

Technique 1: Round the LP solution.

We first solve the LP-relaxation and then we round the fractional values so that we obtain an integral solution.

Set Cover relaxation:

$$\begin{array}{ll} \min & \sum_{i=1}^k w_i x_i \\ \text{s.t.} & \forall u \in U \quad \sum_{i:u \in S_i} x_i \geq 1 \\ & \forall i \in \{1, \dots, k\} \quad x_i \in [0, 1] \end{array}$$

Let f_u be the number of sets that the element u is contained in (the frequency of u). Let $f = \max_u \{f_u\}$ be the maximum frequency.

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Rounding Algorithm:

Set all x_i -values with $x_i \geq \frac{1}{f}$ to 1. Set all other x_i -values to 0.

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Lemma 3

The rounding algorithm gives an f -approximation.

Proof: Every $u \in U$ is covered.

We know that

The sum constraint of every $u \in U$ is satisfied.

Therefore, one of the sets S_i covers u completely.

Therefore, u will be covered with a total weight f .

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- ▶ The sum contains at most $f_u \leq f$ elements.
- ▶ Therefore one of the sets that contain u must have $x_i \geq 1/f$.
- ▶ This set will be selected. Hence, u is covered.

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Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{ll} \min & \sum_{i \in I} w_i x_i \\ \text{s.t. } \forall u & \sum_{i: u \in S_i} x_i \geq 1 \\ & x_i \geq 0 \end{array}$$

Dual:

$$\begin{array}{ll} \max & \sum_{u \in U} y_u \\ \text{s.t. } \forall i & \sum_{u: u \in S_i} y_u \leq w_i \\ & y_u \geq 0 \end{array}$$

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Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u:u \in S_i} y_u = w_i$$

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Lemma 4

The resulting index set is an f -approximation.

Proof:

Every $u \in U$ is covered.

Suppose there is a $u \in U$ that is not covered.

This means that $\sum_{i \in I} x_i a_{ij} < b_j$ for all $j \in J$ that contains u .

But then we could increase the dual solution without violating any constraint. This is a contradiction to the fact

that the dual solution is optimal.

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$$\begin{aligned}\sum_{i \in I} w_i &= \sum_{i \in I} \sum_{u: u \in S_i} y_u \\ &= \sum_u |\{i \in I : u \in S_i\}| \cdot y_u\end{aligned}$$

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Proof:

$$\begin{aligned}\sum_{i \in I} w_i &= \sum_{i \in I} \sum_{u: u \in S_i} \gamma_u \\ &= \sum_u |\{i \in I : u \in S_i\}| \cdot \gamma_u \\ &\leq \sum_u f_u \gamma_u \\ &\leq f \sum_u \gamma_u \\ &\leq f \text{cost}(x^*) \\ &\leq f \cdot \text{OPT}\end{aligned}$$

Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' .$$

This means I' is never better than I .

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Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an f -approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

The solution is dual feasible.

The solution is primal feasible.

The solution is a cover.

Of course, we also need that I is a cover.

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$$\sum_u y_u \leq \text{cost}(x^*) \leq \text{OPT}$$

where x^* is an optimum solution to the primal LP.

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Algorithm 1 PrimalDual

- 1: $y \leftarrow 0$
- 2: $I \leftarrow \emptyset$
- 3: **while** exists $u \notin \bigcup_{i \in I} S_i$ **do**
- 4: increase dual variable y_u until constraint for some
 new set S_ℓ becomes tight
- 5: $I \leftarrow I \cup \{\ell\}$

Technique 4: The Greedy Algorithm

Algorithm 1 Greedy

```
1:  $I \leftarrow \emptyset$ 
2:  $\hat{S}_j \leftarrow S_j$  for all  $j$ 
3: while  $I$  not a set cover do
4:    $\ell \leftarrow \arg \min_{j:\hat{S}_j \neq \emptyset} \frac{w_j}{|\hat{S}_j|}$ 
5:    $I \leftarrow I \cup \{\ell\}$ 
6:    $\hat{S}_j \leftarrow \hat{S}_j - S_\ell$  for all  $j$ 
```

In every round the Greedy algorithm takes the set that covers remaining elements in the most **cost-effective** way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.

Technique 4: The Greedy Algorithm

Lemma 5

Given positive numbers a_1, \dots, a_k and b_1, \dots, b_k , and $S \subseteq \{1, \dots, k\}$ then

$$\min_i \frac{a_i}{b_i} \leq \frac{\sum_{i \in S} a_i}{\sum_{i \in S} b_i} \leq \max_i \frac{a_i}{b_i}$$

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Let n_ℓ denote the number of elements that remain at the beginning of iteration ℓ . $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need s iterations.

In the ℓ -th iteration

since an optimal algorithm can cover the remaining n_ℓ elements with cost OPT .

Let \hat{S}_j be a subset that minimizes this ratio. Hence,
$$w_j / |\hat{S}_j| \leq \frac{\text{OPT}}{n_\ell}.$$

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Adding this set to our solution means $n_{\ell+1} = n_{\ell} - |\hat{S}_j|$.

$$w_j \leq \frac{|\hat{S}_j| \cdot \text{OPT}}{n_{\ell}} = \frac{n_{\ell} - n_{\ell+1}}{n_{\ell}} \cdot \text{OPT}$$

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$$\begin{aligned}\sum_{j \in I} w_j &\leq \sum_{\ell=1}^s \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \text{OPT} \\ &\leq \text{OPT} \sum_{\ell=1}^s \left(\frac{1}{n_\ell} + \frac{1}{n_\ell - 1} + \dots + \frac{1}{n_{\ell+1} + 1} \right)\end{aligned}$$

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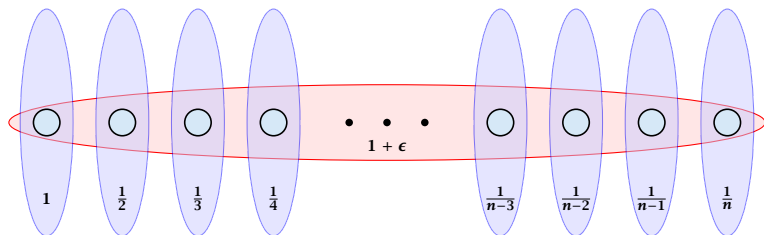
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Technique 4: The Greedy Algorithm

A tight example:



Technique 5: Randomized Rounding

One round of randomized rounding:

Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you nearly have a cover. Cover remaining elements by some simple heuristic.

Version B: Repeat for s rounds. If you have a cover STOP. Otherwise, repeat the whole algorithm.

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$$\begin{aligned} &= \prod_{j:u \in S_j} (1 - x_j) \leq \prod_{j:u \in S_j} e^{-x_j} \\ &= e^{-\sum_{j:u \in S_j} x_j} \end{aligned}$$

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Probability that $u \in U$ is not covered (in one round):

$$\begin{aligned}\Pr[u \text{ not covered in one round}] &= \prod_{j:u \in S_j} (1 - x_j) \leq \prod_{j:u \in S_j} e^{-x_j} \\ &= e^{-\sum_{j:u \in S_j} x_j} \leq e^{-1} .\end{aligned}$$

Probability that $u \in U$ is not covered (after ℓ rounds):

$$\Pr[u \text{ not covered after } \ell \text{ round}] \leq \frac{1}{e^\ell} .$$

$\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}]$

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \end{aligned}$$

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \\ &\leq \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}] \end{aligned}$$

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Lemma 6

With high probability $\mathcal{O}(\log n)$ rounds suffice.

$$\begin{aligned} & \Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\ &= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \dots \vee u_n \text{ not covered}] \\ &\leq \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}] \leq ne^{-\ell}. \end{aligned}$$

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With high probability:

For any constant α the number of rounds is at most $\mathcal{O}(\log n)$ with probability at least $1 - n^{-\alpha}$.

Proof: We have

$$\Pr[\text{\#rounds} \geq (\alpha + 1) \ln n] \leq n e^{-(\alpha+1) \ln n} = n^{-\alpha} .$$

Expected Cost

- ▶ Version A.

Repeat for $s = (\alpha + 1) \ln n$ rounds. If you don't have a cover simply take for each element u the cheapest set that contains u .

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$$E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\ + \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]$$

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for $n \geq 2$ and $\alpha \geq 1$.

Randomized rounding gives an $\mathcal{O}(\log n)$ approximation. The running time is polynomial with high probability.

Theorem 7 (without proof)

There is no approximation algorithm for set cover with approximation guarantee better than $\frac{1}{2} \log n$ unless NP has quasi-polynomial time algorithms (algorithms with running time $2^{\text{poly}(\log n)}$).

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Integrality Gap

The **integrality gap** of the SetCover LP is $\Omega(\log n)$.

- ▶ $n = 2^k - 1$
- ▶ Elements are all vectors \vec{x} over $GF[2]$ of length k (excluding zero vector).
- ▶ Every vector \vec{y} defines a set as follows

$$S_{\vec{y}} := \{\vec{x} \mid \vec{x}^T \vec{y} = 1\}$$

- ▶ each set contains 2^{k-1} vectors; each vector is contained in 2^{k-1} sets
- ▶ $x_i = \frac{1}{2^{k-1}} = \frac{2}{n+1}$ is fractional solution.

Integrality Gap

Every collection of $p < k$ sets does not cover all elements.

Hence, we get a gap of $\Omega(\log n)$.

Techniques:

- ▶ Deterministic Rounding
- ▶ Rounding of the Dual
- ▶ Primal Dual
- ▶ Greedy
- ▶ Randomized Rounding
- ▶ Local Search
- ▶ Rounding Data + Dynamic Programming