Part II

Linear Programming

Brewery Problem

Brewery brews ale and beer.

- Production limited by supply of corn, hops and barley malt
- Recipes for ale and beer require different amounts of resources

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

Brewery Problem

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

How can brewer maximize profits?

- only brew ale: 34 barrels of ale $\Longrightarrow 442$ €
- only brew beer: 32 barrels of beer $\Longrightarrow 736$ €
- 7.5 barrels ale, 29.5 barrels beer $\Longrightarrow 776$ €
- 12 barrels ale, 28 barrels beer $\Longrightarrow 800$ €

Brewery Problem

Linear Program

- Introduce variables a and b that define how much ale and beer to produce.
- Choose the variables in such a way that the objective function (profit) is maximized.
- Make sure that no constraints (due to limited supply) are violated.

$$\begin{array}{lrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \leq 480 \\ & 4a & + & 4b & \leq 160 \\ & 35a & + & 20b & \leq 1190 \\ & & & a,b & \geq 0 \end{array}$$

Standard Form LPs

LP in standard form:

- input: numbers a_{ij}, c_j, b_i
- output: numbers x_j
- n = #decision variables, m = #constraints
- maximize linear objective function subject to linear (in)equalities

$$\begin{array}{lrcll} \max & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\ & x_j & \ge & 0 & 1 \le j \le n \end{array}$$

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

Standard Form LPs

Original LP

$$\begin{array}{lrcrl} \max & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a, b & \ge 0 \end{array}$$

Standard Form

Add a slack variable to every constraint.

$$\begin{array}{lrcrcccccccl} \max & 13a & + & 23b & & & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & & & = 480 \\ & 4a & + & 4b & & & + & s_h & & & & = 160 \\ & 35a & + & 20b & & & & & + & s_m & & = 1190 \\ & a & , & b & , & s_c & , & s_h & , & s_m & & \ge 0 \end{array}$$

Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

standard maximization form

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & \le & b \\ & x & \ge & 0 \end{array}$$

standard minimization form

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \ge & b \\ & x & \ge & 0 \end{array}$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **less or equal to equality:**

$$a - 3b + 5c \le 12 \implies \begin{array}{r} a - 3b + 5c + s = 12 \\ s \ge 0 \end{array}$$

- **greater or equal to equality:**

$$a - 3b + 5c \ge 12 \implies \begin{array}{r} a - 3b + 5c - s = 12 \\ s \ge 0 \end{array}$$

- **min to max:**

$$\min a - 3b + 5c \implies \max -a + 3b - 5c$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **equality to less or equal:**

$$a - 3b + 5c = 12 \implies \begin{array}{r} a - 3b + 5c \le 12 \\ -a + 3b - 5c \le -12 \end{array}$$

- **equality to greater or equal:**

$$a - 3b + 5c = 12 \implies \begin{array}{r} a - 3b + 5c \ge 12 \\ -a + 3b - 5c \ge -12 \end{array}$$

- **unrestricted to nonnegative:**

$$x \text{ unrestricted} \implies x = x^+ - x^-,\ x^+ \ge 0,\ x^- \ge 0$$

Standard Form LPs

Observations:

- a linear program does not contain x^2, $\cos(x)$, etc.
- transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \ge 0$, $c^T x \ge \alpha$?

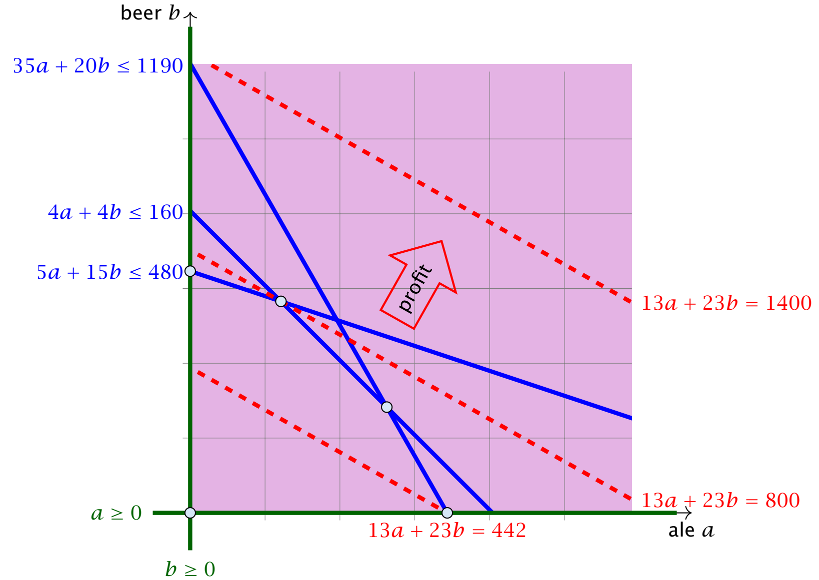
Questions:

- Is LP in NP?
- Is LP in co-NP?
- Is LP in P?

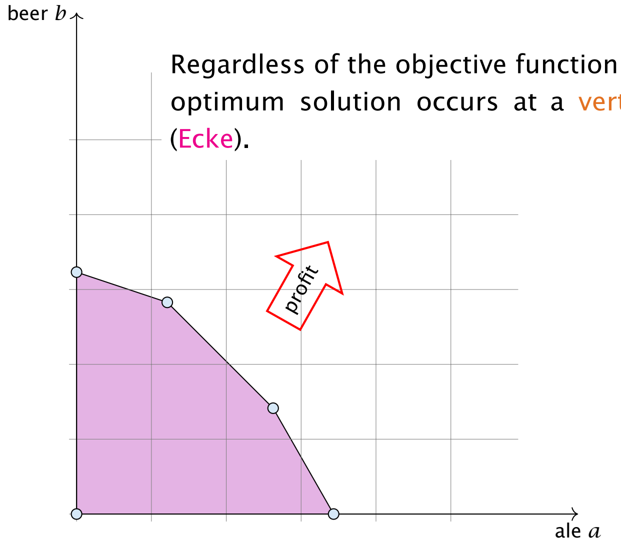
Input size:

- n number of variables, m constraints, L number of bits to encode the input

Geometry of Linear Programming



Geometry of Linear Programming



Regardless of the objective function an optimum solution occurs at a vertex (Ecke).

Definitions

Let for a Linear Program in standard form $P = \{x \mid Ax = b, x \geq 0\}$.

- P is called the feasible region (Lösungsraum) of the LP.
- A point $x \in P$ is called a feasible point (gültige Lösung).
- If $P \neq \emptyset$ then the LP is called feasible (erfüllbar). Otherwise, it is called infeasible (unerfüllbar).
- An LP is bounded (beschränkt) if it is feasible and
  - $c^T x < \infty$ for all $x \in P$ (for maximization problems)
  - $c^T x > -\infty$ for all $x \in P$ (for minimization problems)

Definition 2

Given vectors/points $x_1, \ldots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- linear combination if $\lambda_i \in \mathbb{R}$.
- affine combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- convex combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
- conic combination if $\lambda_i \in \mathbb{R}$ and $\lambda_i \geq 0$.

Note that a combination involves only finitely many vectors.

Definition 3

A set $X \subseteq \mathbb{R}^n$ is called

- a linear subspace if it is closed under linear combinations.
- an affine subspace if it is closed under affine combinations.
- convex if it is closed under convex combinations.
- a convex cone if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

Definition 4
Given a set $X \subseteq \mathbb{R}^n$.

- $\mathrm{span}(X)$ is the set of all linear combinations of X (linear hull, span)
- $\mathrm{aff}(X)$ is the set of all affine combinations of X (affine hull)
- $\mathrm{conv}(X)$ is the set of all convex combinations of X (convex hull)
- $\mathrm{cone}(X)$ is the set of all conic combinations of X (conic hull)

Definition 5
A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

Lemma 6
If $P \subseteq \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ convex then also

$$Q = \{x \in P \mid f(x) \le t\}$$

Dimensions

Definition 7
The dimension $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

Definition 8
The dimension $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension of its affine hull $\mathrm{aff}(X)$.

Definition 9
A set $H \subseteq \mathbb{R}^n$ is a hyperplane if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

Definition 10
A set $H' \subseteq \mathbb{R}^n$ is a (closed) halfspace if $H = \{x \mid a^T x \le b\}$, for $a \neq 0$.

Definitions

Definition 11
A polytop is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a finite set of points, i.e., $P = \mathrm{conv}(X)$ where $|X| = c$.

Definitions

Definition 12
A polyhedron is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of finitely many half-spaces $\{H(a_1, b_1), \dots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} \ .$$

Definition 13
A polyhedron P is bounded if there exists B s.t. $\|x\|_2 \leq B$ for all $x \in P$.

Definitions

Theorem 14
P is a bounded polyhedron iff P is a polytop.

Definition 15
Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a, b) = \{x \in \mathbb{R}^n \mid a^T x = b\}$$

is a supporting hyperplane of P if $\max\{a^T x \mid x \in P\} = b$.

Definition 16
Let $P \subseteq \mathbb{R}^n$. F is a face of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H.

Definition 17
Let $P \subseteq \mathbb{R}^n$.

- a face v is a vertex of P if $\{v\}$ is a face of P.
- a face e is an edge of P if e is a face and $\dim(e) = 1$.
- a face F is a facet of P if F is a face and $\dim(F) = \dim(P) - 1$.

Equivalent definition for vertex:

Definition 18
Given polyhedron P. A point $x \in P$ is a vertex if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

Definition 19
Given polyhedron P. A point $x \in P$ is an extreme point if $\nexists a, b \neq x$, $a, b \in P$, with $\lambda a + (1-\lambda)b = x$ for $\lambda \in [0,1]$.

Lemma 20
A vertex is also an extreme point.

Observation
The feasible region of an LP is a Polyhedron.

Convex Sets

Theorem 21
If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

- suppose x is optimal solution that is not extreme point
- there exists direction $d \neq 0$ such that $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- Wlog. assume $c^T d \geq 0$ (by taking either d or $-d$)
- Consider $x + \lambda d$, $\lambda > 0$

Convex Sets

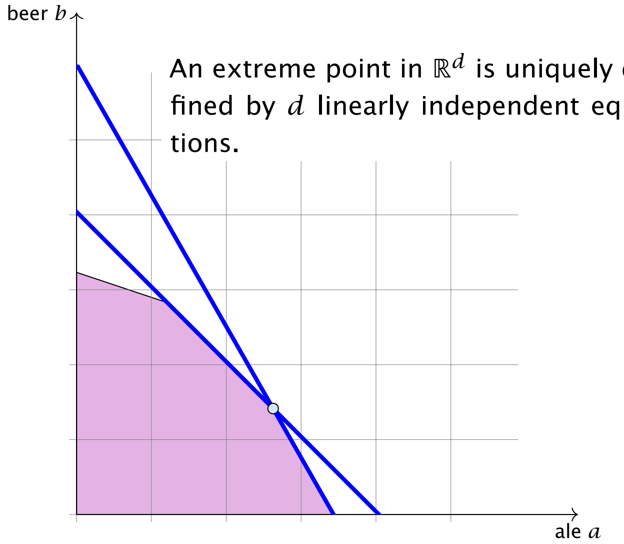
Case 1. [$\exists j$ s.t. $d_j < 0$]

- increase λ to λ' until first component of $x + \lambda d$ hits 0
- $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- $c^T x' = c^T(x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

- $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$
- as $\lambda \to \infty$, $c^T(x + \lambda d) \to \infty$ as $c^T d > 0$

Algebraic View



An extreme point in $\mathbb{R}^d$ is uniquely defined by d linearly independent equations.

Notation

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

Theorem 22

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.*

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*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.*

Proof ($\Leftarrow$)

- assume x is not extreme point
- there exists direction d s.t. $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- define $B' = \{j \mid d_j \neq 0\}$
- $A_{B'}$ has linearly dependent columns as $Ad = 0$
- $d_j = 0$ for all j with $x_j = 0$ as $x \pm d \geq 0$
- Hence, $B' \subseteq B$, $A_{B'}$ is sub-matrix of A_B

Theorem 22

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.*

Proof ($\Rightarrow$)

- assume A_B has linearly dependent columns
- there exists $d \neq 0$ such that $A_B d = 0$
- extend d to $\mathbb{R}^n$ by adding 0-components
- now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$
- for sufficiently small λ we have $x \pm \lambda d \in P$
- hence, x is not extreme point

Theorem 23
Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P.

- define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
- then $c^T x = 0$ and $c^T y \leq 0$ for $y \in P$
- assume $c^T y = 0$; then $y_j = 0$ for all $j \notin B$
- $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
- this means that $x_B = y_B$ since A_B has linearly independent columns
- we get $y = x$
- hence, x is a vertex of P

Observation
For an LP we can assume wlog. that the matrix A has full row-rank. This means $\operatorname{rank}(A) = m$.

- assume that $\operatorname{rank}(A) < m$
- assume wlog. that the first row A_1 lies in the span of the other rows $A_2, \dots, A_m$; this means

$$A_1 = \sum_{i=2}^{m} \lambda_i \cdot A_i, \text{ for suitable } \lambda_i$$

C1 if now $b_1 = \sum_{i=2}^{m} \lambda_i \cdot b_i$ then for all x with $A_i x = b_i$ we also have $A_1 x = b_1$; hence the first constraint is superfluous

C2 if $b_1 \neq \sum_{i=2}^{m} \lambda_i \cdot b_i$ then the LP is infeasible, since for all x that fulfill constraints $A_2, \dots, A_m$ we have

$$A_1 x = \sum_{i=2}^{m} \lambda_i \cdot A_i x = \sum_{i=2}^{m} \lambda_i \cdot b_i \neq b_1$$

From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

Theorem 24
Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- A_B is non-singular
- $x_B = A_B^{-1} b \geq 0$
- $x_N = 0$

where $N = \{1, \dots, n\} \setminus B$.

Proof
Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\operatorname{rank}(A) = m$.

Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and $\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a basic **feasible** solution (gültige Basislösung) if in addition $x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \dots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x_B = b$ and $x_j = 0$ for all $j \notin B$ is the basic solution associated to basis B (die zu B assoziierte Basislösung)

Basic Feasible Solutions

A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i's are zero. The corresponding non-negativity constraint is fulfilled with equality.

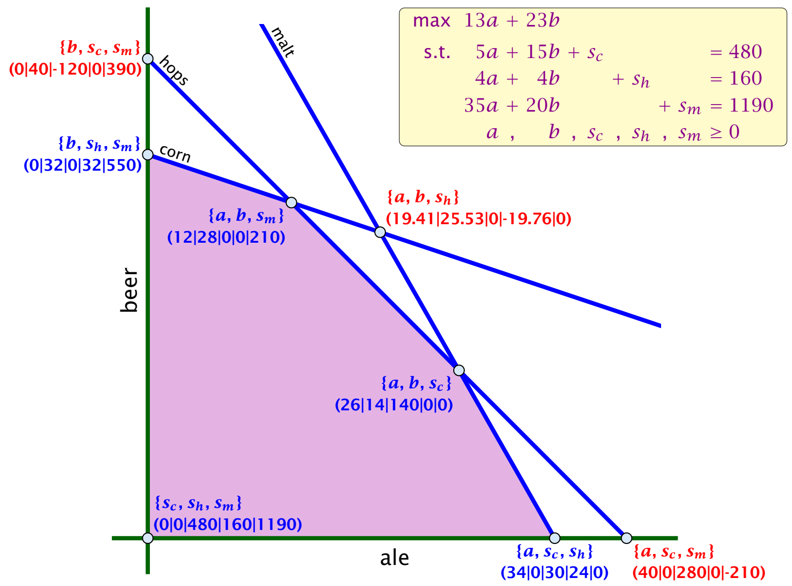
Fact:
In a BFS at least n constraints are fulfilled with equality.

Basic Feasible Solutions

Definition 25

For a general LP ($\max\{c^T x \mid Ax \leq b\}$) with n variables a point x is a basic feasible solution if x is feasible and there exist n (linearly independent) constraints that are tight.

Algebraic View



Fundamental Questions

Linear Programming Problem (LP)
Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:
- Is LP in NP? yes!
- Is LP in co-NP?
- Is LP in P?

Proof:
- Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

Observation
We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \mathrm{poly}(n,m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

What happens if LP is unbounded?

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

4 Simplex Algorithm

$$\begin{array}{lrcrcrcrcrcl} \text{max} & 13a & + & 23b & & & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\ & 4a & + & 4b & & & + & s_h & & & = & 160 \\ & 35a & + & 20b & & & & & + & s_m & = & 1190 \\ & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0 \end{array}$$

$$\begin{array}{rcrcrcrcrcrcl} \multicolumn{13}{l}{\text{max } Z} \\ 13a & + & 23b & & & & & & & - & Z & = & 0 \\ 5a & + & 15b & + & s_c & & & & & & & = & 480 \\ 4a & + & 4b & & & + & s_h & & & & & = & 160 \\ 35a & + & 20b & & & & & + & s_m & & & = & 1190 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

Pivoting Step

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcccccl} 13a & + & 23b & & & & & -Z & = 0 \\ 5a & + & 15b & + & s_c & & & & = 480 \\ 4a & + & 4b & & & + & s_h & & = 160 \\ 35a & + & 20b & & & & + s_m & & = 1190 \\ a & , & b & , & s_c & , & s_h , s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcccccl} 13a & + & 23\boldsymbol{b} & & & & & -Z & = 0 \\ 5a & + & 15\boldsymbol{b} & + & \boldsymbol{s_c} & & & & = 480 \\ 4a & + & 4\boldsymbol{b} & & & + & s_h & & = 160 \\ 35a & + & 20\boldsymbol{b} & & & & + s_m & & = 1190 \\ a & , & \boldsymbol{b} & , & s_c & , & s_h , s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

- Choose variable with coefficient > 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.
- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.
- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcccccl} 13a & + & 23\boldsymbol{b} & & & & & -Z & = 0 \\ 5a & + & 15\boldsymbol{b} & + & s_c & & & & = 480 \\ 4a & + & 4\boldsymbol{b} & & & + & s_h & & = 160 \\ 35a & + & 20\boldsymbol{b} & & & & + s_m & & = 1190 \\ a & , & \boldsymbol{b} & , & s_c & , & s_h , s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$
$s_c = 480$
$s_h = 160$
$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcrcccl} \frac{16}{3}a & & & - & \frac{23}{15}s_c & & & -Z & = -736 \\ \frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & = 32 \\ \frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & = 32 \\ \frac{85}{3}a & & & - & \frac{4}{3}s_c & & + s_m & & = 550 \\ a & , & b & , & s_c & , & s_h , s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcrcccl} \frac{16}{3}\boldsymbol{a} & & & - & \frac{23}{15}s_c & & & -Z & = -736 \\ \frac{1}{3}\boldsymbol{a} & + & b & + & \frac{1}{15}s_c & & & & = 32 \\ \frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & = 32 \\ \frac{85}{3}\boldsymbol{a} & & & - & \frac{4}{3}s_c & & + s_m & & = 550 \\ \boldsymbol{a} & , & b & , & s_c & , & s_h , s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$
$b = 32$
$s_h = 32$
$s_m = 550$

Choose variable a to bring into basis.
Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.
Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcrcrcccl} & & & - & s_c & - & 2s_h & & & -Z & = -800 \\ & & b & + & \frac{1}{10}s_c & - & \frac{1}{8}s_h & & & & = 28 \\ a & & & - & \frac{1}{10}s_c & + & \frac{3}{8}s_h & & & & = 12 \\ & & & & \frac{3}{2}s_c & - & \frac{85}{8}s_h & + & s_m & & = 210 \\ a & , & b & , & s_c & , & s_h & , & s_m & & \geq 0 \end{array} \end{array}$$

basis $= \{a, b, s_m\}$
$s_c = s_h = 0$
$Z = 800$
$b = 28$
$a = 12$
$s_m = 210$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0, s_h \geq 0$
- hence optimum solution value is at most 800
- the current solution has value 800

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

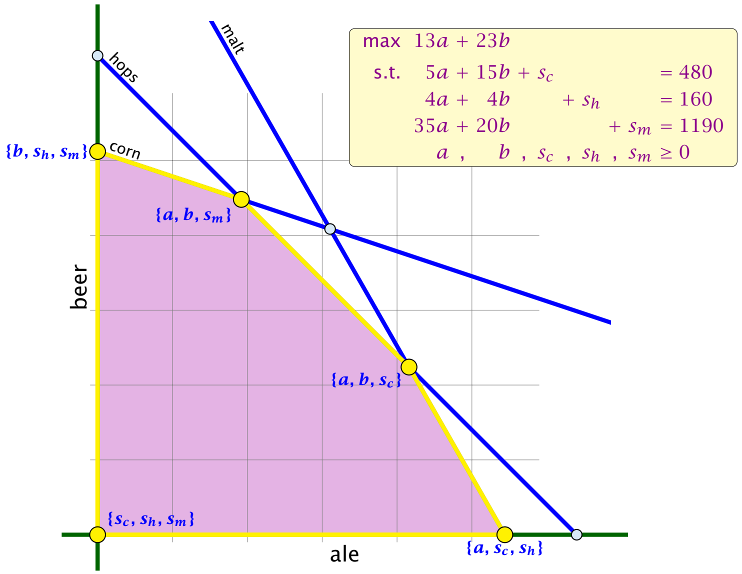
The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Geometric View of Pivoting



Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
  - Other non-basis variables should stay at 0.
  - Basis variables change to maintain feasibility.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

- $d_j = 1$ (normalization)
- $d_\ell = 0$, $\ell \notin B$, $\ell \neq j$
- $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

Algebraic Definition of Pivoting

Definition 26 (j-th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1} A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

Algebraic Definition of Pivoting

Definition 27 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the reduced cost for variable x_j.

Note that this is defined for every j. If $j \in B$ then the above term is 0.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{cccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

4 Simplex Algorithm

Questions:

- What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- How do we find the initial basic feasible solution?
- Is there always a basis B such that

  $$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 \;\; ?$$

  Then we can terminate because we know that the solution is optimal.
- If yes how do we make sure that we reach such a basis?

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

Termination

The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

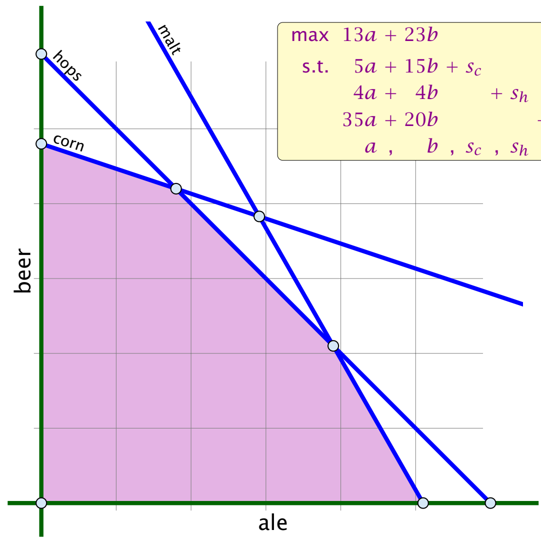
The set of inequalities is degenerate (also the basis is degenerate).

Definition 28 (Degeneracy)

A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

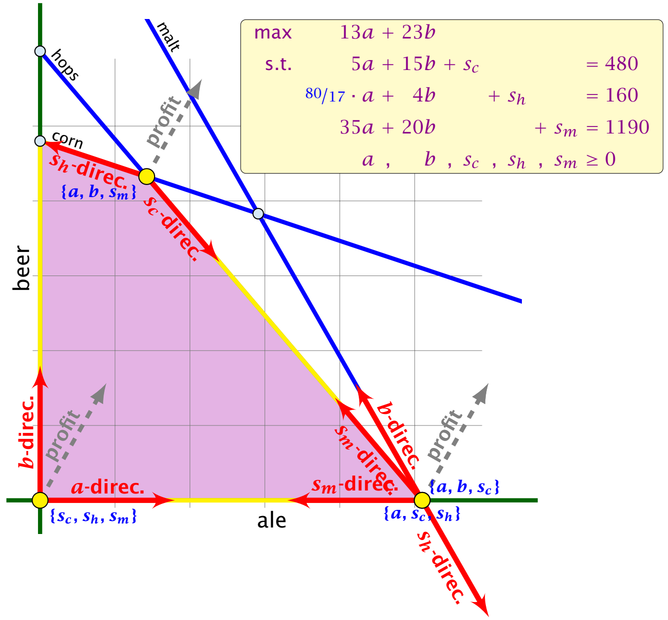
It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

Non Degenerate Example



$$
\begin{array}{llll}
\max & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & & = 480 \\
& 4a + 4b + s_h & & = 160 \\
& 35a + 20b + s_m & & = 1190 \\
& a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & & \geq 0
\end{array}
$$

Degenerate Example



$$
\begin{array}{llr}
\text{max} & 13a + 23b & \\
\text{s.t.} & 5a + 15b + s_c & = 480 \\
 & {}^{80}/_{17} \cdot a + 4b + s_h & = 160 \\
 & 35a + 20b + s_m & = 1190 \\
 & a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m & \geq 0
\end{array}
$$

Summary: How to choose pivot-elements

- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \leq 0$ for all $i \in \{1, \ldots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.
- Depending on the choice of ℓ it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

Termination

What do we have so far?
Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

How do we come up with an initial solution?

- $Ax \leq b, x \geq 0$, and $\boldsymbol{b} \geq \boldsymbol{0}$.
- The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- Then $s = b$, $x = 0$ is a basic feasible solution (how?).
- We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1.
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b$, $x \geq 0$, $v \geq 0$ using Simplex. $x = 0$, $v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

Optimality

Lemma 29
Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.

Duality

How do we get an upper bound to a maximization LP?

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \leq 480 \\ & 4a & + & 4b & \leq 160 \\ & 35a & + & 20b & \leq 1190 \\ & & & a, b & \geq 0 \end{array}$$

Note that a lower bound is easy to derive. Every choice of $a, b \geq 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \geq 0$) such that $\sum_i y_i a_{ij} \geq c_j$ then $\sum_i y_i b_i$ will be an upper bound.

Duality

Definition 30
Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the dual problem.

Duality

Lemma 31
The dual of the dual problem is the primal problem.

Proof:

- $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$
- $w = -\max\{-b^T y \mid -A^T y \leq -c, y \geq 0\}$

The dual problem is

- $z = -\min\{-c^T x \mid -Ax \geq -b, x \geq 0\}$
- $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$

Weak Duality

Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ and
$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \leq b, x \geq 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \geq c, y \geq 0\}$.

Theorem 32 (Weak Duality)
Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T \hat{x} \leq z \leq w \leq b^T \hat{y} \ .$$

Weak Duality

$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \ (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b \ (\hat{y} \geq 0)$

This gives
$$c^T \hat{x} \leq \hat{y}^T A\hat{x} \leq b^T \hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\begin{aligned}&\max\{c^Tx \mid Ax = b, x \geq 0\}\\ &\quad = \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\\ &\quad = \max\{c^Tx \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}\end{aligned}$$

Dual:

$$\begin{aligned}&\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \geq c, y \geq 0\}\\ &\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}\\ &\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\}\\ &\quad = \min\left\{b^Ty' \;\middle|\; A^Ty' \geq c\right\}\end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T(A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^Ty \mid A^Ty \geq c\}$.

$$\begin{aligned}b^Ty^* &= (Ax^*)^Ty^* = (A_Bx_B^*)^Ty^*\\ &= (A_Bx_B^*)^T(A_B^{-1})^Tc_B = (x_B^*)^TA_B^T(A_B^{-1})^Tc_B\\ &= c^Tx^*\end{aligned}$$

Hence, the solution is optimal.

5.3 Strong Duality

$P = \max\{c^Tx \mid Ax \leq b, x \geq 0\}$

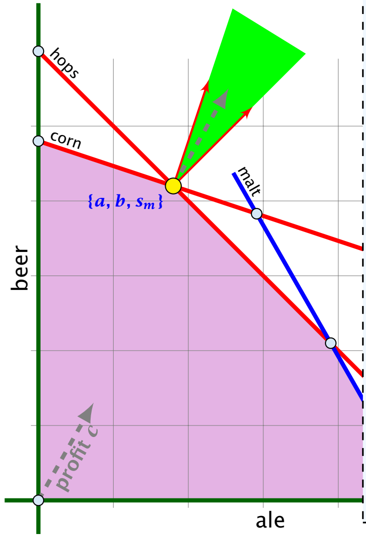
n_A: number of variables, m_A: number of constraints

We can put the non-negativity constraints into A (which gives us unrestricted variables): $\bar{P} = \max\{c^Tx \mid \bar{A}x \leq \bar{b}\}$

$n_{\bar{A}} = n_A$, $m_{\bar{A}} = m_A + n_A$

Dual $D = \min\{\bar{b}^Ty \mid \bar{A}^Ty = c, y \geq 0\}$.

5.3 Strong Duality



If we have a conic combination y of c then b^Ty is an upper bound of the profit we can obtain (weak duality):

$$c^Tx = (\bar{A}^Ty)^Tx = y^T\bar{A}x \leq y^T\bar{b}$$

If x and y are optimal then the duality gap is 0 (strong duality). This means

$$\begin{aligned}0 &= c^Tx - y^T\bar{b}\\ &= (\bar{A}^Ty)^Tx - y^T\bar{b}\\ &= y^T(\bar{A}x - \bar{b})\end{aligned}$$

The last term can only be 0 if y_i is 0 whenever the i-th constraint is not tight. This means we have a conic combination of c by normals (columns of $\bar{A}^T$) of *tight* constraints.

Conversely, if we have x such that the normals of tight constraint (at x) give rise to a conic combination of c, we know that x is optimal.

The profit vector c lies in the cone generated by the normals for the hops and the corn constraint (the tight constraints).

Strong Duality

Theorem 33 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

$$z^* = w^*$$

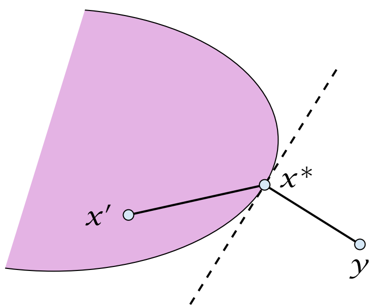
Lemma 34 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.

(without proof)

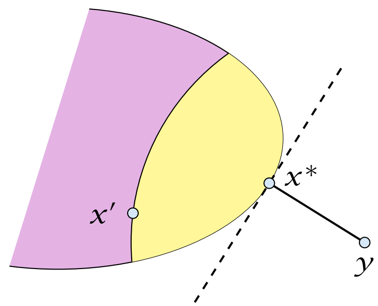
Lemma 35 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y. Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \leq 0$.



Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
- We want to apply Weierstrass but X may not be bounded.
- $X \neq \emptyset$. Hence, there exists $x' \in X$.
- Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- Applying Weierstrass gives the existence.



Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2\|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

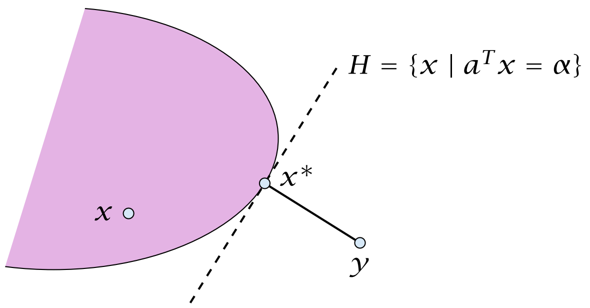
Hence, $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\epsilon\|x - x^*\|^2$.

Letting $\epsilon \to 0$ gives the result.

Theorem 36 (Separating Hyperplane)
Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)

Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to y in X.
- By previous lemma $(y - x^*)^T(x - x^*) \leq 0$ for all $x \in X$.
- Choose $a = (x^* - y)$ and $\alpha = a^T x^*$.
- For $x \in X : a^T(x - x^*) \geq 0$, and, hence, $a^T x \geq \alpha$.
- Also, $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$



Lemma 37 (Farkas Lemma)
Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

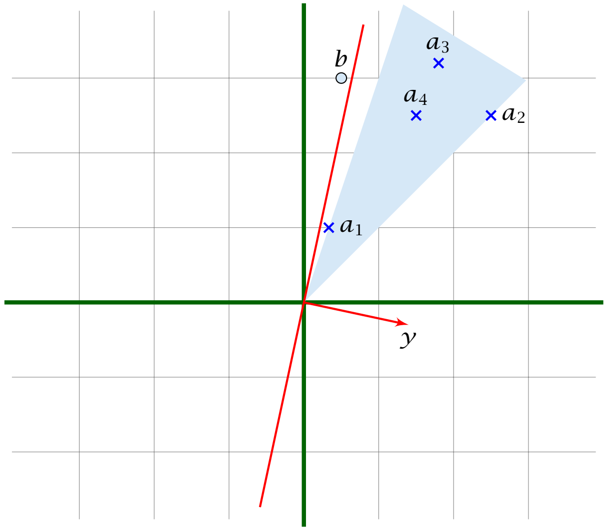
1. $\exists x \in \mathbb{R}^n$ *with* $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^T y \geq 0$, $b^T y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^T b = y^T Ax \geq 0$$

Hence, at most one of the statements can hold.

Farkas Lemma



If b is not in the cone generated by the columns of A, there exists a hyperplane y that separates b from the cone.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^T y \geq 0$, $b^T y < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

Lemma 38 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ *with* $Ax \leq b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^T y \geq 0$, $b^T y < 0$, $y \geq 0$

Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \geq 0$, $s \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^T \\ I \end{bmatrix} y \geq 0$, $b^T y < 0$

Proof of Strong Duality

P: $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$

D: $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$

Theorem 39 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

$$z = w\ .$$

Proof of Strong Duality

$z \leq w$: follows from weak duality

$z \geq w$:
We show $z < \alpha$ implies $w < \alpha$.

$$\begin{array}{lrcl} \exists x \in \mathbb{R}^n & & & \\ \text{s.t.} & Ax & \leq & b \\ & -c^T x & \leq & -\alpha \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \exists y \in \mathbb{R}^m; v \in \mathbb{R} & & & \\ \text{s.t.} & A^T y - cv & \geq & 0 \\ & b^T y - \alpha v & < & 0 \\ & y, v & \geq & 0 \end{array}$$

From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

$$\begin{array}{lrcl} \exists y \in \mathbb{R}^m; v \in \mathbb{R} & & & \\ \text{s.t.} & A^T y - cv & \geq & 0 \\ & b^T y - \alpha v & < & 0 \\ & y, v & \geq & 0 \end{array}$$

If the solution y, v has $v = 0$ we have that

$$\begin{array}{lrcl} \exists y \in \mathbb{R}^m & & & \\ \text{s.t.} & A^T y & \geq & 0 \\ & b^T y & < & 0 \\ & y & \geq & 0 \end{array}$$

is feasible. By Farkas lemma this gives that LP P is infeasible. Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

Then y is feasible for the dual but $b^T y < \alpha$. This means that $w < \alpha$.

Fundamental Questions

Definition 40 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP? yes!
- Is LP in P?

Proof:

- Given a primal maximization problem P and a parameter α. Suppose that $\alpha > \text{opt}(P)$.
- We can prove this by providing an optimal basis for the dual.
- A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost $< \alpha$.

Complementary Slackness

Lemma 41
Assume a linear program $P = \max\{c^T x \mid Ax \le b; x \ge 0\}$ has solution x^* and its dual $D = \min\{b^T y \mid A^T y \ge c; y \ge 0\}$ has solution y^*.

1. *If $x_j^* > 0$ then the j-th constraint in D is tight.*
2. *If the j-th constraint in D is not tight than $x_j^* = 0$.*
3. *If $y_i^* > 0$ then the i-th constraint in P is tight.*
4. *If the i-th constraint in P is not tight than $y_i^* = 0$.*

If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \le y^{*T} A x^* \le b^T y^*$$

Because of strong duality we then get

$$c^T x^* = y^{*T} A x^* = b^T y^*$$

This gives e.g.

$$\sum_j (y^T A - c^T)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^T A \ge c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$\begin{array}{rrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a,b & \ge 0 \end{array}$$

- Entrepeneur: buy resources from brewer at minimum cost
  C, H, M: unit price for corn, hops and malt.

$$\begin{array}{rrcrcrl} \text{min} & 480C & + & 160H & + & 1190M & \\ \text{s.t.} & 5C & + & 4H & + & 35M & \ge 13 \\ & 15C & + & 4H & + & 20M & \ge 23 \\ & & & & & C,H,M & \ge 0 \end{array}$$

Note that brewer won't sell (at least not all) if e.g.
$5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

Interpretation of Dual Variables

Marginal Price:

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ε_C, ε_H, and ε_M, respectively.

The profit increases to $\max\{c^T x \mid Ax \le b + \varepsilon; x \ge 0\}$. Because of strong duality this is equal to

$$\begin{array}{lrcl} \text{min} & (b^T + \epsilon^T)y & & \\ \text{s.t.} & A^T y & \ge & c \\ & y & \ge & 0 \end{array}$$

Interpretation of Dual Variables

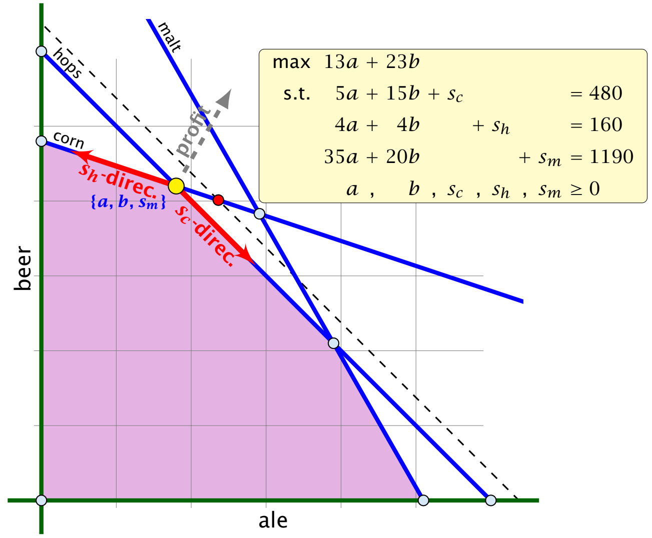
If ϵ is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

- If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
- If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.

Example



The change in profit when increasing hops by one unit is $= \underbrace{c_B^T A_B^{-1}}_{y^*} e_h$.

Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

Flows

Definition 42

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)
$$0 \leq f_{xy} \leq c_{xy} \ .$$
(capacity constraints)
2. For each $v \in V \setminus \{s,t\}$
$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$
(flow conservation constraints)

Flows

Definition 43
The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \ .$$

Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

LP-Formulation of Maxflow

$$\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V\times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
 & \forall w \ne s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
 & & f_{zw} & \ge & 0 &
\end{array}$$

$$\begin{array}{lllcr}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \ne s,t): & 1\ell_{sy} \qquad + 1p_y & \ge & 1 \\
 & f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
 & f_{ty}\ (y \ne s,t): & 1\ell_{ty} \qquad + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
 & f_{st}: & 1\ell_{st} & \ge & 1 \\
 & f_{ts}: & 1\ell_{ts} & \ge & -1 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$

LP-Formulation of Maxflow

$$\begin{array}{lllcr}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \ne s,t): & 1\ell_{sy} - 1 + 1p_y & \ge & 0 \\
 & f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x + 1 & \ge & 0 \\
 & f_{ty}\ (y \ne s,t): & 1\ell_{ty} - 0 + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x + 0 & \ge & 0 \\
 & f_{st}: & 1\ell_{st} - 1 + 0 & \ge & 0 \\
 & f_{ts}: & 1\ell_{ts} - 0 + 1 & \ge & 0 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$

LP-Formulation of Maxflow

$$\begin{array}{lllcr}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \ne s,t): & 1\ell_{sy} - p_s + 1p_y & \ge & 0 \\
 & f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x + p_s & \ge & 0 \\
 & f_{ty}\ (y \ne s,t): & 1\ell_{ty} - p_t + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x + p_t & \ge & 0 \\
 & f_{st}: & 1\ell_{st} - p_s + p_t & \ge & 0 \\
 & f_{ts}: & 1\ell_{ts} - p_t + p_s & \ge & 0 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$

with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

$$
\begin{array}{lrrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & & \ell_{xy} & \geq & 0 \\
 & & p_s & = & 1 \\
 & & p_t & = & 0
\end{array}
$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.

Degeneracy Revisited

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

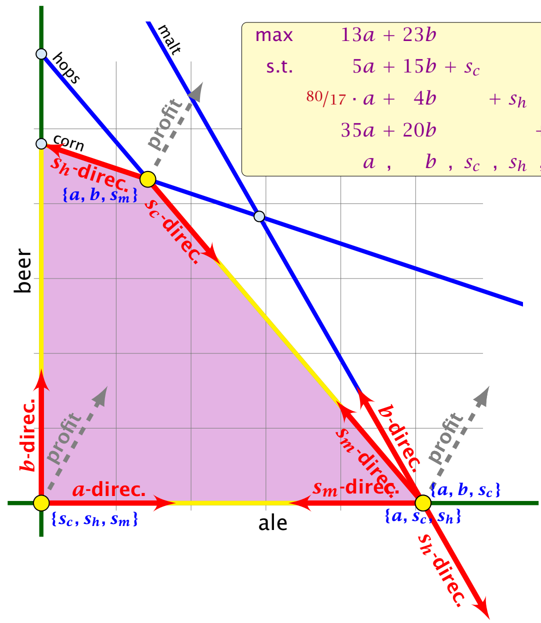
Idea:
Change $\text{LP} := \max\{c^Tx, Ax = b; x \geq 0\}$ into
$\text{LP}' := \max\{c^Tx, Ax = b', x \geq 0\}$ such that

I. LP is feasible

II. If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \not\geq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).

III. LP has no degenerate basic solutions

Degenerate Example



$$
\begin{array}{lrcl}
\max & 13a + 23b & & \\
\text{s.t.} & 5a + 15b + s_c & = & 480 \\
 & 80/17 \cdot a + 4b + s_h & = & 160 \\
 & 35a + 20b + s_m & = & 1190 \\
 & a, b, s_c, s_h, s_m & \geq & 0
\end{array}
$$

Degeneracy Revisited

If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

Idea:
Given feasible $\mathrm{LP} := \max\{c^T x, Ax = b; x \geq 0\}$. Change it into $\mathrm{LP}' := \max\{c^T x, Ax = b', x \geq 0\}$ such that

- **I.** LP' is feasible
- **II.** If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1} b \not\geq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).
- **III.** LP' has no degenerate basic solutions

Perturbation

Let B be index set of some basis with basic solution

$$x_B^* = A_B^{-1} b \geq 0, x_N^* = 0 \quad \text{(i.e. } B \text{ is feasible)}$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \text{ for } \varepsilon > 0 \ .$$

This is the perturbation that we are using.

Property I

The new LP is feasible because the set B of basis variables provides a feasible basis:

$$A_B^{-1}\left(b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}\right) = x_B^* + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \geq 0 \ .$$

Property II

Let $\tilde{B}$ be a non-feasible basis. This means $(A_{\tilde{B}}^{-1} b)_i < 0$ for some row i.

Then for small enough $\epsilon > 0$

$$\left(A_{\tilde{B}}^{-1}\left(b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}\right)\right)_i = (A_{\tilde{B}}^{-1} b)_i + \left(A_{\tilde{B}}^{-1} A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}\right)_i < 0$$

Hence, $\tilde{B}$ is not feasible.

Property III

Let $\tilde{B}$ be a basis. It has an associated solution

$$x^*_{\tilde{B}} = A^{-1}_{\tilde{B}} b + A^{-1}_{\tilde{B}} A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynom with variable ε of degree at most m.

$A^{-1}_{\tilde{B}} A_B$ has rank m. Therefore no polynom is 0.

A polynom of degree at most m has at most m roots (Nullstellen).

Hence, $\epsilon > 0$ small enough gives that no component of the above vector is 0. Hence, no degeneracies.

Since, there are no degeneracies Simplex will terminate when run on LP'.

- If it terminates because the reduced cost vector fulfills

$$\tilde{c} = (c^T - c^T_B A^{-1}_B A) \le 0$$

  then we have found an optimal basis. Note that this basis is also optimal for LP, as the above constraint does not depend on b.
- If it terminates because it finds a variable x_j with $\tilde{c}_j > 0$ for which the j-th basis direction d, fulfills $d \ge 0$ we know that LP' is unbounded. The basis direction does not depend on b. Hence, we also know that LP is unbounded.

Lexicographic Pivoting

Doing calculations with perturbed instances may be costly. Also the right choice of ε is difficult.

Idea:
Simulate behaviour of LP' without explicitly doing a perturbation.

Lexicographic Pivoting

We choose the entering variable arbitrarily as before ($\tilde{c}_e > 0$, of course).

If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

Otherwise we have to be careful.

Lexicographic Pivoting

In the following we assume that $b \geq 0$. This can be obtained by replacing the initial system $(A \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where B is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

Then the perturbed instance is

$$b' = b + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1}b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

Lexicographic Pivoting

LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_\ell = \frac{\hat{b}_\ell}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_\ell}{(A_B^{-1}A_{*e})_\ell} \; .$$

ℓ is the index of a leaving variable within B. This means if e.g. $B = \{1, 3, 7, 14\}$ and leaving variable is 3 then $\ell = 2$.

Lexicographic Pivoting

Definition 44
$u \leq_{\text{lex}} v$ if and only if the first component in which u and v differ fulfills $u_i \leq v_i$.

Lexicographic Pivoting

LP' chooses an index that minimizes

$$\theta_\ell = \frac{\left(A_B^{-1}\left(b + \begin{pmatrix}\varepsilon \\ \vdots \\ \varepsilon^m\end{pmatrix}\right)\right)_\ell}{(A_B^{-1}A_{*e})_\ell} = \frac{\left(A_B^{-1}(b \mid I)\begin{pmatrix}1 \\ \varepsilon \\ \vdots \\ \varepsilon^m\end{pmatrix}\right)_\ell}{(A_B^{-1}A_{*e})_\ell}$$

$$= \frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}\begin{pmatrix}1 \\ \varepsilon \\ \vdots \\ \varepsilon^m\end{pmatrix}$$

Lexicographic Pivoting

This means you can choose the variable/row ℓ for which the vector

$$\frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}$$

is lexicographically minimal.

Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

Number of Simplex Iterations

Each iteration of Simplex can be implemented in polynomial time.

If we use lexicographic pivoting we know that Simplex requires at most $\binom{n}{m}$ iterations, because it will not visit a basis twice.

The input size is $L \cdot n \cdot m$, where n is the number of variables, m is the number of constraints, and L is the length of the binary representation of the largest coefficient in the matrix A.

If we really require $\binom{n}{m}$ iterations then Simplex is not a polynomial time algorithm.

Can we obtain a better analysis?

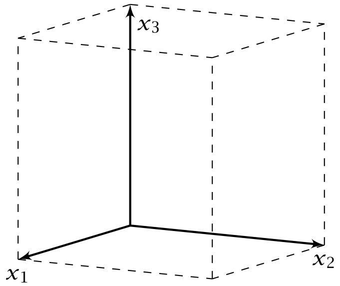
Number of Simplex Iterations

Observation
Simplex visits every feasible basis at most once.

However, also the number of feasible bases can be very large.

Example

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & 0 \le x_1 \le 1 \\ & 0 \le x_2 \le 1 \\ & \vdots \\ & 0 \le x_n \le 1 \end{aligned}$$

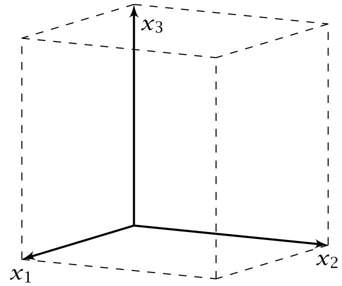


$2n$ constraint on n variables define an n-dimensional hypercube as feasible region.

The feasible region has 2^n vertices.

Example

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & 0 \le x_1 \le 1 \\ & 0 \le x_2 \le 1 \\ & \vdots \\ & 0 \le x_n \le 1 \end{aligned}$$



However, Simplex may still run quickly as it usually does not visit all feasible bases.

In the following we give an example of a feasible region for which there is a bad Pivoting Rule.

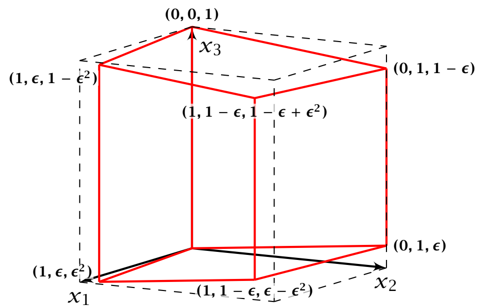
Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

Klee Minty Cube

$$\begin{aligned} \max\ & x_n \\ \text{s.t.}\ & 0 \le x_1 \le 1 \\ & \epsilon x_1 \le x_2 \le 1 - \epsilon x_1 \\ & \epsilon x_2 \le x_3 \le 1 - \epsilon x_2 \\ & \vdots \\ & \epsilon x_{n-1} \le x_n \le 1 - \epsilon x_{n-1} \\ & x_i \ge 0 \end{aligned}$$



Observations

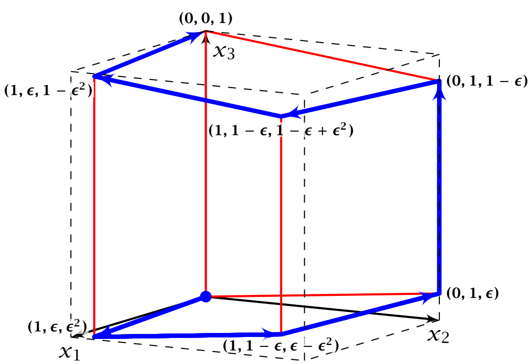
- We have $2n$ constraints, and $3n$ variables (after adding slack variables to every constraint).
- Every basis is defined by $2n$ variables, and n non-basic variables.
- There exist degenerate vertices.
- The degeneracies come from the non-negativity constraints, which are superfluous.
- In the following all variables x_i stay in the basis at all times.
- Then, we can uniquely specify a basis by choosing for each variable whether it should be equal to its lower bound, or equal to its upper bound (the slack variable corresponding to the non-tight constraint is part of the basis).
- We can also simply identify each basis/vertex with the corresponding hypercube vertex obtained by letting $\epsilon \to 0$.

Analysis

- In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
- The basis $(0, \dots, 0, 1)$ is the unique optimal basis.
- Our sequence S_n starts at $(0, \dots, 0)$ ends with $(0, \dots, 0, 1)$ and visits every node of the hypercube.
- An unfortunate Pivoting Rule may choose this sequence, and, hence, require an exponential number of iterations.

Klee Minty Cube

$$\begin{array}{lrcl} \text{max } x_n & & & \\ \text{s.t.} & 0 \le & x_1 & \le 1 \\ & \epsilon x_1 \le & x_2 & \le 1 - \epsilon x_1 \\ & \epsilon x_2 \le & x_3 & \le 1 - \epsilon x_2 \end{array}$$



Analysis

The sequence S_n that visits every node of the hypercube is defined recursively

$$\begin{tikzcd} (0,\dots,0,0,0) \arrow[d, squiggly, "S_{n-1}"] \arrow[ddd, squiggly, bend left=60, "S_n"] \\ (0,\dots,0,1,0) \arrow[d] \\ (0,\dots,0,1,1) \arrow[d, squiggly, "S_{n-1}^{\text{rev}}"] \\ (0,\dots,0,0,1) \end{tikzcd}$$

The non-recursive case is $S_1 = 0 \to 1$

Analysis

Lemma 45

The objective value x_n is increasing along path S_n.

Proof by induction:

$n = 1$: obvious, since $S_1 = 0 \to 1$, and $1 > 0$.

$n - 1 \to n$

- For the first part the value of $x_n = \epsilon x_{n-1}$.
- By induction hypothesis x_{n-1} is increasing along S_{n-1}, hence, also x_n.
- Going from $(0, \ldots, 0, 1, 0)$ to $(0, \ldots, 0, 1, 1)$ increases x_n for small enough ϵ.
- For the remaining path S_{n-1}^{rev} we have $x_n = 1 - \epsilon x_{n-1}$.
- By induction hypothesis x_{n-1} is increasing along S_{n-1}, hence $-\epsilon x_{n-1}$ is increasing along S_{n-1}^{rev}.

Remarks about Simplex

Observation

The simplex algorithm takes at most $\binom{n}{m}$ iterations. Each iteration can be implemented in time $\mathcal{O}(mn)$.

In practise it usually takes a linear number of iterations.

Remarks about Simplex

Theorem

For almost all known deterministic pivoting rules (rules for choosing entering and leaving variables) there exist lower bounds that require the algorithm to have exponential running time ($\Omega(2^{\Omega(n)})$) (e.g. Klee Minty 1972).

Remarks about Simplex

Theorem

For some standard randomized pivoting rules there exist subexponential lower bounds ($\Omega(2^{\Omega(n^\alpha)})$ for $\alpha > 0$) (Friedmann, Hansen, Zwick 2011).

Remarks about Simplex

Conjecture (Hirsch 1957)
The edge-vertex graph of an m-facet polytope in d-dimensional Euclidean space has diameter no more than $m - d$.

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form $\mathcal{O}(\text{poly}(m,d))$ is open.

8 Seidels LP-algorithm

- Suppose we want to solve $\min\{c^Tx \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- If d is much smaller than m one can do a lot better.
- In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., linear in m.

8 Seidels LP-algorithm

Setting:

- We assume an LP of the form

$$\begin{array}{lrcl} \min & c^Tx & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

- We assume that the LP is bounded.

Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{lrcl} \min & c^Tx & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on c^Tx for any basic feasible solution.**

Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b.

Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables to A; denote the resulting matrix with $\bar{A}$.

If B is an optimal basis then x_B with $\bar{A}_B x_B = \bar{b}$, gives an optimal assignment to the basis variables (non-basic variables are 0).

Theorem 46 (Cramers Rule)

Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_i = \frac{\det(M_j)}{\det(M)} ,$$

where M_i is the matrix obtained from M by replacing the i-th column by the vector b.

Proof:

- Define
$$X_i = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{i-1} & x & e_{i+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$
Note that expanding along the i-th column gives that $\det(X_i) = x_i$.
- Further, we have
$$MX_i = \begin{pmatrix} | & & | & | & | & & | \\ Me_1 & \cdots & Me_{i-1} & Mx & Me_{i+1} & \cdots & Me_n \\ | & & | & | & | & & | \end{pmatrix} = M_i$$
- Hence,
$$x_i = \det(X_i) = \frac{\det(M_i)}{\det(M)}$$

Bounding the Determinant

Let Z be the maximum absolute entry occuring in $\bar{A}, \bar{b}$ or c. Let C denote the matrix obtained from $\bar{A}_B$ by replacing the j-th column with vector $\bar{b}$ (for some j).

Observe that

$$\begin{aligned} |\det(C)| &= \left| \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \prod_{1 \le i \le m} C_{i\pi(i)} \right| \\ &\le \sum_{\pi \in S_m} \prod_{1 \le i \le m} |C_{i\pi(i)}| \\ &\le m! \cdot Z^m . \end{aligned}$$

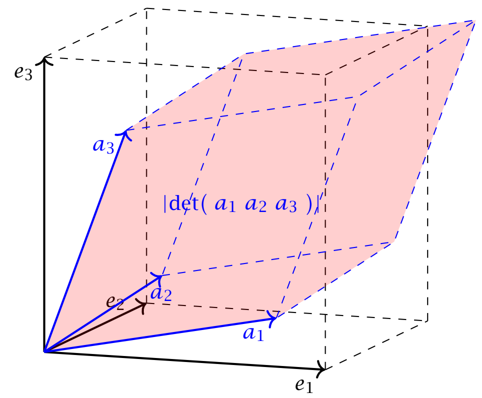
Here $\operatorname{sgn}(\pi)$ denotes the sign of the permutation, which is 1 if the permutation can be generated by an even number of transpositions (exchanging two elements), and -1 if the number of transpositions is odd.
The first identity is known as Leibniz formula.

Bounding the Determinant

Alternatively, Hadamards inequality gives

$$|\det(C)| \le \prod_{i=1}^{m} \|C_{*i}\| \le \prod_{i=1}^{m} (\sqrt{m}Z)$$
$$\le m^{m/2} Z^m \ .$$

Hadamards Inequality



Hadamards inequality says that the volume of the red parallelepiped (Spat) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \ge & b \\ & x & \ge & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^T x$ for any basic feasible solution.** Add the constraint $c^T x \ge -dZ(m! \cdot Z^m) - 1$. Note that this constraint is superfluous unless the LP is unbounded.

Ensuring Conditions

Compute an optimum basis for the new LP.

- If the cost is $c^T x = -(dZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
- Otw. we have an optimum basis.

In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^T x \geq -dZ(m! \cdot Z^m) - 1$.

We give a routine SeidelLP$(\mathcal{H}, d)$ that is given a set $\mathcal{H}$ of explicit, non-degenerate constraints over d variables, and minimizes $c^T x$ over all feasible points.

In addition it obeys the implicit constraint $c^T x \geq -(dZ)(m! \cdot Z^m) - 1$.

Algorithm 1 SeidelLP$(\mathcal{H}, d)$

```
if d = 1 then solve 1-dimensional problem and return;
if H = ∅ then return x on implicit constraint hyperplane
choose random constraint h ∈ H
Ĥ ← H \ {h}
x̂* ← SeidelLP(Ĥ, d)
if x̂* = infeasible then return infeasible
if x̂* fulfills h then return x̂*
// optimal solution fulfills h with equality, i.e., a_h^T x = b_h
solve a_h^T x = b_h for some variable x_ℓ;
eliminate x_ℓ in constraints from Ĥ and in implicit constr.;
x̂* ← SeidelLP(Ĥ, d − 1)
if x̂* = infeasible then
      return infeasible
else
      add the value of x_ℓ to x̂* and return the solution
```

8 Seidels LP-algorithm

Note that for the case $d = 1$, the asymptotic bound $\mathcal{O}(\max\{m, 1\})$ is valid also for the case $m = 0$.

- If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(\max\{m, 1\})$.
- If $d > 1$ and $m = 0$ we take time $\mathcal{O}(d)$ to return d-dimensional vector x.
- The first recursive call takes time $T(m-1, d)$ for the call plus $\mathcal{O}(d)$ for checking whether the solution fulfills h.
- If we are unlucky and $\hat{x}^*$ does not fulfill h we need time $\mathcal{O}(d(m+1)) = \mathcal{O}(dm)$ to eliminate x_ℓ. Then we make a recursive call that takes time $T(m-1, d-1)$.
- The probability of being unlucky is at most d/m as there are at most d constraints whose removal will decrease the objective function

8 Seidels LP-algorithm

This gives the recurrence

$$T(m,d) = \begin{cases} \mathcal{O}(\max\{1, m\}) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1, d) + \\ \frac{d}{m}(\mathcal{O}(dm) + T(m-1, d-1)) & \text{otw.} \end{cases}$$

Note that $T(m, d)$ denotes the expected running time.

8 Seidels LP-algorithm

Let C be the largest constant in the $\mathcal{O}$-notations.

$$T(m,d) = \begin{cases} C\max\{1,m\} & \text{if } d = 1 \\ Cd & \text{if } d > 1 \text{ and } m = 0 \\ Cd + T(m-1,d) + & \\ \frac{d}{m}(Cdm + T(m-1,d-1)) & \text{otw.} \end{cases}$$

Note that $T(m,d)$ denotes the expected running time.

8 Seidels LP-algorithm

Let C be the largest constant in the $\mathcal{O}$-notations.

We show $T(m,d) \le Cf(d)\max\{1,m\}$.

$d = 1$:

$T(m,1) \le C\max\{1,m\} \le Cf(1)\max\{1,m\}$ for $f(1) \ge 1$

$d > 1; m = 0$:

$T(0,d) \le \mathcal{O}(d) \le Cd \le Cf(d)\max\{1,m\}$ for $f(d) \ge d$

$d > 1; m = 1$:

$$\begin{aligned} T(1,d) &= \mathcal{O}(d) + T(0,d) + d\Big(\mathcal{O}(d) + T(0,d-1)\Big) \\ &\le Cd + Cd + Cd^2 + dCf(d-1) \\ &\le Cf(d)\max\{1,m\} \text{ for } f(d) \ge 3d^2 + df(d-1) \end{aligned}$$

8 Seidels LP-algorithm

$d > 1; m > 1$:
(by induction hypothesis statm. true for $d' < d, m' \ge 0$;
and for $d' = d$, $m' < m$)

$$\begin{aligned} T(m,d) &= \mathcal{O}(d) + T(m-1,d) + \frac{d}{m}\Big(\mathcal{O}(dm) + T(m-1,d-1)\Big) \\ &\le Cd + Cf(d)(m-1) + Cd^2 + \frac{d}{m}Cf(d-1)(m-1) \\ &\le 2Cd^2 + Cf(d)(m-1) + dCf(d-1) \\ &\le Cf(d)m \end{aligned}$$

if $f(d) \ge df(d-1) + 2d^2$.

8 Seidels LP-algorithm

- Define $f(1) = 3 \cdot 1^2$ and $f(d) = df(d-1) + 3d^2$ for $d > 1$.

Then

$$\begin{aligned} f(d) &= 3d^2 + df(d-1) \\ &= 3d^2 + d\Big[3(d-1)^2 + (d-1)f(d-2)\Big] \\ &= 3d^2 + d\Big[3(d-1)^2 + (d-1)\Big[3(d-2)^2 + (d-2)f(d-3)\Big]\Big] \\ &= 3d^2 + 3d(d-1)^2 + 3d(d-1)(d-2)^2 + \dots \\ &\quad + 3d(d-1)(d-2)\cdot\dots\cdot 4\cdot 3\cdot 2\cdot 1^2 \\ &= 3d!\left(\frac{d^2}{d!} + \frac{(d-1)^2}{(d-1)!} + \frac{(d-2)^2}{(d-2)!} + \dots\right) \\ &= \mathcal{O}(d!) \end{aligned}$$

since $\sum_{i\ge 1}\frac{i^2}{i!}$ is a constant.

$$\sum_{i\ge 1}\frac{i^2}{i!} = \sum_{i\ge 0}\frac{i+1}{i!} = e + \sum_{i\ge 1}\frac{i}{i!} = 2e$$

Complexity

LP Feasibility Problem (LP feasibility A)
Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}^n$ with $Ax \le b$, $x \ge 0$?

LP Feasiblity Problem (LP feasibility B)
Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. Find $x \in \mathbb{R}^n$ with $Ax \le b$, $x \ge 0$!

LP Optimization A
Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m, c \in \mathbb{Z}^n$. What is the maximum value of $c^T x$ for a feasible point $x \in \mathbb{R}^n$?

LP Optimization B
Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m, c \in \mathbb{Z}^n$. Return feasible point $x \in \mathbb{R}^n$ with maximum value of $c^T x$?

Note that allowing A, b to contain rational numbers does not make a difference, as we can multiply every number by a suitable large constant so that everything becomes integral but the feasible region does not change.

The Bit Model

Input size

- The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

$$\langle M \rangle := \sum_{i,j} \lceil \log_2(|m_{ij}|) + 1 \rceil$$

- In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
- Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

- In the following we sometimes refer to $L := \langle A \rangle + \langle b \rangle$ as the input size (even though the real input size is something in $\Theta(\langle A \rangle + \langle b \rangle)$).
- Sometimes we may also refer to $L := \langle A \rangle + \langle b \rangle + n \log_2 n$ as the input size. Note that $n \log_2 n = \Theta(\langle A \rangle + \langle b \rangle)$.
- In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in L).

Note that $m \log_2 m$ may be much larger than $\langle A \rangle + \langle b \rangle$.

Suppose that $\bar{A}x = b$; $x \ge 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

$$x_B = \bar{A}_B^{-1} b$$

and all other entries in x are 0.

In the following we show that this x has small encoding length and we give an explicit bound on this length. So far we have only been handwaving and have said that we can compute x via Gaussian elimination and it will be short...

Size of a Basic Feasible Solution

Note that n in the theorem denotes the number of columns in A which may be much smaller than m.

- A: original input matrix
- $\bar{A}$: transformation of A into standard form
- $\bar{A}_B$: submatrix of $\bar{A}$ corresponding to basis B

Lemma 47

Let $\bar{A}_B \in \mathbb{Z}^{m\times m}$ and $b \in \mathbb{Z}^m$. Define $L = \langle A\rangle + \langle b\rangle + n\log_2 n$. Then a solution to $\bar{A}_B x_B = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \le 2^L$ and $|D| \le 2^L$.

Proof:

Cramers rules says that we can compute x_j as

$$x_j = \frac{\det(\bar{A}_B^j)}{\det(\bar{A}_B)}$$

where $\bar{A}_B^j$ is the matrix obtained from $\bar{A}_B$ by replacing the j-th column by the vector b.

Bounding the Determinant

Let $X = \bar{A}_B$. Then

$$\begin{aligned}
|\det(X)| &= |\det(\bar{X})| \\
&= \left| \sum_{\pi\in S_{\tilde{n}}} \operatorname{sgn}(\pi) \prod_{1\le i\le \tilde{n}} \bar{X}_{i\pi(i)} \right| \\
&\le \sum_{\pi\in S_{\tilde{n}}} \prod_{1\le i\le \tilde{n}} |\bar{X}_{i\pi(i)}| \\
&\le n! \cdot 2^{\langle A\rangle + \langle b\rangle} \le 2^L
\end{aligned}$$

Here $\bar{X}$ is an $\tilde{n}\times\tilde{n}$ submatrix of A with $\tilde{n} \le n$.

Analogously for $\det(A_B^j)$.

When computing the determinant of $X = \bar{A}_B$ we first do expansions along columns that were introduced when transforming A into standard form, i.e., into $\bar{A}$.

Such a column contains a single 1 and the remaining entries of the column are 0. Therefore, these expansions do not increase the absolute value of the determinant. After we did expansions for all these columns we are left with a square sub-matrix of A of size at most $n\times n$.

Reducing LP-solving to LP decision.

Given an LP $\max\{c^Tx \mid Ax \le b; x \ge 0\}$ do a binary search for the optimum solution

(Add constraint $c^Tx \ge M$). Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2\left(\frac{2n2^{2L'}}{1/2^{L'}}\right) = \mathcal{O}(L') \ ,$$

as the range of the search is at most $-n2^{2L'}, \dots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \ge \frac{1}{2^{L'}}$.

Here we use $L' = \langle A\rangle + \langle b\rangle + \langle c\rangle + n\log_2 n$ (it also includes the encoding size of c).

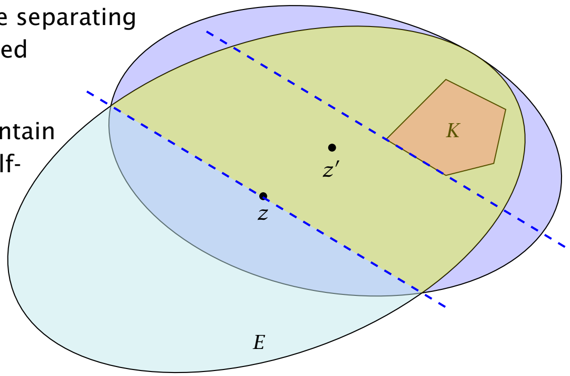
How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

We can add a constraint $c^Tx \ge M_{\max} + 1$ and check for feasibility.

Ellipsoid Method

- Let K be a convex set.
- Maintain ellipsoid E that is guaranteed to contain K provided that K is non-empty.
- If center $z \in K$ STOP.
- Otw. find a hyperplane separating K from z (e.g. a violated constraint in the LP).
- Shift hyperplane to contain node z. H denotes half-space that contains K.
- Compute (smallest) ellipsoid E' that contains $E \cap H$.
- REPEAT



Issues/Questions:

- How do you choose the first Ellipsoid? What is its volume?
- How do you measure progress? By how much does the volume decrease in each iteration?
- When can you stop? What is the minimum volume of a non-empty polytop?

Definition 48

A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an affine transformation.

Definition 49

A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$\begin{aligned} B(c,r) &= \{x \mid (x-c)^T(x-c) \le r^2\} \\ &= \{x \mid \sum_i (x-c)_i^2/r^2 \le 1\} \end{aligned}$$

$B(0,1)$ is called the unit ball.

Definition 50
An affine transformation of the unit ball is called an ellipsoid.

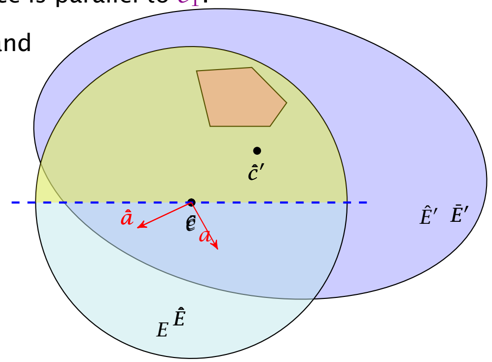
From $f(x) = Lx + t$ follows $x = L^{-1}(f(x) - t)$.

$$\begin{aligned} f(B(0,1)) &= \{f(x) \mid x \in B(0,1)\} \\ &= \{y \in \mathbb{R}^n \mid L^{-1}(y-t) \in B(0,1)\} \\ &= \{y \in \mathbb{R}^n \mid (y-t)^T {L^{-1}}^T L^{-1}(y-t) \le 1\} \\ &= \{y \in \mathbb{R}^n \mid (y-t)^T Q^{-1}(y-t) \le 1\} \end{aligned}$$

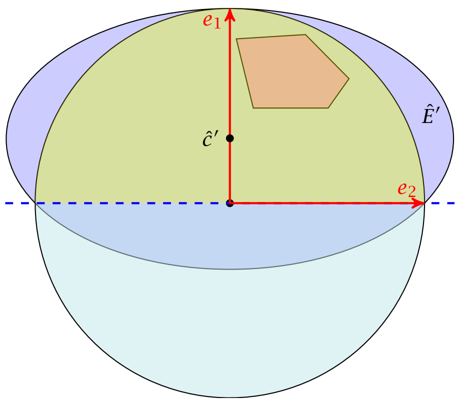
where $Q = LL^T$ is an invertible matrix.

How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1.
- Compute the new center $\hat{c}'$ and the new matrix $\hat{Q}'$ for this simplified setting.
- Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E.



The Easy Case



- The new center lies on axis x_1. Hence, $\hat{c}' = te_1$ for $t > 0$.
- The vectors $e_1, e_2, \dots$ have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^T \hat{Q}'^{-1}(e_i - \hat{c}') = 1$.

The Easy Case

- To obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is axis-parallel.
- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.
- The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

The Easy Case

- As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix $\hat{Q}'^{-1}$ is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- $(e_1 - \hat{c}')^T \hat{Q}'^{-1}(e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $(1-t)^2 = a^2$.

The Easy Case

- For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1}(e_i - \hat{c}') = 1$ looks like (here $i = 2$)

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2} = 1 - \frac{t^2}{(1-t)^2} = \frac{1-2t}{(1-t)^2}$$

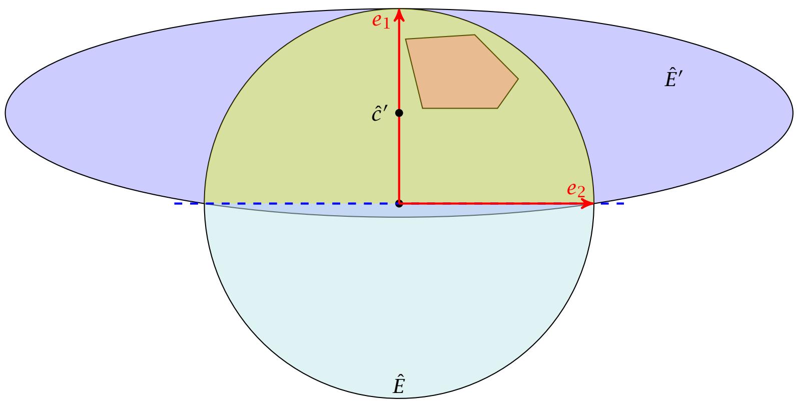
Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1-t}{\sqrt{1-2t}}$$

The Easy Case

We still have many choices for t:



Choose t such that the volume of $\hat{E}'$ is minimal!!!

The Easy Case

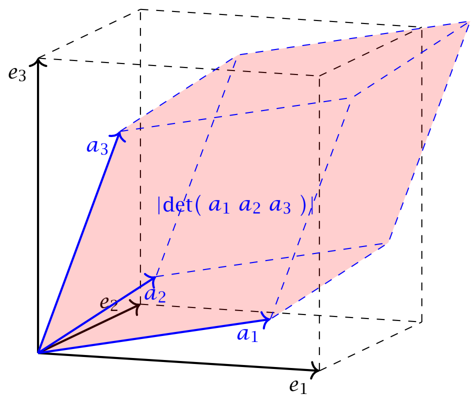
We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 51

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\mathrm{vol}(L(K)) = |\det(L)| \cdot \mathrm{vol}(K) \ .$$

n-dimensional volume



The Easy Case

- We want to choose t such that the volume of $\hat{E}'$ is minimal.

$$\mathrm{vol}(\hat{E}') = \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')| \ ,$$

- Recall that

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

- Note that a and b in the above equations depend on t, by the previous equations.

The Easy Case

$$
\begin{aligned}
\mathrm{vol}(\hat{E}') &= \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')| \\
&= \mathrm{vol}(B(0,1)) \cdot ab^{n-1} \\
&= \mathrm{vol}(B(0,1)) \cdot (1-t) \cdot \left(\frac{1-t}{\sqrt{1-2t}}\right)^{n-1} \\
&= \mathrm{vol}(B(0,1)) \cdot \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}
\end{aligned}
$$

We use the shortcut $\Phi := \mathrm{vol}(B(0,1))$.

The Easy Case

$$
\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Phi \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right) \\
&= \frac{\Phi}{N^2} \cdot \Bigg(\underbrace{(-1) \cdot n\cancel{(1-t)^{n-1}}}_{\text{derivative of numerator}} \cdot \underbrace{\cancelto{1-2t}{(\sqrt{1-2t})^{n-1}}}_{\text{denominator}} \\
&\qquad \cancel{-}\underbrace{(n-1)\cancel{(\sqrt{1-2t})^{n-2}}}_{\text{outer derivative}} \cdot \underbrace{\frac{1}{\cancel{2}\cancel{\sqrt{1-2t}}}}_{\text{inner derivative}} \cdot \cancel{(-2)} \cdot \underbrace{\cancelto{1-t}{(1-t)^n}}_{\text{numerator}}\Bigg) \\
&= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \\
&\qquad \cdot \Big((n-1)(1-t) - n(1-2t)\Big) \\
&= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \Big((n+1)t - 1\Big)
\end{aligned}
$$

N = denominator

The Easy Case

- We obtain the minimum for $t = \frac{1}{n+1}$.
- For this value we obtain

$$
a = 1 - t = \frac{n}{n+1} \text{ and } b = \frac{1-t}{\sqrt{1-2t}} = \frac{n}{\sqrt{n^2-1}}
$$

To see the equation for b, observe that

$$
b^2 = \frac{(1-t)^2}{1-2t} = \frac{(1-\frac{1}{n+1})^2}{1-\frac{2}{n+1}} = \frac{(\frac{n}{n+1})^2}{\frac{n-1}{n+1}} = \frac{n^2}{n^2-1}
$$

The Easy Case

Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

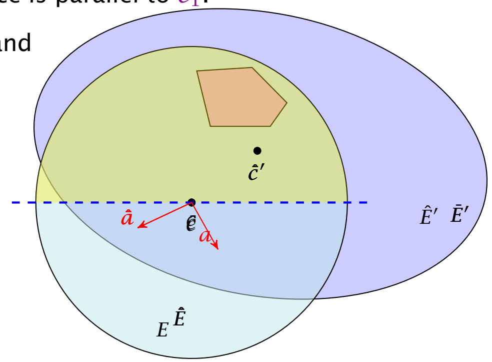
$$
\begin{aligned}
\gamma_n^2 &= \left(\frac{n}{n+1}\right)^2\left(\frac{n^2}{n^2-1}\right)^{n-1} \\
&= \left(1 - \frac{1}{n+1}\right)^2\left(1 + \frac{1}{(n-1)(n+1)}\right)^{n-1} \\
&\leq e^{-2\frac{1}{n+1}} \cdot e^{\frac{1}{n+1}} \\
&= e^{-\frac{1}{n+1}}
\end{aligned}
$$

where we used $(1+x)^a \leq e^{ax}$ for $x \in \mathbb{R}$ and $a > 0$.

This gives $\gamma_n \leq e^{-\frac{1}{2(n+1)}}$.

How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to translate/distort the ellipsoid (back) into the unit ball.
- Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1.
- Compute the new center $\hat{c}'$ and the new matrix $\hat{Q}'$ for this simplified setting.
- Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E.



Our progress is the same:

$$
\begin{aligned}
e^{-\frac{1}{2(n+1)}} \geq \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} &= \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(\hat{E})} = \frac{\mathrm{vol}(R(\hat{E}'))}{\mathrm{vol}(R(\hat{E}))} \\
&= \frac{\mathrm{vol}(\bar{E}')}{\mathrm{vol}(\bar{E})} = \frac{\mathrm{vol}(f(\bar{E}'))}{\mathrm{vol}(f(\bar{E}))} = \frac{\mathrm{vol}(E')}{\mathrm{vol}(E)}
\end{aligned}
$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to compute the new parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \leq 0\}$;

$$
\begin{aligned}
f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x - c) \leq 0\} \\
&= \{f^{-1}(f(y)) \mid a^T(f(y) - c) \leq 0\} \\
&= \{y \mid a^T(f(y) - c) \leq 0\} \\
&= \{y \mid a^T(Ly + c - c) \leq 0\} \\
&= \{y \mid (a^T L)y \leq 0\}
\end{aligned}
$$

This means $\bar{a} = L^T a$.

The center $\bar{c}$ is of course at the origin.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$
R^{-1}\Big(\frac{L^T a}{\|L^T a\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1
$$

Hence,

$$
\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1}\frac{L^T a}{\|L^T a\|}
$$

$$
\begin{aligned}
c' &= f(\bar{c}') = L \cdot \bar{c}' + c \\
&= -\frac{1}{n+1} L \frac{L^T a}{\|L^T a\|} + c \\
&= c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^T Q a}}
\end{aligned}
$$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)$$

Note that $e_1e_1^T$ is a matrix M that has $M_{11} = 1$ and all other entries equal to 0.

because for $a^2 = {}^{n^2}/_{(n+1)^2}$ and $b^2 = {}^{n^2}/_{n^2-1}$

$$\begin{aligned} b^2 - b^2\frac{2}{n+1} &= \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2} \\ &= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2 \end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned} \bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T\hat{Q}'^{-1}x \le 1\} \\ &= \{y \mid (R^{-1}y)^T\hat{Q}'^{-1}R^{-1}y \le 1\} \\ &= \{y \mid y^T(R^T)^{-1}\hat{Q}'^{-1}R^{-1}y \le 1\} \\ &= \{y \mid y^T(\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1}y \le 1\} \end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} \bar{Q}' &= R\hat{Q}'R^T \\ &= R\cdot\frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big)\cdot R^T \\ &= \frac{n^2}{n^2-1}\Big(R\cdot R^T - \frac{2}{n+1}(Re_1)(Re_1)^T\Big) \\ &= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{\|L^Ta\|^2}\Big) \end{aligned}$$

Here we used the equation for Re_1 proved before, and the fact that $RR^T = I$, which holds for any rotation matrix. To see this observe that the length of a rotated vector x should not change, i.e.,

$$x^TIx = (Rx)^T(Rx) = x^T(R^TR)x$$

which means $x^T(I - R^TR)x = 0$ for every vector x. It is easy to see that this can only be fulfilled if $I - R^TR = 0$.

9 The Ellipsoid Algorithm

$$
\begin{aligned}
E' &= L(\bar{E}') \\
&= \{L(x) \mid x^T \bar{Q}'^{-1} x \le 1\} \\
&= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1} y \le 1\} \\
&= \{y \mid y^T (\underbrace{L\bar{Q}'L^T}_{Q'})^{-1} y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$
\begin{aligned}
Q' &= L\bar{Q}'L^T \\
&= L \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{a^T Q a}\Big) \cdot L^T \\
&= \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Q a a^T Q}{a^T Q a}\Big)
\end{aligned}
$$

Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
output: point $x \in K$ or "K is empty"
$Q \leftarrow$???
repeat
  if $c \in K$ **then return** c
  else
    choose a violated hyperplane a
    $c \leftarrow c - \dfrac{1}{n+1}\dfrac{Qa}{\sqrt{a^T Q a}}$
    $Q \leftarrow \dfrac{n^2}{n^2-1}\Big(Q - \dfrac{2}{n+1}\dfrac{Qaa^TQ}{a^TQa}\Big)$
  endif
until ???
return "K is empty"

Repeat: Size of basic solutions

Lemma 52

Let $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ be a bounded polyhedron. Let $L := 2\langle A\rangle + \langle b\rangle + 2n(1 + \log_2 n)$. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \le 2^L$.

In the following we use $\delta := 2^L$.

Proof:
We can replace P by $P' := \{x \mid A'x \le b; x \ge 0\}$ where $A' = \begin{bmatrix} A & -A \end{bmatrix}$. The lemma follows by applying Lemma 47, and observing that $\langle A'\rangle = 2\langle A\rangle$ and $n' = 2n$.

How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \le \delta$.

Hence, P is contained in the cube $-\delta \le x_i \le \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \operatorname{vol}(B(0,1)) \le (n\delta)^n \operatorname{vol}(B(0,1))$.

When can we terminate?

Let $P := \{x \mid Ax \le b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \le b + \frac{1}{\lambda}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \right\} ,$$

where $\lambda = \delta^2 + 1$.

Note that the volume of P_λ cannot be 0

Making P full-dimensional

Lemma 53
P_λ is feasible if and only if P is feasible.

$\Longleftarrow$: obvious!

Making P full-dimensional

$\Longrightarrow$:

Consider the polyhedrons

$$\bar{P} = \left\{ x \mid \begin{bmatrix} A & -A & I_m \end{bmatrix} x = b; x \ge 0 \right\}$$

and

$$\bar{P}_\lambda = \left\{ x \mid \begin{bmatrix} A & -A & I_m \end{bmatrix} x = b + \frac{1}{\lambda}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}; x \ge 0 \right\} .$$

P is feasible if and only if $\bar{P}$ is feasible, and P_λ feasible if and only if $\bar{P}_\lambda$ feasible.

$\bar{P}_\lambda$ is bounded since P_λ and P are bounded.

Making P full-dimensional

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$.

$\bar{P}_\lambda$ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1} b + \frac{1}{\lambda} \bar{A}_B^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1} b)_i < 0 \le (\bar{A}_B^{-1} b)_i + \frac{1}{\lambda} (\bar{A}_B^{-1} \vec{1})_i$$

Making P full-dimensional

By Cramers rule we get

$$(\bar{A}_B^{-1} b)_i < 0 \quad \Longrightarrow \quad (\bar{A}_B^{-1} b)_i \le -\frac{1}{\det(\bar{A}_B)} \le -1/\delta$$

and

$$(\bar{A}_B^{-1} \vec{1})_i \le \det(\bar{A}_B^j) \le \delta \ ,$$

where $\bar{A}_B^j$ is obtained by replacing the j-th column of $\bar{A}_B$ by $\vec{1}$.

But then

$$(\bar{A}_B^{-1} b)_i + \frac{1}{\lambda} (\bar{A}_B^{-1} \vec{1})_i \le -1/\delta + \delta/\lambda < 0 \ ,$$

as we chose $\lambda = \delta^2 + 1$. Contradiction.

Lemma 54
If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \mathrm{vol}(B(0,1)) = \frac{1}{\delta^{3n}} \mathrm{vol}(B(0,1))$.

Proof:
If P_λ feasible then also P. Let x be feasible for P.
This means $Ax \le b$.

Let $\vec{\ell}$ with $\|\vec{\ell}\| \le r$. Then

$$\begin{aligned}(A(x + \vec{\ell}))_i &= (Ax)_i + (A\vec{\ell})_i \le b_i + \vec{a}_i^T \vec{\ell} \\ &\le b_i + \|\vec{a}_i\| \cdot \|\vec{\ell}\| \le b_i + \sqrt{n} \cdot 2^{\langle a_{\max} \rangle} \cdot r \\ &\le b_i + \frac{\sqrt{n} \cdot 2^{\langle a_{\max} \rangle}}{\delta^3} \le b_i + \frac{1}{\delta^2 + 1} \le b_i + \frac{1}{\lambda}\end{aligned}$$

Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

How many iterations do we need until the volume becomes too small?

$$e^{-\frac{i}{2(n+1)}} \cdot \mathrm{vol}(B(0,R)) < \mathrm{vol}(B(0,r))$$

Hence,

$$\begin{aligned} i &> 2(n+1) \ln\left(\frac{\mathrm{vol}(B(0,R))}{\mathrm{vol}(B(0,r))}\right) \\ &= 2(n+1) \ln\left(n^n \delta^n \cdot \delta^{3n}\right) \\ &= 8n(n+1) \ln(\delta) + 2(n+1) n \ln(n) \\ &= \mathcal{O}(\mathrm{poly}(n) \cdot L) \end{aligned}$$

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
    with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
output: point $x \in K$ or "K is empty"
$Q \leftarrow \mathrm{diag}(R^2, \dots, R^2)$ // i.e., $L = \mathrm{diag}(R, \dots, R)$
repeat
    if $c \in K$ **then return** c
    else
        choose a violated hyperplane a

$$c \leftarrow c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}}$$

$$Q \leftarrow \frac{n^2}{n^2-1}\left(Q - \frac{2}{n+1}\frac{Qaa^TQ}{a^TQa}\right)$$

    endif
until $\det(Q) \le r^{2n}$ // i.e., $\det(L) \le r^n$
return "K is empty"

Separation Oracle

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- certifies that $x \in K$,
- or finds a hyperplane separating x from K.

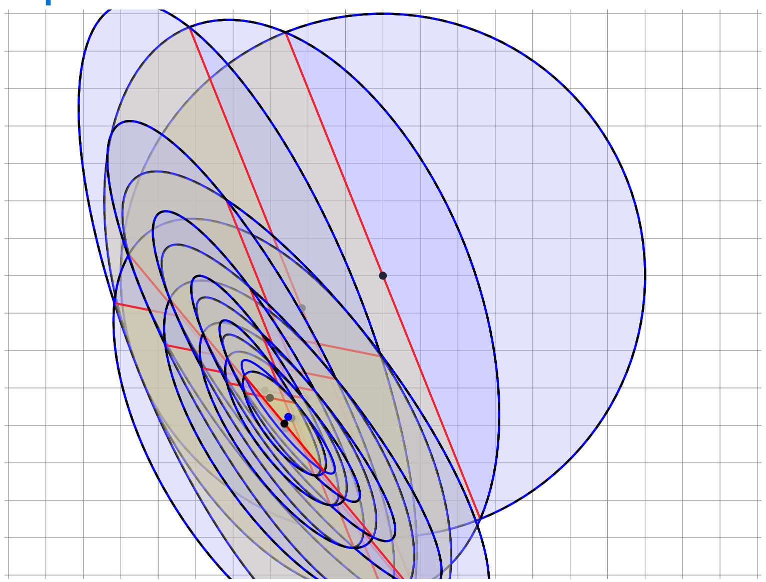
We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- a guarantee that a ball of radius r is contained in K,
- an initial ball $B(c, R)$ with radius R that contains K,
- a separation oracle for K.

The Ellipsoid algorithm requires $\mathcal{O}(\mathrm{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

Example



10 Karmarkars Algorithm

- inequalities $Ax \le b$; $m \times n$ matrix A with rows a_i^T
- $P = \{x \mid Ax \le b\}$; $P^\circ := \{x \mid Ax < b\}$
- interior point algorithm: $x \in P^\circ$ throughout the algorithm
- for $x \in P^\circ$ define
$$s_i(x) := b_i - a_i^T x$$
as the slack of the i-th constraint

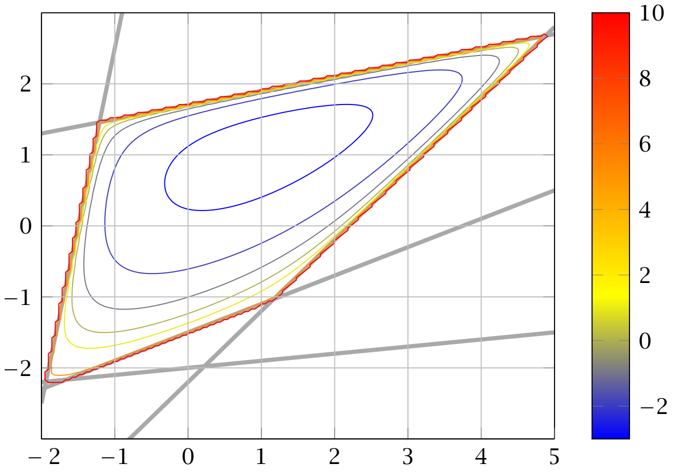
logarithmic barrier function:

$$\phi(x) = -\sum_{i=1}^{m} \ln(s_i(x))$$

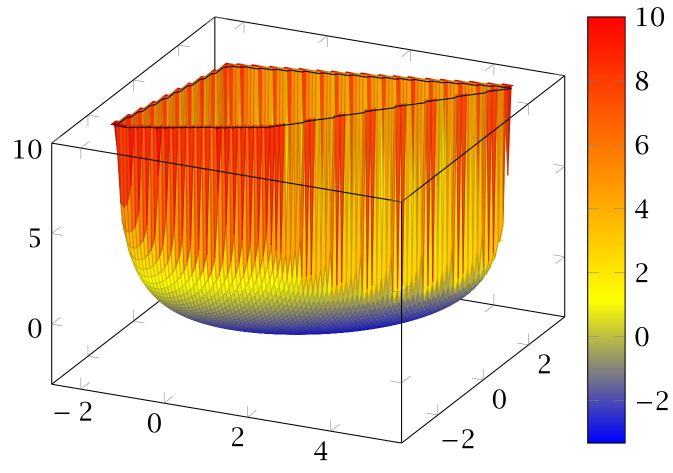
Penalty for point x; points close to the boundary have a very large penalty.

Throughout this section a_i denotes the i-th row as a column vector.

Penalty Function



Penalty Function



Gradient and Hessian

Taylor approximation:

$$\phi(x+\epsilon) \approx \phi(x) + \nabla\phi(x)^T\epsilon + \frac{1}{2}\epsilon^T\nabla^2\phi(x)\epsilon$$

Gradient:

$$\nabla\phi(x) = \sum_{i=1}^{m}\frac{1}{s_i(x)}\cdot a_i = A^T d_x$$

where $d_x^T = (1/s_1(x),\dots,1/s_m(x))$. ($d_x$ vector of inverse slacks)

Hessian:

$$H_x := \nabla^2\phi(x) = \sum_{i=1}^{m}\frac{1}{s_i(x)^2}a_i a_i^T = A^T D_x^2 A$$

with $D_x = \mathrm{diag}(d_x)$.

Proof for Gradient

$$\begin{aligned}\frac{\partial\phi(x)}{\partial x_i} &= \frac{\partial}{\partial x_i}\Big(-\sum_r \ln(s_r(x))\Big)\\ &= -\sum_r \frac{\partial}{\partial x_i}\Big(\ln(s_r(x))\Big) = -\sum_r\frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(s_r(x)\Big)\\ &= -\sum_r\frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(b_r - a_r^T x\Big) = \sum_r\frac{1}{s_r(x)}\frac{\partial}{\partial x_i}\Big(a_r^T x\Big)\\ &= \sum_r\frac{1}{s_r(x)}A_{ri}\end{aligned}$$

The i-th entry of the gradient vector is $\sum_r 1/s_r(x)\cdot A_{ri}$. This gives that the gradient is

$$\nabla\phi(x) = \sum_r 1/s_r(x)a_r = A^T d_x$$

Proof for Hessian

$$\frac{\partial}{\partial x_j}\Big(\sum_r \frac{1}{s_r(x)} A_{ri}\Big) = \sum_r A_{ri}\Big(-\frac{1}{s_r(x)^2}\Big)\cdot\frac{\partial}{\partial x_j}\Big(s_r(x)\Big)$$
$$= \sum_r A_{ri}\frac{1}{s_r(x)^2}A_{rj}$$

Note that $\sum_r A_{ri}A_{rj} = (A^TA)_{ij}$. Adding the additional factors $1/s_r(x)^2$ can be done with a diagonal matrix.

Hence the Hessian is

$$H_x = A^TD^2A$$

Properties of the Hessian

H_x is positive semi-definite for $x \in P^\circ$

$$u^TH_xu = u^TA^TD_x^2Au = \|D_xAu\|_2^2 \geq 0$$

This gives that $\phi(x)$ is convex.

If $\mathrm{rank}(A) = n$, H_x is positive definite for $x \in P^\circ$

$$u^TH_xu = \|D_xAu\|_2^2 > 0 \text{ for } u \neq 0$$

This gives that $\phi(x)$ is strictly convex.

$\|u\|_{H_x} := \sqrt{u^TH_xu}$ is a (semi-)norm; the unit ball w.r.t. this norm is an ellipsoid.

Dikin Ellipsoid

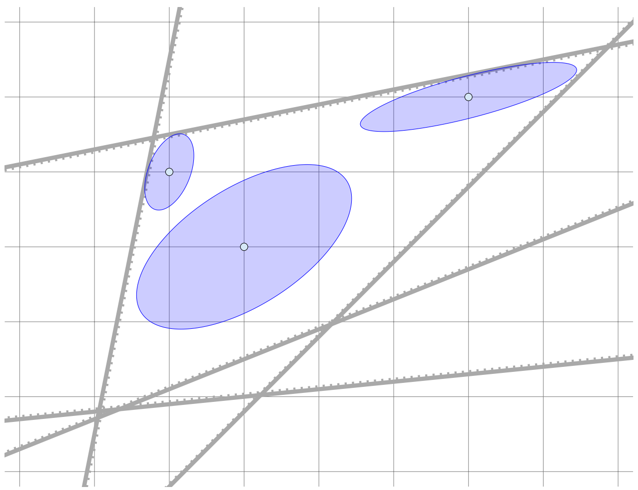
$$E_x = \{y \mid (y-x)^TH_x(y-x) \leq 1\} = \{y \mid \|y-x\|_{H_x} \leq 1\}$$

Points in E_x are feasible!!!

$$(y-x)^TH_x(y-x) = (y-x)^TA^TD_x^2A(y-x)$$
$$= \sum_{i=1}^m \frac{(a_i^T(y-x))^2}{s_i(x)^2}$$
$$= \sum_{i=1}^m \frac{(\text{change of distance to } i\text{-th constraint going from } x \text{ to } y)^2}{(\text{distance of } x \text{ to } i\text{-th constraint})^2}$$
$$\leq 1$$

In order to become infeasible when going from x to y one of the terms in the sum would need to be larger than 1.

Dikin Ellipsoids



Analytic Center

$$x_{\text{ac}} := \arg\min_{x \in P^\circ} \phi(x)$$

- x_{ac} is solution to

$$\nabla\phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)} a_i = 0$$

- depends on the description of the polytope
- x_{ac} exists and is unique iff P° is nonempty and bounded

Central Path

In the following we assume that the LP and its dual are strictly feasible and that $\operatorname{rank}(A) = n$.

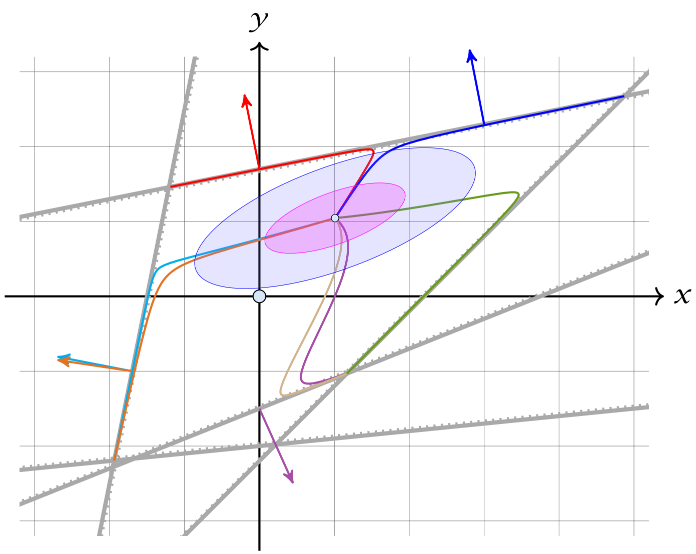
Central Path:
Set of points $\{x^*(t) \mid t > 0\}$ with

$$x^*(t) = \operatorname{argmin}_x \{tc^T x + \phi(x)\}$$

- $t = 0$: analytic center
- $t = \infty$: optimum solution

$x^*(t)$ exists and is unique for all $t \geq 0$.

Different Central Paths



Central Path

Intuitive Idea:
Find point on central path for large value of t. Should be close to optimum solution.

Questions:

- Is this really true? How large a t do we need?
- How do we find corresponding point $x^*(t)$ on central path?

The Dual

primal-dual pair:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax \le b \end{aligned}$$

$$\begin{aligned} \max\ & -b^T z \\ \text{s.t.}\ & A^T z + c = 0 \\ & z \ge 0 \end{aligned}$$

Assumptions

- primal and dual problems are strictly feasible;
- $\mathrm{rank}(A) = n$.

Note that the right LP in standard form is equal to $\max\{-b^T y \mid -A^T y = c, x \ge 0\}$. The dual of this is $\min\{c^T x \mid -Ax \ge -b\}$ (variables x are unrestricted).

Force Field Interpretation

Point $x^*(t)$ on central path is solution to $tc + \nabla\phi(x) = 0$

- We can view each constraint as generating a repelling force. The combination of these forces is represented by $\nabla\phi(x)$.
- In addition there is a force tc pulling us towards the optimum solution.

The "gravitational force" actually pulls us in direction $-\nabla\Phi(x)$. We are minimizing, hence, optimizing in direction $-c$.

How large should t be?

Point $x^*(t)$ on central path is solution to $tc + \nabla\phi(x) = 0$.

This means

$$tc + \sum_{i=1}^{m} \frac{1}{s_i(x^*(t))} a_i = 0$$

or

$$c + \sum_{i=1}^{m} z_i^*(t) a_i = 0 \ \text{ with } \ z_i^*(t) = \frac{1}{t s_i(x^*(t))}$$

- $z^*(t)$ is strictly dual feasible: ($A^T z^* + c = 0$; $z^* > 0$)
- duality gap between $x := x^*(t)$ and $z := z^*(t)$ is

$$c^T x + b^T z = (b - Ax)^T z = \frac{m}{t}$$

- if gap is less than $1/2^{\Omega(L)}$ we can snap to optimum point

How to find $x^*(t)$

First idea:

- start somewhere in the polytope
- use iterative method (Newtons method) to minimize $f_t(x) := tc^T x + \phi(x)$

Newton Method

Quadratic approximation of f_t

$$f_t(x+\epsilon) \approx f_t(x) + \nabla f_t(x)^T\epsilon + \frac{1}{2}\epsilon^T H_{f_t}(x)\,\epsilon$$

Suppose this were exact:

$$f_t(x+\epsilon) = f_t(x) + \nabla f_t(x)^T\epsilon + \frac{1}{2}\epsilon^T H_{f_t}(x)\,\epsilon$$

Then gradient is given by:

$$\nabla f_t(x+\epsilon) = \nabla f_t(x) + H_{f_t}(x)\cdot\epsilon$$

Note that for the one-dimensional case $g(\epsilon) = f(x) + f'(x)\epsilon + \frac{1}{2}f''(x)\epsilon^2$, then $g'(\epsilon) = f'(x) + f''(x)\epsilon$.

Newton Method

Observe that $H_{f_t}(x) = H(x)$, where $H(x)$ is the Hessian for the function $\phi(x)$ (adding a linear term like tc^Tx does not affect the Hessian).

Also $\nabla f_t(x) = tc + \nabla\phi(x)$.

We want to move to a point where this gradient is 0:

Newton Step at $x \in P^\circ$

$$\begin{aligned}\Delta x_{\text{nt}} &= -H_{f_t}^{-1}(x)\nabla f_t(x)\\ &= -H_{f_t}^{-1}(x)(tc + \nabla\phi(x))\\ &= -(A^TD_x^2A)^{-1}(tc + A^Td_x)\end{aligned}$$

Newton Iteration:

$$x := x + \Delta x_{\text{nt}}$$

Measuring Progress of Newton Step

Newton decrement:

$$\begin{aligned}\lambda_t(x) &= \|D_xA\Delta x_{\text{nt}}\|\\ &= \|\Delta x_{\text{nt}}\|_{H_x}\end{aligned}$$

Square of Newton decrement is linear estimate of reduction if we do a Newton step:

$$-\lambda_t(x)^2 = \nabla f_t(x)^T\Delta x_{\text{nt}}$$

- $\lambda_t(x) = 0$ iff $x = x^*(t)$
- $\lambda_t(x)$ is measure of proximity of x to $x^*(t)$

Recall that Δx_{nt} fulfills $-H(x)\Delta x_{\text{nt}} = \nabla f_t(x)$.

Convergence of Newtons Method

Theorem 55
If $\lambda_t(x) < 1$ then

- $x_+ := x + \Delta x_{nt} \in P^\circ$ *(new point feasible)*
- $\lambda_t(x_+) \le \lambda_t(x)^2$

This means we have quadratic convergence. Very fast.

Convergence of Newtons Method

feasibility:

- $\lambda_t(x) = \|\Delta x_{\text{nt}}\|_{H_x} < 1$; hence x_+ lies in the Dikin ellipsoid around x.

Convergence of Newtons Method

bound on $\lambda_t(x^+)$:
we use $D := D_x = \text{diag}(d_x)$ and $D_+ := D_{x^+} = \text{diag}(d_{x^+})$

$$\begin{aligned}\lambda_t(x^+)^2 &= \|D_+ A\Delta x_{\text{nt}}^+\|^2\\ &\le \|D_+ A\Delta x_{\text{nt}}^+\|^2 + \|D_+ A\Delta x_{\text{nt}}^+ + (I - D_+^{-1}D)DA\Delta x_{\text{nt}}\|^2\\ &= \|(I - D_+^{-1}D)DA\Delta x_{\text{nt}}\|^2\end{aligned}$$

To see the last equality we use Pythagoras

$$\|a\|^2 + \|a+b\|^2 = \|b\|^2$$

if $a^T(a+b) = 0$.

Convergence of Newtons Method

$$\begin{aligned}DA\Delta x_{\text{nt}} &= DA(x^+ - x)\\ &= D(b - Ax - (b - Ax^+))\\ &= D(D^{-1}\vec{1} - D_+^{-1}\vec{1})\\ &= (I - D_+^{-1}D)\vec{1}\end{aligned}$$

$$\begin{aligned}&a^T(a+b)\\ &\quad= \Delta x_{\text{nt}}^{+T} A^T D_+ \Big(D_+ A\Delta x_{\text{nt}}^+ + (I - D_+^{-1}D)DA\Delta x_{\text{nt}}\Big)\\ &\quad= \Delta x_{\text{nt}}^{+T}\Big(A^T D_+^2 A\Delta x_{\text{nt}}^+ - A^T D^2 A\Delta x_{\text{nt}} + A^T D_+ DA\Delta x_{\text{nt}}\Big)\\ &\quad= \Delta x_{\text{nt}}^{+T}\Big(H_+\Delta x_{\text{nt}}^+ - H\Delta x_{\text{nt}} + A^T D_+\vec{1} - A^T D\vec{1}\Big)\\ &\quad= \Delta x_{\text{nt}}^{+T}\Big(-\nabla f_t(x^+) + \nabla f_t(x) + \nabla\phi(x^+) - \nabla\phi(x)\Big)\\ &\quad= 0\end{aligned}$$

Convergence of Newtons Method

bound on $\lambda_t(x^+)$:
we use $D := D_x = \text{diag}(d_x)$ and $D_+ := D_{x^+} = \text{diag}(d_{x^+})$

$$\begin{aligned}\lambda_t(x^+)^2 &= \|D_+ A\Delta x_{\text{nt}}^+\|^2\\ &\le \|D_+ A\Delta x_{\text{nt}}^+\|^2 + \|D_+ A\Delta x_{\text{nt}}^+ + (I - D_+^{-1}D)DA\Delta x_{\text{nt}}\|^2\\ &= \|(I - D_+^{-1}D)DA\Delta x_{\text{nt}}\|^2\\ &= \|(I - D_+^{-1}D)^2\vec{1}\|^2\\ &\le \|(I - D_+^{-1}D)\vec{1}\|^4\\ &= \|DA\Delta x_{\text{nt}}\|^4\\ &= \lambda_t(x)^4\end{aligned}$$

The second inequality follows from $\sum_i y_i^4 \le \left(\sum_i y_i^2\right)^2$

If $\lambda_t(x)$ is large we do not have a guarantee.

Try to avoid this case!!!

Path-following Methods

Try to slowly travel along the central path.

Algorithm 1 PathFollowing
1: start at analytic center
2: **while** solution not good enough **do**
3: make step to improve objective function
4: recenter to return to central path

Short Step Barrier Method

simplifying assumptions:

- a first central point $x^*(t_0)$ is given
- $x^*(t)$ is computed exactly in each iteration

ϵ is approximation we are aiming for

start at $t = t_0$, repeat until $m/t \le \epsilon$

- compute $x^*(\mu t)$ using Newton starting from $x^*(t)$
- $t := \mu t$

where $\mu = 1 + 1/(2\sqrt{m})$

Short Step Barrier Method

gradient of f_{t^+} at ($x = x^*(t)$)

$$\begin{aligned}\nabla f_{t^+}(x) &= \nabla f_t(x) + (\mu - 1)tc \\ &= -(\mu - 1)A^T D_x \vec{1}\end{aligned}$$

This holds because $0 = \nabla f_t(x) = tc + A^T D_x \vec{1}$.

The Newton decrement is

$$\begin{aligned}\lambda_{t^+}(x)^2 &= \nabla f_{t^+}(x)^T H^{-1} \nabla f_{t^+}(x) \\ &= (\mu - 1)^2 \vec{1}^T B(B^T B)^{-1} B^T \vec{1} \qquad B = D_x^T A \\ &\le (\mu - 1)^2 m \\ &= 1/4\end{aligned}$$

This means we are in the range of quadratic convergence!!!

Number of Iterations

Explanation for previous slide

$P = B(B^TB)^{-1}B^T$ is a symmetric real-valued matrix; it has n linearly independent Eigenvectors. Since it is a projection matrix ($P^2 = P$) it can only have Eigenvalues 0 and 1 (because the Eigenvalues of P^2 are λ_i^2, where λ_i is Eigenvalue of P).

The expression

$$\max_v \frac{v^TPv}{v^Tv}$$

gives the largest Eigenvalue for P. Hence, $\vec{1}^TP\vec{1} \le \vec{1}^T\vec{1} = m$

the number of Newton iterations per outer iteration is very small; in practise only 1 or 2

Number of outer iterations:

We need $t_k = \mu^k t_0 \ge m/\epsilon$. This holds when

$$k \ge \frac{\log(m/(\epsilon t_0))}{\log(\mu)}$$

We get a bound of

$$\mathcal{O}\Big(\sqrt{m}\log\frac{m}{\epsilon t_0}\Big)$$

We show how to get a starting point with $t_0 = 1/2^L$. Together with $\epsilon \approx 2^{-L}$ we get $\mathcal{O}(L\sqrt{m})$ iterations.

Damped Newton Method

We assume that the polytope (not just the LP) is bounded. Then $Av \le 0$ is not possible.

For $x \in P^\circ$ and direction $v \neq 0$ define

$$\sigma_x(v) := \max_i \frac{a_i^Tv}{s_i(x)}$$

a_i^Tv is the change on the left hand side of the i-th constraint when moving in direction of v.

If $\sigma_x(v) > 1$ then for one coordinate this change is larger than the slack in the constraint at position x.

By downscaling v we can ensure to stay in the polytope.

Observation:

$$x + \alpha v \in P \qquad \text{for } \alpha \in \{0, 1/\sigma_x(v)\}$$

Damped Newton Method

Suppose that we move from x to $x + \alpha v$. The linear estimate says that $f_t(x)$ should change by $\nabla f_t(x)^T\alpha v$.

The following argument shows that f_t is well behaved. For small α the reduction of $f_t(x)$ is close to linear estimate.

$$f_t(x+\alpha v) - f_t(x) = tc^T\alpha v + \phi(x+\alpha v) - \phi(x)$$

$$\begin{aligned}\phi(x+\alpha v) - \phi(x) &= -\textstyle\sum_i \log(s_i(x+\alpha v)) + \sum_i \log(s_i(x))\\ &= -\textstyle\sum_i \log(s_i(x+\alpha v)/s_i(x))\\ &= -\textstyle\sum_i \log(1 - a_i^T\alpha v/s_i(x))\end{aligned}$$

$s_i(x+\alpha v) = b_i - a_i^Tx - a_i^T\alpha v = s_i(x) - a_i^T\alpha v$

Damped Newton Method

$$\begin{aligned}&\nabla f_t(x)^T\alpha v\\ &\quad = \big(tc^T + \textstyle\sum_i a_i^T/s_i(x)\big)\alpha v\\ &\quad = tc^T\alpha v + \textstyle\sum_i \alpha w_i\end{aligned}$$

Note that $\|w\| = \|v\|_{H_x}$.

Define $w_i = a_i^Tv/s_i(x)$ and $\sigma = \max_i w_i$. Then

$$\begin{aligned}&f_t(x+\alpha v) - f_t(x) - \nabla f_t(x)^T\alpha v\\ &\quad = -\textstyle\sum_i(\alpha w_i + \log(1-\alpha w_i))\\ &\quad \le -\sum_{w_i>0}(\alpha w_i + \log(1-\alpha w_i)) + \sum_{w_i\le 0}\frac{\alpha^2 w_i^2}{2}\\ &\quad \le -\sum_{w_i>0}\frac{w_i^2}{\sigma^2}\Big(\alpha\sigma + \log(1-\alpha\sigma)\Big) + \frac{(\alpha\sigma)^2}{2}\sum_{w_i\le 0}\frac{w_i^2}{\sigma^2}\end{aligned}$$

For $|x| < 1$, $x \le 0$:

$$x + \log(1-x) = -\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \ge -\frac{x^2}{2} = -\frac{y^2}{2}\frac{x^2}{y^2}$$

For $|x| < 1$, $0 < x \le y$:

$$\begin{aligned}x + \log(1-x) &= -\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots = \frac{x^2}{y^2}\Big(-\frac{y^2}{2} - \frac{y^2x}{3} - \frac{y^2x^2}{4} - \dots\Big)\\ &\ge \frac{x^2}{y^2}\Big(-\frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \dots\Big) = \frac{x^2}{y^2}(y + \log(1-y))\end{aligned}$$

Damped Newton Method

For $x \geq 0$
$\frac{x^2}{2} \leq \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = -\big(x + \log(1-x)\big)$

$$\leq -\sum_i \frac{w_i^2}{\sigma^2}\Big(\alpha\sigma + \log(1-\alpha\sigma)\Big)$$
$$= -\frac{1}{\sigma^2}\|v\|_{H_x}^2\Big(\alpha\sigma + \log(1-\alpha\sigma)\Big)$$

Damped Newton Iteration:
In a damped Newton step we choose

$$x_+ = x + \frac{1}{1+\sigma_x(\Delta x_{\text{nt}})}\Delta x_{\text{nt}}$$

This means that in the above expressions we choose $\alpha = \frac{1}{1+\sigma}$ and $v = \Delta x_{\text{nt}}$. Note that it wouldn't make sense to choose α larger than 1 as this would mean that our real target $(x + \Delta x_{\text{nt}})$ is inside the polytope but we overshoot and go further than this target.

Damped Newton Method

Theorem:
In a damped Newton step the cost decreases by at least

$$\lambda_t(x) - \log(1+\lambda_t(x))$$

Proof: The decrease in cost is

$$-\alpha\nabla f_t(x)^T v + \frac{1}{\sigma^2}\|v\|_{H_x}^2(\alpha\sigma + \log(1-\alpha\sigma))$$

Choosing $\alpha = \frac{1}{1+\sigma}$ and $v = \Delta x_{\text{nt}}$ gives

$$\frac{1}{1+\sigma}\lambda_t(x)^2 + \frac{\lambda_t(x)^2}{\sigma^2}\left(\frac{\sigma}{1+\sigma} + \log\left(1 - \frac{\sigma}{1+\sigma}\right)\right)$$
$$= \frac{\lambda_t(x)^2}{\sigma^2}\Big(\sigma - \log(1+\sigma)\Big)$$

With $v = \Delta x_{\text{nt}}$ we have $\|w\|_2 = \|v\|_{H_x} = \lambda_t(x)$; further recall that $\sigma = \|w\|_\infty$; hence $\sigma \leq \lambda_t(x)$.

Damped Newton Method

The first inequality follows since the function $\frac{1}{x^2}(x - \log(1+x))$ is monotonically decreasing.

$$\geq \lambda_t(x) - \log(1+\lambda_t(x))$$
$$\geq 0.09$$

for $\lambda_t(x) \geq 0.5$

Centering Algorithm:
Input: precision δ; starting point x
1. compute Δx_{nt} and $\lambda_t(x)$
2. if $\lambda_t(x) \leq \delta$ return x
3. set $x := x + \alpha\Delta x_{\text{nt}}$ with

$$\alpha = \begin{cases} \frac{1}{1+\sigma_x(\Delta x_{\text{nt}})} & \lambda_t \geq 1/2 \\ 1 & \text{otw.} \end{cases}$$

Centering

Lemma 56
The centering algorithm starting at x_0 reaches a point with $\lambda_t(x) \leq \delta$ after

$$\frac{f_t(x_0) - \min_y f_t(y)}{0.09} + \mathcal{O}(\log\log(1/\delta))$$

iterations.

This can be very, very slow...

How to get close to analytic center?

Let $P = \{Ax \le b\}$ be our (feasible) polyhedron, and x_0 a feasible point.

We change $b \to b + \frac{1}{\lambda} \cdot \vec{1}$, where $L = \langle A \rangle + \langle b \rangle + \langle c \rangle$ (encoding length) and $\lambda = 2^{2L}$. Recall that a basis is feasible in the old LP iff it is feasible in the new LP.

Lemma [without proof]
The inverse of a matrix M can be represented with rational numbers that have denominators $z_{ij} = \det(M)$.

For two basis solutions $x_B, x_{\bar{B}}$, the cost-difference $c^T x_B - c^T x_{\bar{B}}$ can be represented by a rational number that has denominator $z = \det(A_B) \cdot \det(A_{\bar{B}})$.

This means that in the perturbed LP it is sufficient to decrease the duality gap to $1/2^{4L}$ (i.e., $t \approx 2^{4L}$). This means the previous analysis essentially also works for the perturbed LP.

For a point x from the polytope (not necessarily BFS) the objective value $\bar{c}^T x$ is at most $n 2^M 2^L$, where $M \le L$ is the encoding length of the largest entry in $\bar{c}$.

How to get close to analytic center?

Start at x_0.

Choose $\hat{c} := -\nabla \phi(x)$.

Note that an entry in $\hat{c}$ fulfills $|\hat{c}_i| \le 2^{2L}$. This holds since the slack in every constraint at x_0 is at least $\lambda = 1/2^{2L}$, and the gradient is the vector of inverse slacks.

$x_0 = x^*(1)$ is point on central path for $\hat{c}$ and $t = 1$.

You can travel the central path in both directions. Go towards 0 until $t \approx 1/2^{\Omega(L)}$. This requires $O(\sqrt{m}L)$ outer iterations.

Let $x_{\hat{c}}$ denote this point.

Let x_c denote the point that minimizes

$$t \cdot c^T x + \phi(x)$$

(i.e., same value for t but different c, hence, different central path).

How to get close to analytic center?

Clearly,

$$t \cdot \hat{c}^T x_{\hat{c}} + \phi(x_{\hat{c}}) \le t \cdot \hat{c}^T x_c + \phi(x_c)$$

The difference between $f_t(x_{\hat{c}})$ and $f_t(x_c)$ is

$$\begin{aligned} & t c^T x_{\hat{c}} + \phi(x_{\hat{c}}) - t c^T x_c - \phi(x_c) \\ & \le t(c^T x_{\hat{c}} + \hat{c}^T x_c - \hat{c}^T x_{\hat{c}} - c^T x_c) \\ & \le 4tn2^{3L} \end{aligned}$$

For $t = 1/2^{\Omega(L)}$ the last term becomes constant. Hence, using damped Newton we can move from $x_{\hat{c}}$ to x_c quickly.

In total for this analysis we require $\mathcal{O}(\sqrt{m}L)$ outer iterations for the whole algorithm.

One iteration can be implemented in $\tilde{\mathcal{O}}(m^3)$ time.