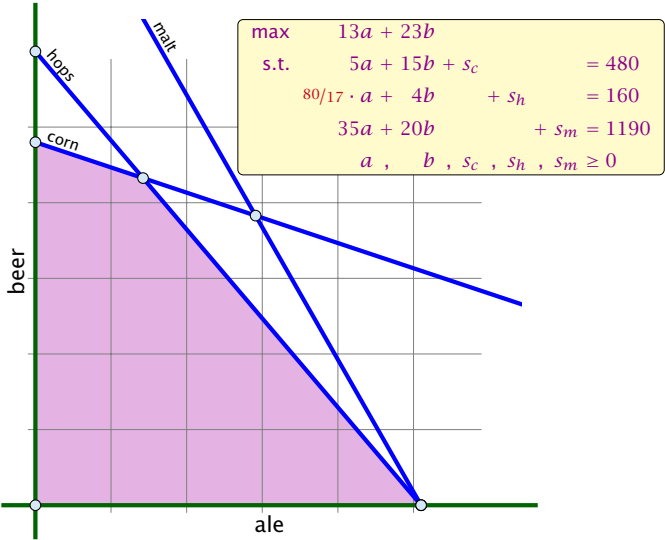


# Degeneracy Revisited

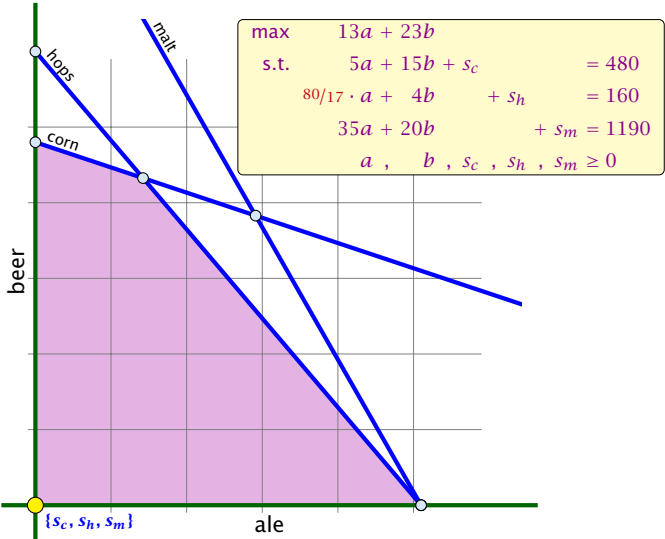
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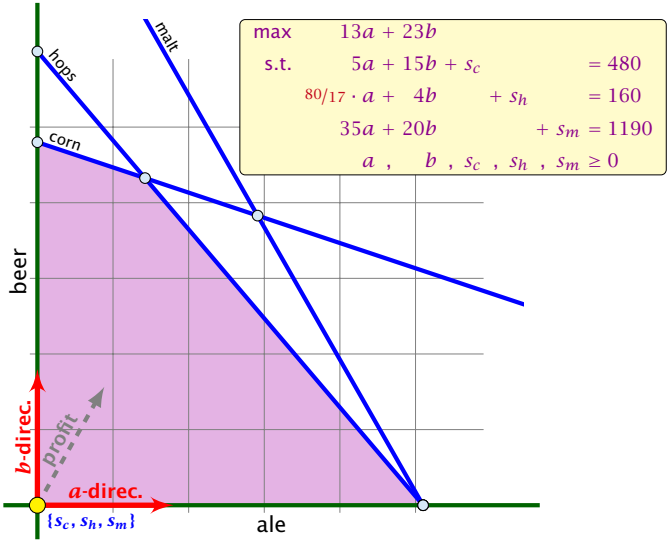
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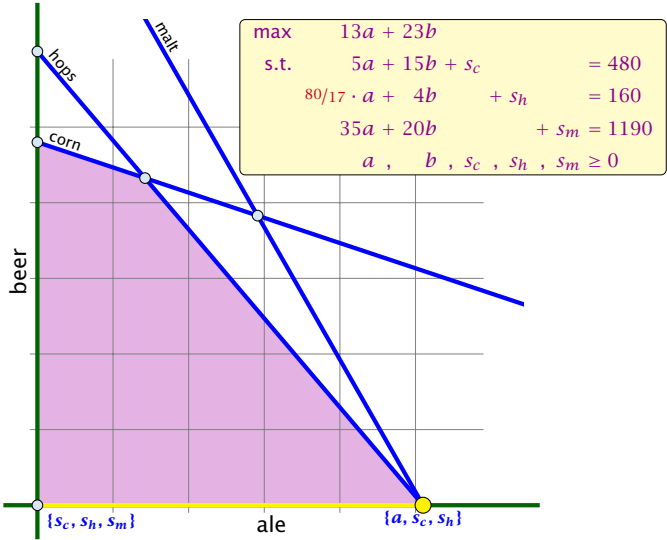
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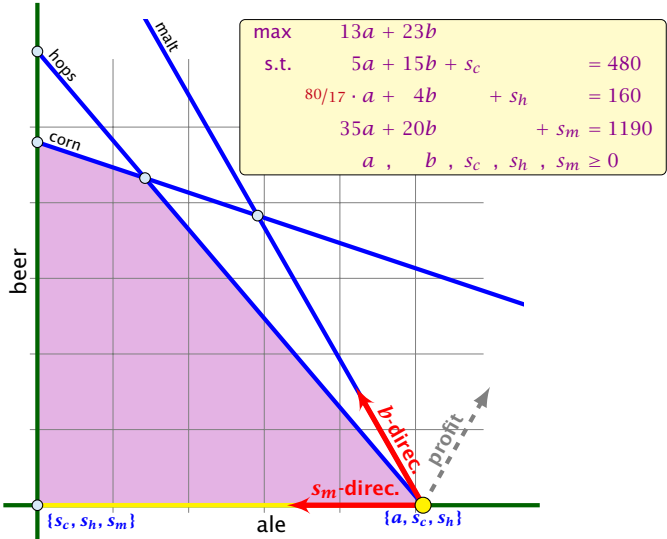
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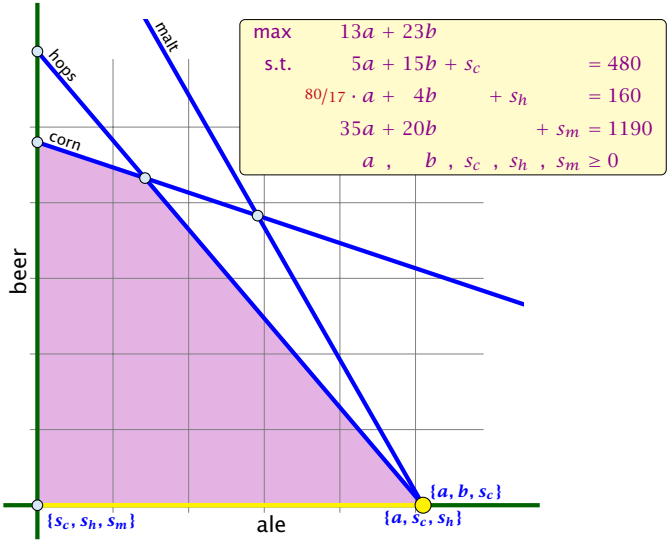
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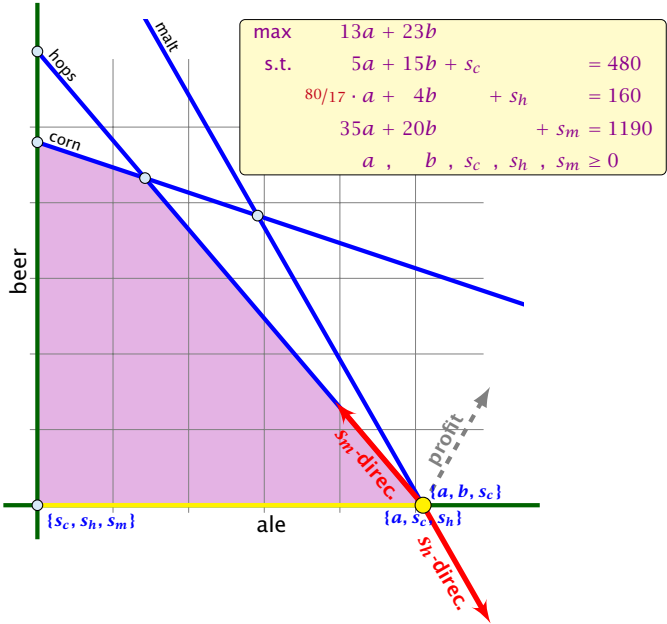


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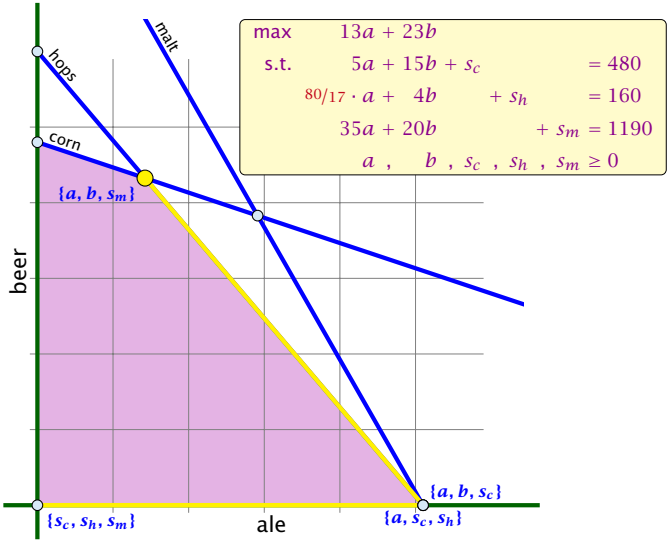


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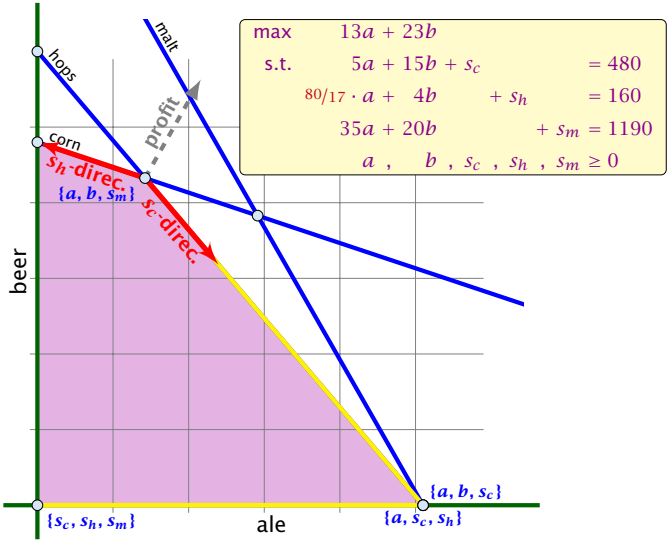


$$\begin{aligned}
 \max \quad & 13a + 23b \\
 \text{s.t.} \quad & 5a + 15b + s_c = 480 \\
 & 80/17 \cdot a + 4b + s_h = 160 \\
 & 35a + 20b + s_m = 1190 \\
 & a, b, s_c, s_h, s_m \geq 0
 \end{aligned}$$

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Idea:

Given feasible LP :=  $\max\{c^T x, Ax = b; x \geq 0\}$ . Change it into LP' :=  $\max\{c^T x, Ax = b', x \geq 0\}$  such that

is feasible

and a set of basic variables corresponds to an optimal solution of LP. (The new RHS  $b'$  corresponds to an artificial basis for LP')

Since the columns in  $A$  are linearly independent,

there is always a unique basic solution.

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- II. If a set  $B$  of basis variables corresponds to an infeasible basis (i.e.  $A_B^{-1}b \not\geq 0$ ) then  $B$  corresponds to an infeasible basis in  $LP'$  (note that columns in  $A_B$  are linearly independent).
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# Perturbation

Let  $B$  be index set of **some** basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad (\text{i.e. } B \text{ is feasible})$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \quad \text{for } \varepsilon > 0 .$$

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# Property I

The new LP is feasible because the set  $B$  of basis variables provides a feasible basis:

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Hence,  $\tilde{B}$  is not feasible.



## Property III

Let  $\tilde{B}$  be a basis. It has an associated solution

$$x_{\tilde{B}}^* = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynomial with variable  $\varepsilon$  of degree at most  $m$ .

$A_{\tilde{B}}^{-1}A_B$  has rank  $m$ . Therefore no polynomial is 0.

A polynomial of degree at most  $m$  has at most  $m$  roots (Nullstellen).

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- ▶ If it terminates because it finds a variable  $x_j$  with  $\tilde{c}_j > 0$  for which the  $j$ -th basis direction  $d$ , fulfills  $d \geq 0$  we know that  $LP'$  is unbounded. The basis direction **does not depend on  $b$ .** Hence, we also know that  $LP$  is unbounded.

# Lexicographic Pivoting

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Simulate behaviour of  $LP'$  without explicitly doing a perturbation.

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We choose the entering variable arbitrarily as before ( $\tilde{c}_e > 0$ , of course).

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In the following we assume that  $b \geq 0$ . This can be obtained by replacing the initial system  $(A \mid b)$  by  $(A_B^{-1}A \mid A_B^{-1}b)$  where  $B$  is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

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## Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis  $B$  is

$$\begin{aligned} & (c_N^T - c_B^T A_B^{-1} A_N) x_N = Z - c_B^T A_B^{-1} b \\ I x_B + & A_B^{-1} A_N x_N = A_B^{-1} b \\ x_B, & x_N \geq 0\end{aligned}$$

The BFS is given by  $x_N = 0, x_B = A_B^{-1} b$ .

If  $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$  we know that we have an optimum solution.

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LP chooses an arbitrary leaving variable that has  $\hat{A}_{\ell e} > 0$  and minimizes

$$\theta_{\ell} = \frac{\hat{b}_{\ell}}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_{\ell}}{(A_B^{-1}A_{*e})_{\ell}}.$$

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## Definition 3

$u \leq_{\text{lex}} v$  if and only if the first component in which  $u$  and  $v$  differ fulfills  $u_i \leq v_i$ .

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This means you can choose the variable/row  $\ell$  for which the vector

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