

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

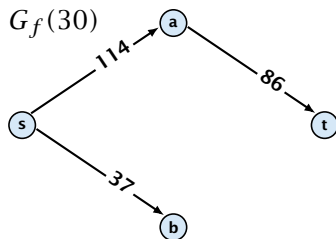
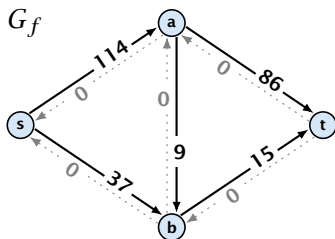
Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter Δ .
- ▶ $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ .



Capacity Scaling

Algorithm 1 $\text{maxflow}(G, s, t, c)$

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
```

Capacity Scaling

Assumption:

All capacities are integers between 1 and C .

Invariant:

All flows and capacities are/remain integral throughout the algorithm.

Correctness:

The algorithm computes a maxflow:

- ▶ because of integrality we have $G_f(1) = G_f$
- ▶ therefore after the last phase there are no augmenting paths anymore
- ▶ this means we have a maximum flow.

Capacity Scaling

Lemma 59

There are $\lceil \log C \rceil + 1$ iterations over Δ .

Proof: obvious.

Lemma 60

Let f be the flow at the end of a Δ -phase. Then the maximum flow is smaller than $\text{val}(f) + m\Delta$.

Proof: less obvious, but simple:

- ▶ There must exist an s - t cut in $G_f(\Delta)$ of zero capacity.
- ▶ In G_f this cut can have capacity at most $m\Delta$.
- ▶ This gives me an upper bound on the flow that I can still add.

Capacity Scaling

Lemma 61

There are at most $2m$ augmentations per scaling-phase.

Proof:

- ▶ Let f be the flow at the end of the previous phase.
- ▶ $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$
- ▶ Each augmentation increases flow by Δ .

Theorem 62

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.