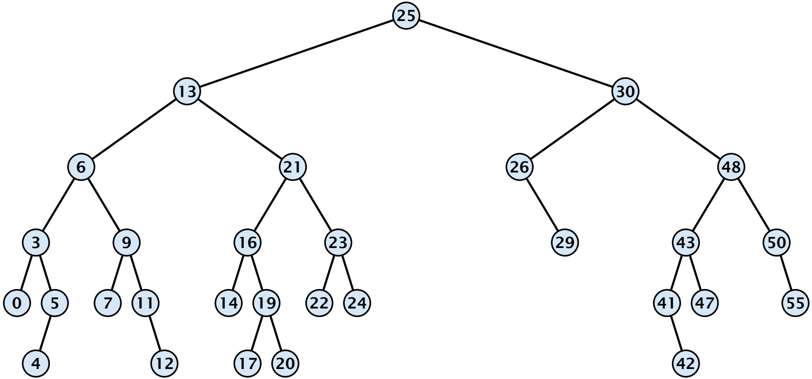

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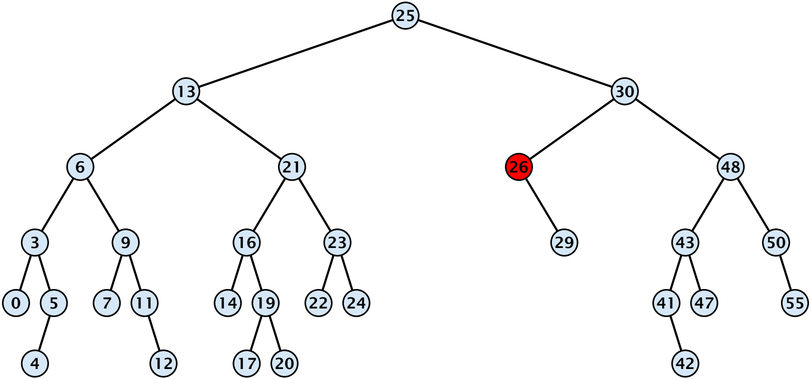
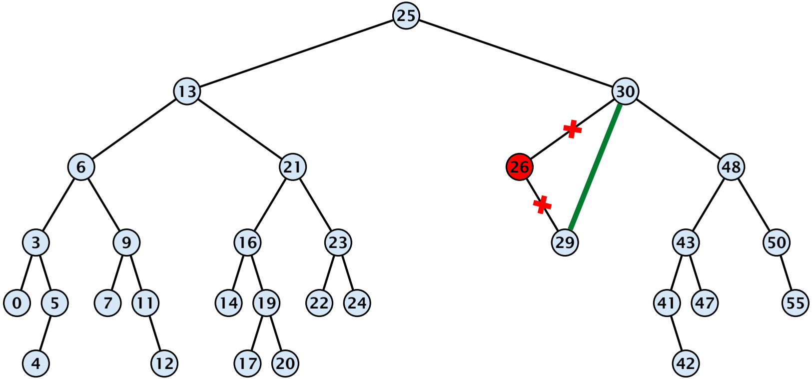


Case 2:

Element has exactly one child

- Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete



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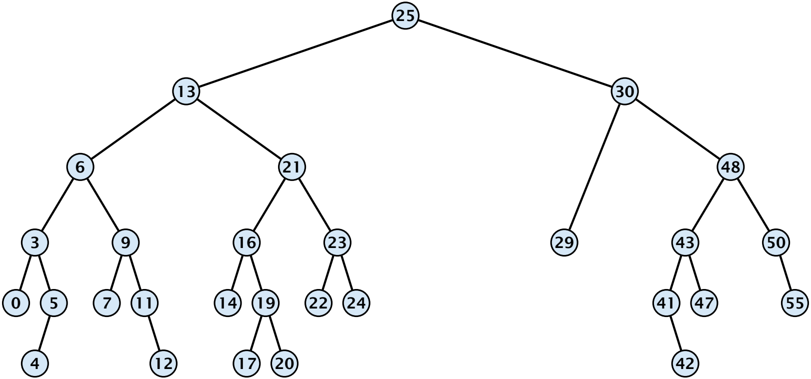

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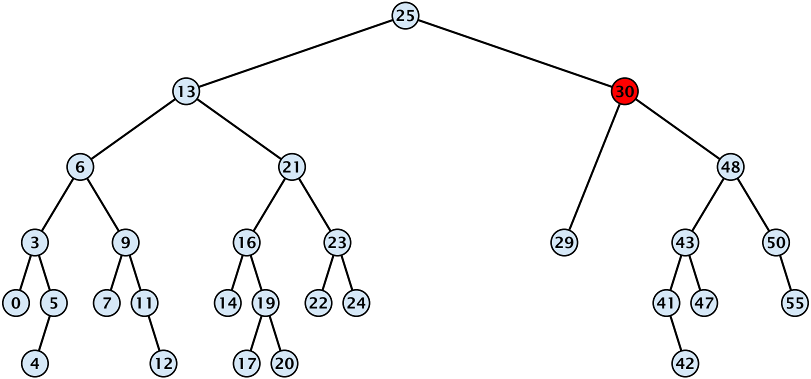
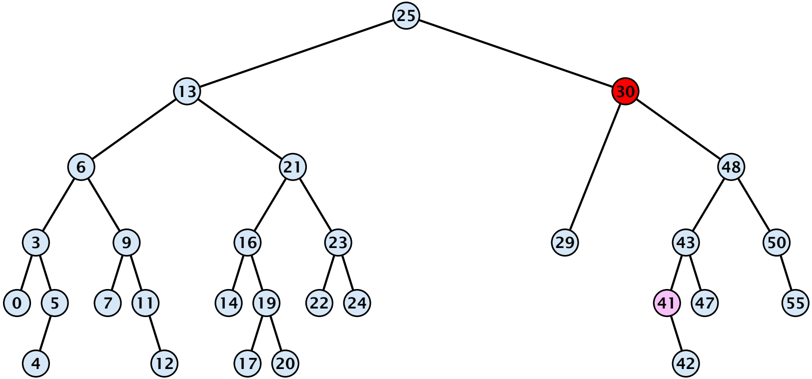


Case 3:

Element has two children

- Find the successor of the element
- Splice successor out of the tree
- Replace content of element by content of successor

Binary Search Trees: Delete



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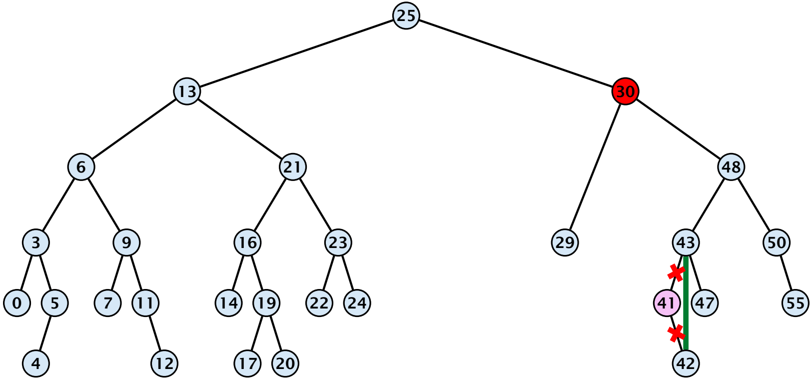

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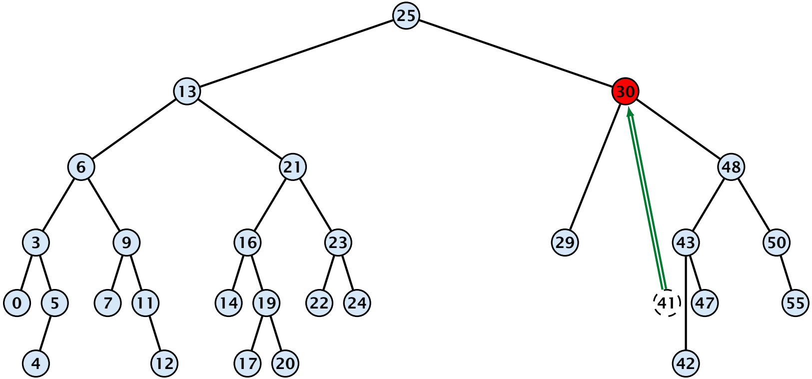

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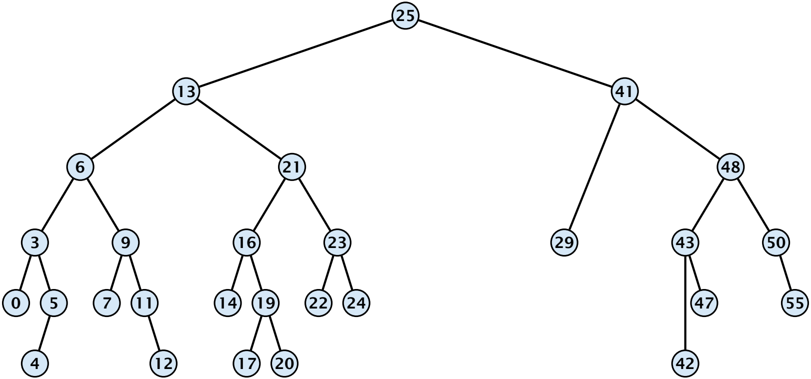

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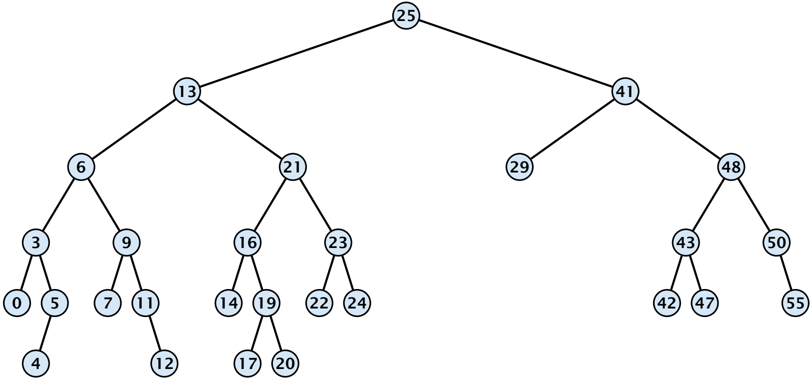

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Element has two children

- Find the successor of the element
- Splice successor out of the tree
- Replace content of element by content of successor

Binary Search Trees: Delete

Algorithm 5 TreeDelete(z)

if left[z] = null **or** right[z] = null
    then $y \leftarrow z$ **else** $y \leftarrow$ TreeSucc(z); select y to splice out
if left[y] $\neq$ null
    then $x \leftarrow$ left[y] **else** $x \leftarrow$ right[y]; x is child of y (or null)
if $x \neq$ null **then** parent[x] $\leftarrow$ parent[y]; parent[x] is correct
if parent[y] = null **then**
    root[T] $\leftarrow x$
else
    if y = left[parent[y]] **then**
        left[parent[y]] $\leftarrow x$
    else
        right[parent[y]] $\leftarrow x$
if $y \neq z$ **then** copy y-data to z

(lines 6–12: fix pointer to x)

Balanced Binary Search Trees

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Balanced Binary Search Trees
With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.