

## 7.5 Skip Lists

**Why do we not use a list for implementing the ADT Dynamic Set?**

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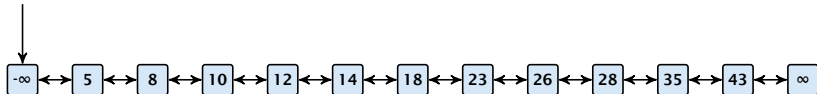
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- ▶ time for search  $\Theta(n)$
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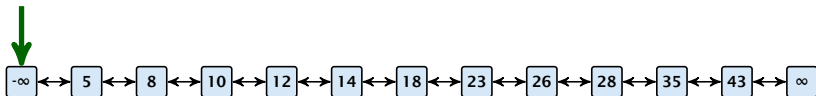
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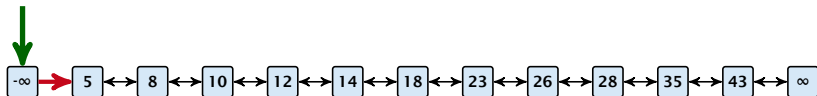
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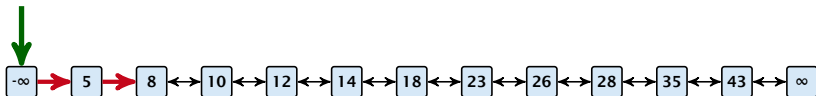
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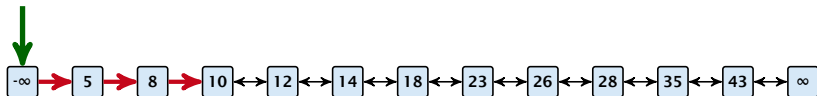
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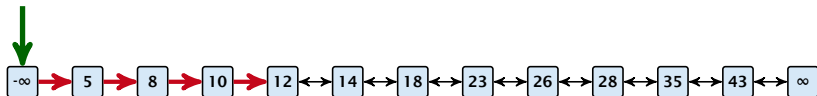
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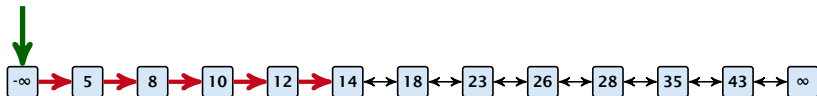




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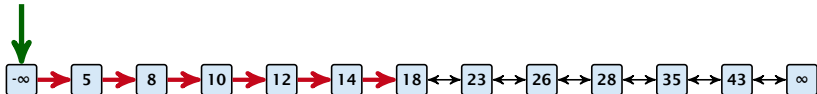
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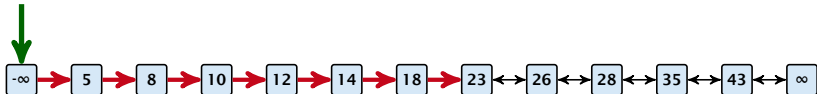
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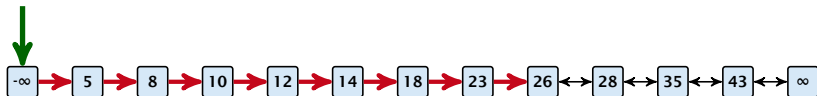
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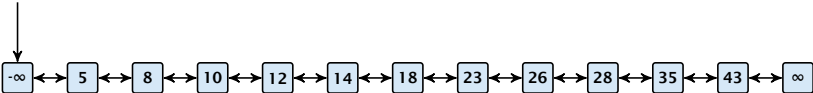
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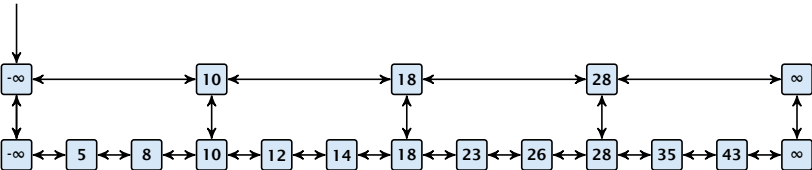
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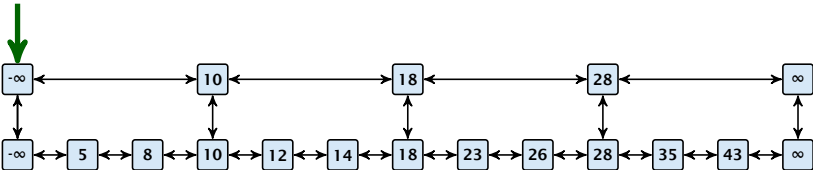




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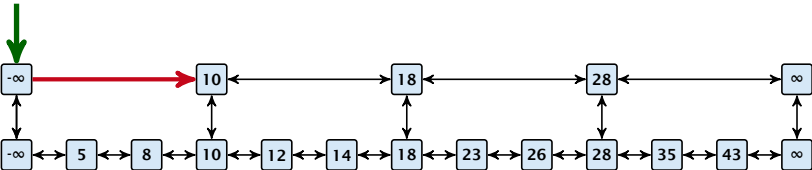
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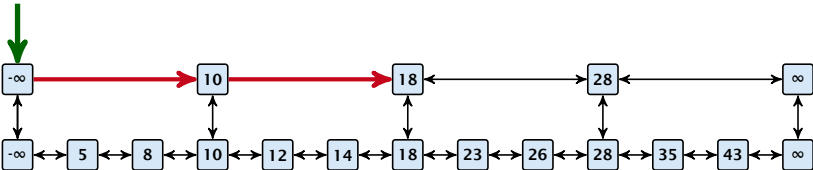
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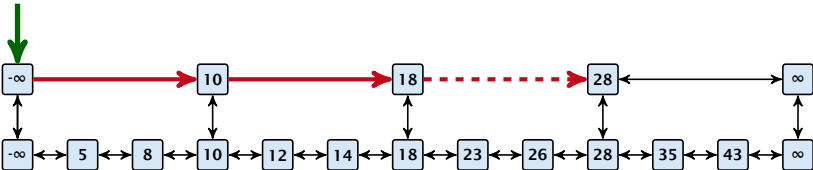
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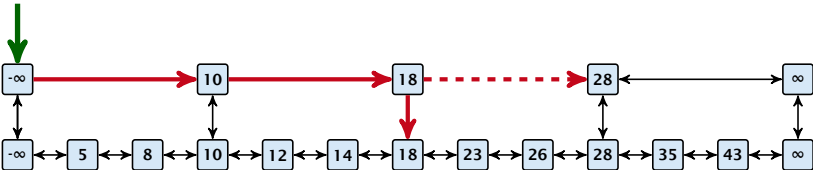
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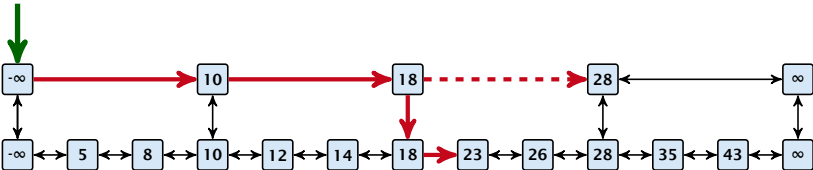
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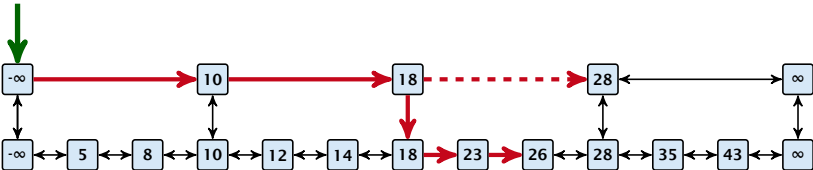
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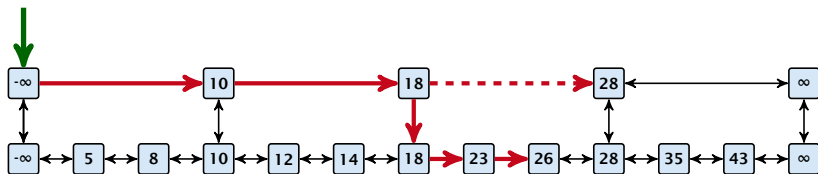
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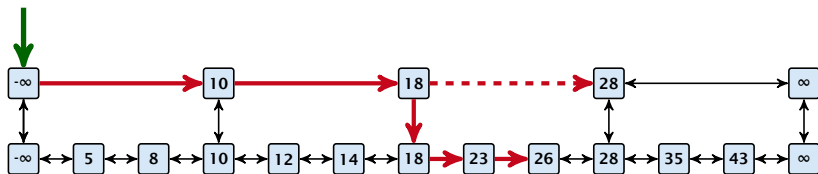
Let  $|L_1|$  denote the number of elements in the “express lane”, and  $|L_0| = n$  the number of all elements (ignoring dummy elements).



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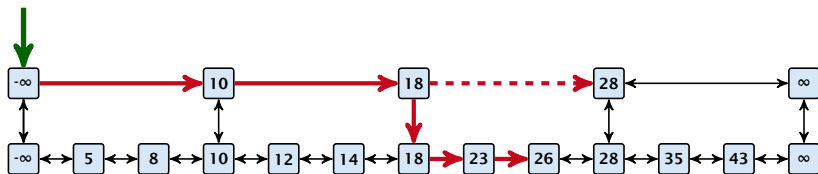
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Choose  $|L_1| = \sqrt{n}$ . Then search time  $\Theta(\sqrt{n})$ .

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Add more express lanes. Lane  $L_i$  contains roughly every  $\frac{L_{i-1}}{L_i}$ -th item from list  $L_{i-1}$ .

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- ▶ At most  $|L_k| + \sum_{i=1}^k \frac{L_{i-1}}{L_i} + 3(k + 1)$  steps.

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Choosing  $k = \Theta(\log n)$  gives a logarithmic running time.



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**Use randomization instead!**

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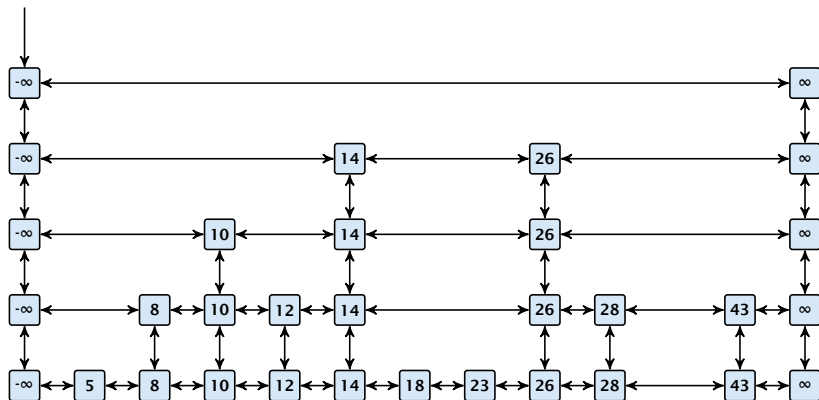
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- ▶ You get all predecessors via backward pointers.
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**The time for both operations is dominated by the search time.**

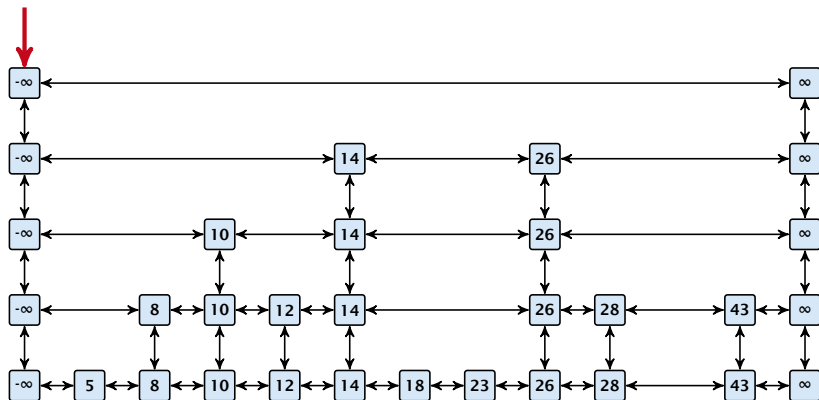
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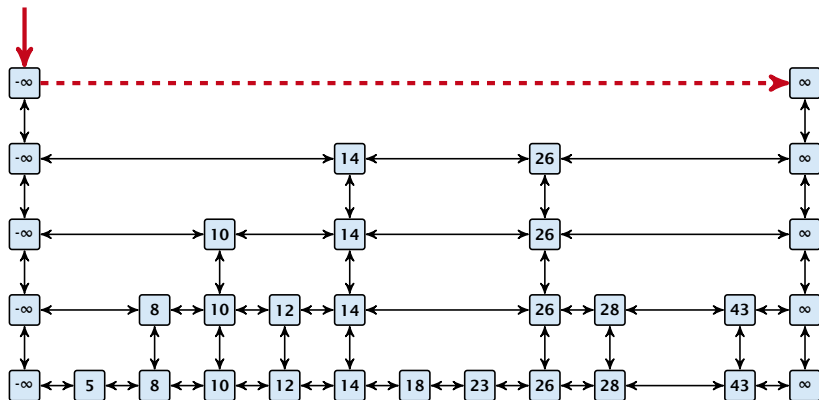
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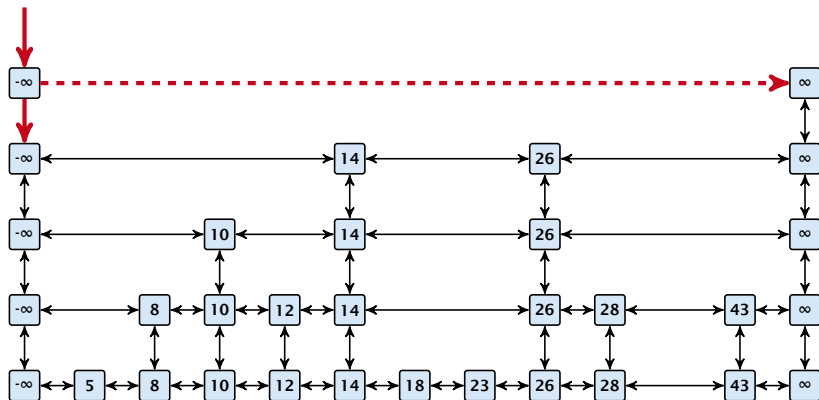
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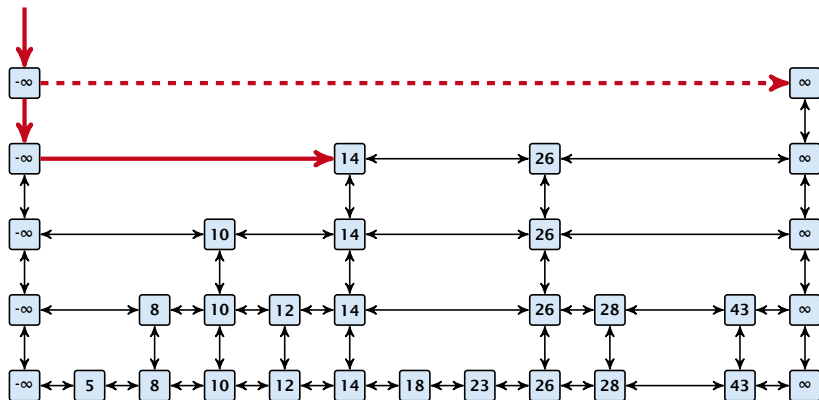
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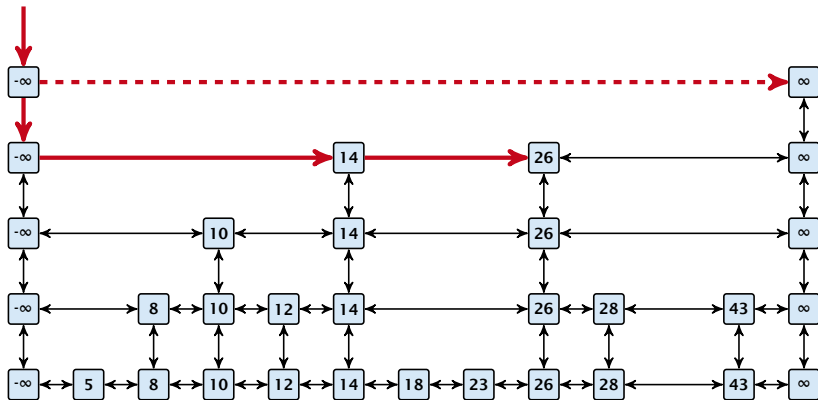
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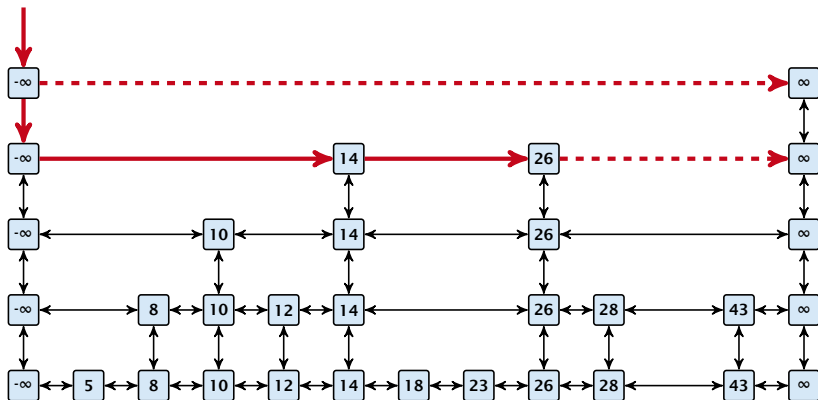
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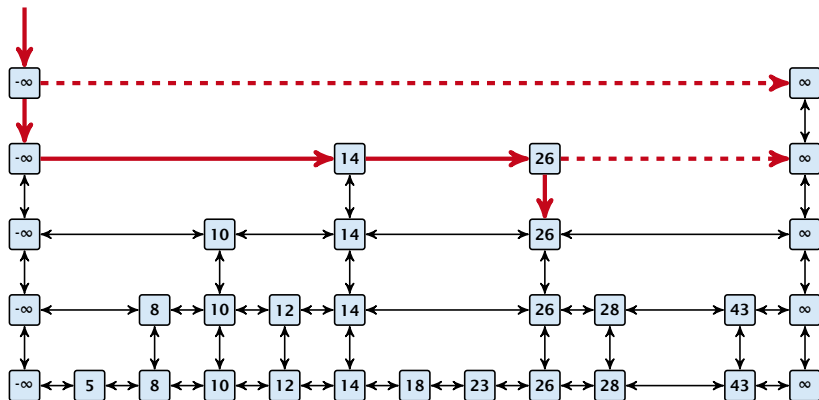
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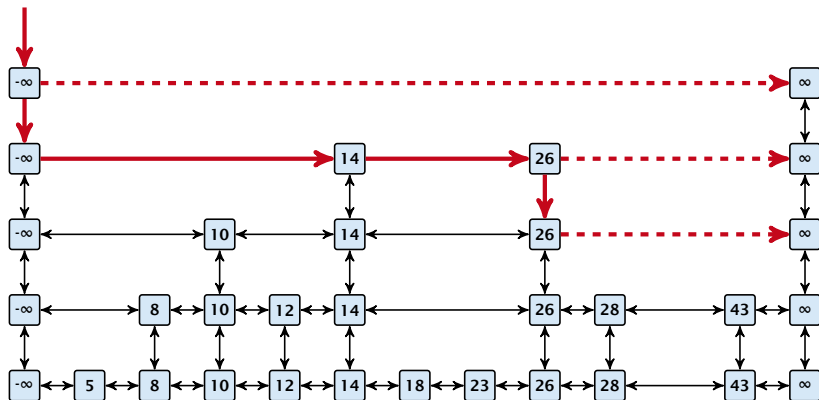
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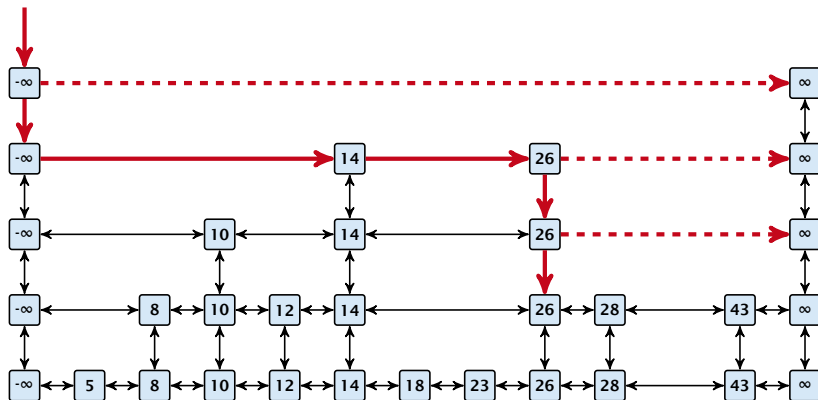
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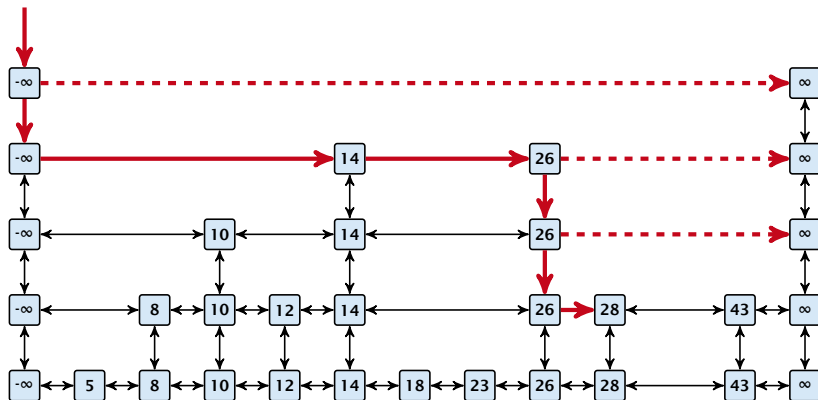
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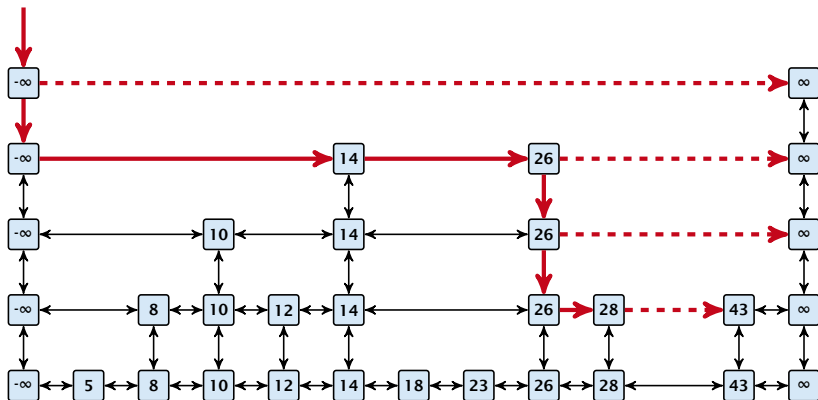
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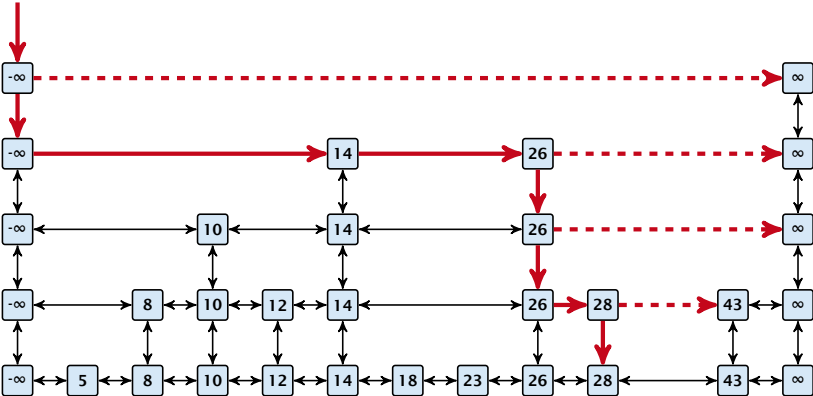
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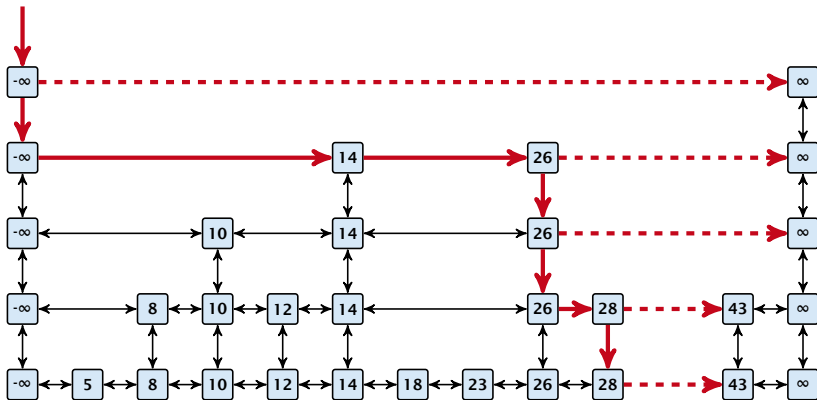
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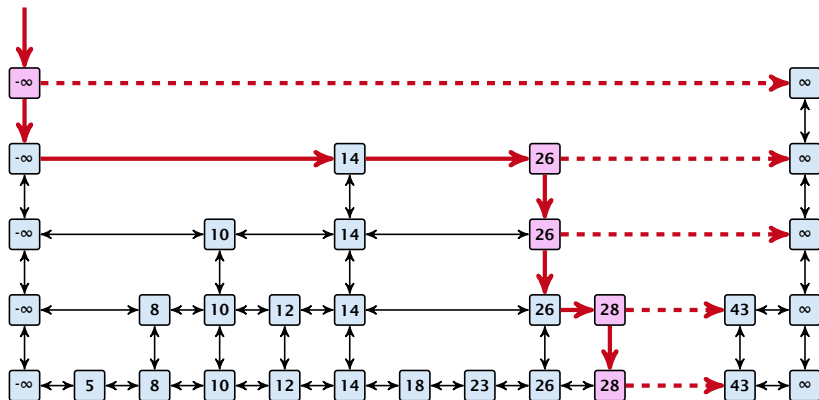
## 7.5 Skip Lists

Insert (35):



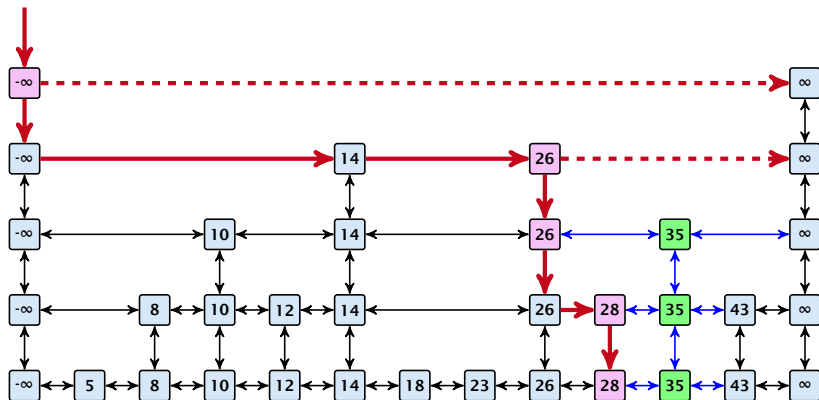
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# High Probability

## Definition 18 (High Probability)

We say a **randomized** algorithm has running time  $\mathcal{O}(\log n)$  with **high probability** if for any constant  $\alpha$  the running time is at most  $\mathcal{O}(\log n)$  with probability at least  $1 - \frac{1}{n^\alpha}$ .

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Here the  $\mathcal{O}$ -notation hides a constant that may depend on  $\alpha$ .

# High Probability

Suppose there are **polynomially** many events  $E_1, E_2, \dots, E_\ell$ ,  $\ell = n^c$  each holding with high probability (e.g.  $E_i$  may be the event that the  $i$ -th search in a skip list takes time at most  $\mathcal{O}(\log n)$ ).

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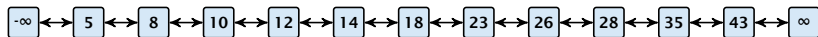
## 7.5 Skip Lists

### Lemma 19

*A search (and, hence, also insert and delete) in a skip list with  $n$  elements takes time  $\mathcal{O}(\log n)$  with high probability (w. h. p.).*

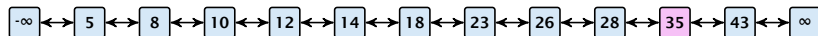
## 7.5 Skip Lists

Backward analysis:



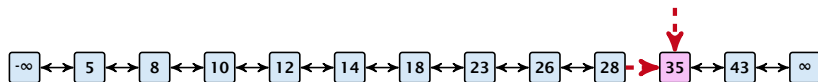
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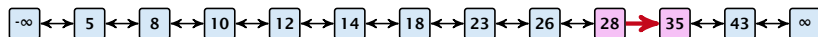
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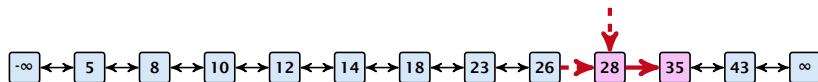
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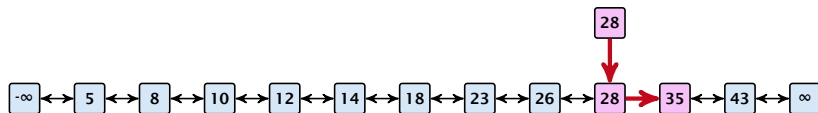
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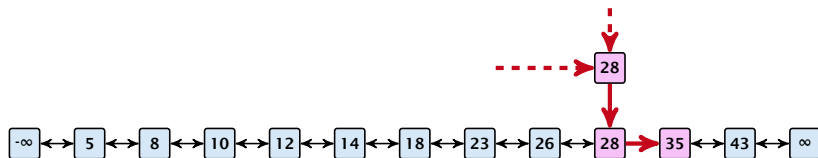
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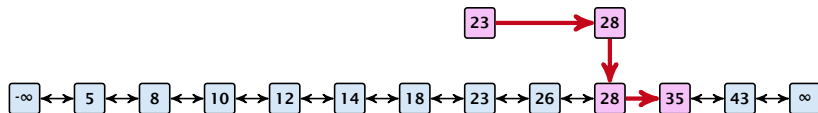
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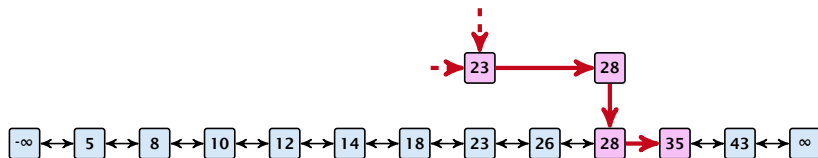
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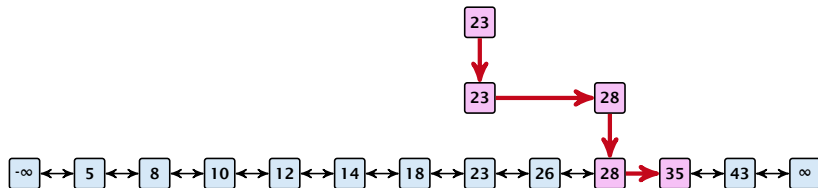
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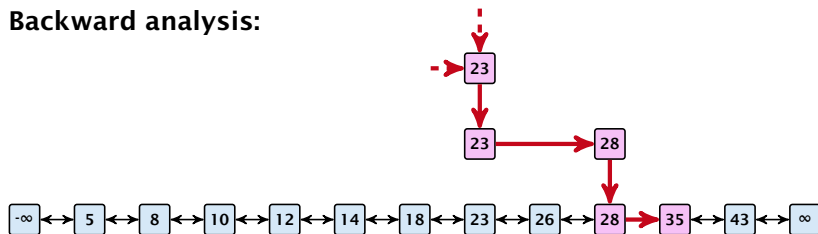
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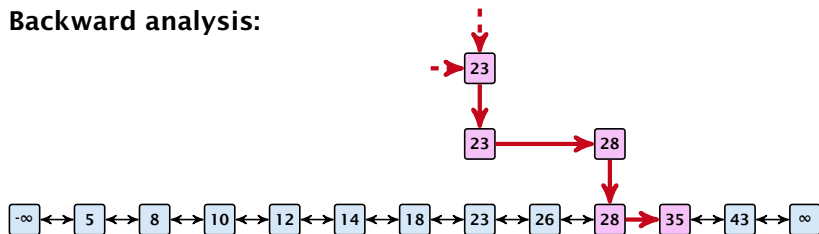
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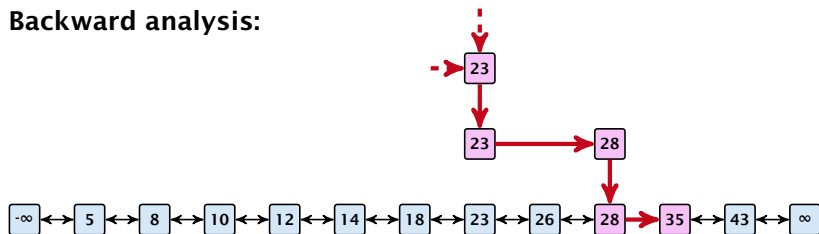


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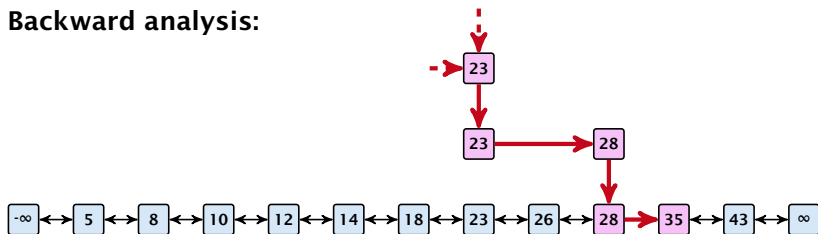
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We show that w.h.p:

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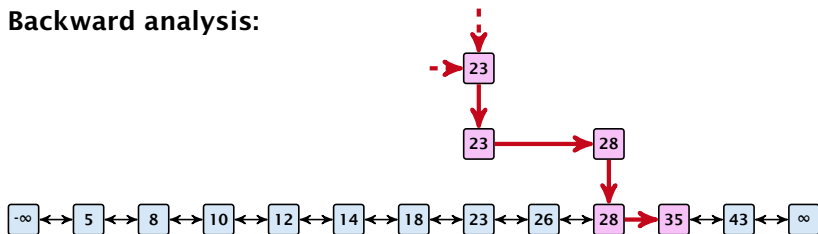
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We show that w.h.p:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

## 7.5 Skip Lists

### Estimation for Binomial Coefficients

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

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## 7.5 Skip Lists



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In particular, this means that during the construction in the backward analysis we see at most  $k$  heads (i.e., coin flips that tell you to go up) in  $z$  trials.

## 7.5 Skip Lists

$$\Pr[E_{z,k}]$$

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

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For the search to take at least  $z = 7\alpha\gamma \log n$  steps either the event  $E_{z,k}$  or the event  $A_{k+1}$  must hold.

Hence,

$$\Pr[\text{search requires } z \text{ steps}]$$

## 7.5 Skip Lists

So far we fixed  $k = \gamma \log n$ ,  $\gamma \geq 1$ , and  $z = 7\alpha\gamma \log n$ ,  $\alpha \geq 1$ .

This means that a search path of length  $\Omega(\log n)$  visits a list on a level  $\Omega(\log n)$ , w.h.p.

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This means, the search requires at most  $z$  steps, w. h. p.