

## Duality

How do we get an upper bound to a maximization LP?

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

Note that a lower bound is easy to derive. Every choice of  $a, b \geq 0$  gives us a lower bound (e.g.  $a = 12, b = 28$  gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the  $i$ -th row with  $y_i \geq 0$ ) such that  $\sum_i y_i a_{ij} \geq c_j$  then  $\sum_i y_i b_i$  will be an upper bound.

## Duality

### Definition 2

Let  $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$  be a linear program  $P$  (called the primal linear program).

The linear program  $D$  defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the **dual problem**.

## Duality

### Lemma 3

*The dual of the dual problem is the primal problem.*

**Proof:**

- ▶  $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$
- ▶  $w = -\max\{-b^T y \mid -A^T y \leq -c, y \geq 0\}$

The dual problem is

- ▶  $z = -\min\{-c^T x \mid -Ax \geq -b, x \geq 0\}$
- ▶  $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$

## Weak Duality

Let  $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$  and  $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$  be a primal dual pair.

$x$  is primal feasible iff  $x \in \{x \mid Ax \leq b, x \geq 0\}$

$y$  is dual feasible, iff  $y \in \{y \mid A^T y \geq c, y \geq 0\}$ .

### Theorem 4 (Weak Duality)

Let  $\hat{x}$  be primal feasible and let  $\hat{y}$  be dual feasible. Then

$$c^T \hat{x} \leq z \leq w \leq b^T \hat{y} .$$

## Weak Duality

$$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \quad (\hat{x} \geq 0)$$

$$A \hat{x} \leq b \Rightarrow y^T A \hat{x} \leq y^T b \quad (y \geq 0)$$

This gives

$$c^T \hat{x} \leq y^T A \hat{x} \leq b^T y .$$

Since, there exists primal feasible  $\hat{x}$  with  $c^T \hat{x} = z$ , and dual feasible  $\hat{y}$  with  $b^T \hat{y} = w$  we get  $z \leq w$ .

If  $P$  is unbounded then  $D$  is infeasible.

## 5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$

$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

## Proof

**Primal:**

$$\begin{aligned} & \max\{c^T x \mid Ax = b, x \geq 0\} \\ & = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ & = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\} \end{aligned}$$

**Dual:**

$$\begin{aligned} & \min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \geq c, y \geq 0\} \\ & = \min \left\{ [b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \mid [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0 \right\} \\ & = \min \{ b^T \cdot (y^+ - y^-) \mid A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0 \} \\ & = \min \{ b^T y' \mid A^T y' \geq c \} \end{aligned}$$

## Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with **reduced cost**

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to  $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$  is solution to the **dual**  $\min\{b^T y \mid A^T y \geq c\}$ .

$$\begin{aligned} b^T y^* & = (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ & = (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ & = c^T x^* \end{aligned}$$

Hence, the solution is optimal.

## 5.3 Strong Duality

$$P = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

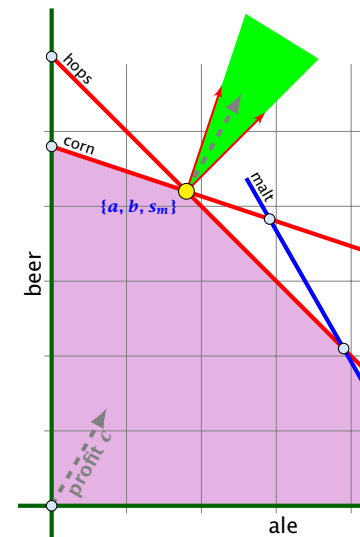
$n_A$ : number of variables,  $m_A$ : number of constraints

We can put the non-negativity constraints into  $A$  (which gives us unrestricted variables):  $\bar{P} = \max\{c^T x \mid \bar{A}x \leq \bar{b}\}$

$$n_{\bar{A}} = n_A, m_{\bar{A}} = m_A + n_A$$

$$\text{Dual } D = \min\{\bar{b}^T y \mid \bar{A}^T y = c, y \geq 0\}.$$

## 5.3 Strong Duality



If we have a conic combination  $y$  of  $c$  then  $b^T y$  is an upper bound of the profit we can obtain (**weak duality**):

$$c^T x = (\bar{A}^T y)^T x = y^T \bar{A}x \leq y^T \bar{b}$$

If  $x$  and  $y$  are optimal then the **duality gap** is 0 (**strong duality**). This means

$$\begin{aligned} 0 &= c^T x - y^T \bar{b} \\ &= (\bar{A}^T y)^T x - y^T \bar{b} \\ &= y^T (\bar{A}x - \bar{b}) \end{aligned}$$

The last term can only be 0 if  $y_i$  is 0 whenever the  $i$ -th constraint is not tight. This means we have a conic combination of  $c$  by normals (columns of  $\bar{A}^T$ ) of **tight** constraints.

Conversely, if we have  $x$  such that the normals of tight constraint (at  $x$ ) give rise to a conic combination of  $c$ , we know that  $x$  is optimal.

The profit vector  $c$  lies in the cone generated by the normals for the hops and the corn constraint (the tight constraints).

## Strong Duality

### Theorem 5 (Strong Duality)

Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z^*$  and  $w^*$  denote the optimal solution to  $P$  and  $D$ , respectively.

Then

$$z^* = w^*$$

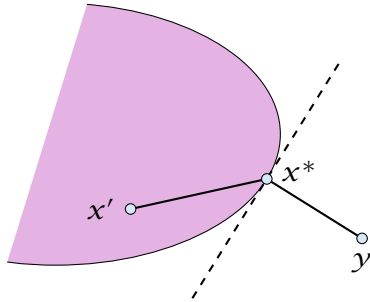
### Lemma 6 (Weierstrass)

Let  $X$  be a compact set and let  $f(x)$  be a continuous function on  $X$ . Then  $\min\{f(x) : x \in X\}$  exists.

(without proof)

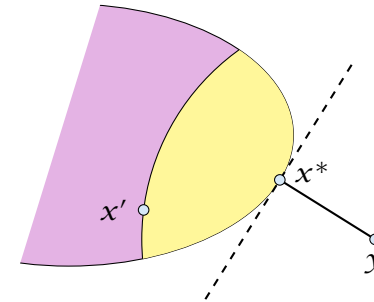
### Lemma 7 (Projection Lemma)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty convex set, and let  $y \notin X$ . Then there exist  $x^* \in X$  with minimum distance from  $y$ . Moreover for all  $x \in X$  we have  $(y - x^*)^T(x - x^*) \leq 0$ .



### Proof of the Projection Lemma

- ▶ Define  $f(x) = \|y - x\|$ .
- ▶ We want to apply Weierstrass but  $X$  may not be bounded.
- ▶  $X \neq \emptyset$ . Hence, there exists  $x' \in X$ .
- ▶ Define  $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$ . This set is closed and bounded.
- ▶ Applying Weierstrass gives the existence.



### Proof of the Projection Lemma (continued)

$x^*$  is minimum. Hence  $\|y - x^*\|^2 \leq \|y - x\|^2$  for all  $x \in X$ .

By **convexity**:  $x \in X$  then  $x^* + \epsilon(x - x^*) \in X$  for all  $0 \leq \epsilon \leq 1$ .

$$\begin{aligned} \|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*) \end{aligned}$$

Hence,  $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\epsilon \|x - x^*\|^2$ .

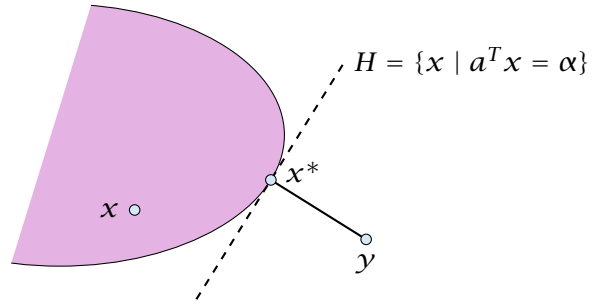
Letting  $\epsilon \rightarrow 0$  gives the result.

### Theorem 8 (Separating Hyperplane)

Let  $X \subseteq \mathbb{R}^m$  be a non-empty closed convex set, and let  $y \notin X$ . Then there exists a **separating hyperplane**  $\{x \in \mathbb{R}^m : a^T x = \alpha\}$  where  $a \in \mathbb{R}^m$ ,  $\alpha \in \mathbb{R}$  that **separates**  $y$  from  $X$ . ( $a^T y < \alpha$ ;  $a^T x \geq \alpha$  for all  $x \in X$ )

## Proof of the Hyperplane Lemma

- ▶ Let  $x^* \in X$  be closest point to  $y$  in  $X$ .
- ▶ By previous lemma  $(y - x^*)^T(x - x^*) \leq 0$  for all  $x \in X$ .
- ▶ Choose  $a = (x^* - y)$  and  $\alpha = a^T x^*$ .
- ▶ For  $x \in X$ :  $a^T(x - x^*) \geq 0$ , and, hence,  $a^T x \geq \alpha$ .
- ▶ Also,  $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$



## Lemma 9 (Farkas Lemma)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then *exactly one* of the following statements holds.

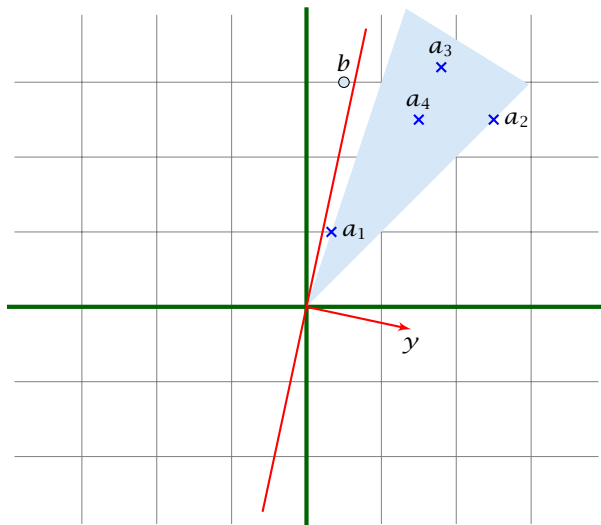
1.  $\exists x \in \mathbb{R}^n$  with  $Ax = b, x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^T y \geq 0, b^T y < 0$

Assume  $\hat{x}$  satisfies 1. and  $\hat{y}$  satisfies 2. Then

$$0 > \hat{y}^T b = \hat{y}^T A \hat{x} \geq 0$$

Hence, at most one of the statements can hold.

## Farkas Lemma



If  $b$  is not in the cone generated by the columns of  $A$ , there exists a hyperplane  $y$  that separates  $b$  from the cone.

## Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider  $S = \{Ax : x \geq 0\}$  so that  $S$  closed, convex,  $b \notin S$ .

We want to show that there is  $y$  with  $A^T y \geq 0, b^T y < 0$ .

Let  $y$  be a hyperplane that separates  $b$  from  $S$ . Hence,  $y^T b < \alpha$  and  $y^T s \geq \alpha$  for all  $s \in S$ .

$$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$$

$y^T Ax \geq \alpha$  for all  $x \geq 0$ . Hence,  $y^T A \geq 0$  as we can choose  $x$  arbitrarily large.

### Lemma 10 (Farkas Lemma; different version)

Let  $A$  be an  $m \times n$  matrix,  $b \in \mathbb{R}^m$ . Then exactly one of the following statements holds.

1.  $\exists x \in \mathbb{R}^n$  with  $Ax \leq b, x \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $A^T y \geq 0, b^T y < 0, y \geq 0$

Rewrite the conditions:

1.  $\exists x \in \mathbb{R}^n$  with  $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b, x \geq 0, s \geq 0$
2.  $\exists y \in \mathbb{R}^m$  with  $\begin{bmatrix} A^T \\ I \end{bmatrix} y \geq 0, b^T y < 0$

### Proof of Strong Duality

$$P: z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$D: w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

### Theorem 11 (Strong Duality)

Let  $P$  and  $D$  be a primal dual pair of linear programs, and let  $z$  and  $w$  denote the optimal solution to  $P$  and  $D$ , respectively (i.e.,  $P$  and  $D$  are non-empty). Then

$$z = w .$$

### Proof of Strong Duality

$z \leq w$ : follows from weak duality

$z \geq w$ :

We show  $z < \alpha$  implies  $w < \alpha$ .

$$\begin{array}{l} \exists x \in \mathbb{R}^n \\ \text{s.t. } Ax \leq b \\ \quad -c^T x \leq -\alpha \\ \quad x \geq 0 \end{array}$$

$$\begin{array}{l} \exists y \in \mathbb{R}^m; v \in \mathbb{R} \\ \text{s.t. } A^T y - cv \geq 0 \\ \quad b^T y - \alpha v < 0 \\ \quad y, v \geq 0 \end{array}$$

From the definition of  $\alpha$  we know that the first system is infeasible; hence the second must be feasible.

### Proof of Strong Duality

$$\begin{array}{l} \exists y \in \mathbb{R}^m; v \in \mathbb{R} \\ \text{s.t. } A^T y - cv \geq 0 \\ \quad b^T y - \alpha v < 0 \\ \quad y, v \geq 0 \end{array}$$

If the solution  $y, v$  has  $v = 0$  we have that

$$\begin{array}{l} \exists y \in \mathbb{R}^m \\ \text{s.t. } A^T y \geq 0 \\ \quad b^T y < 0 \\ \quad y \geq 0 \end{array}$$

is feasible. By Farkas lemma this gives that LP  $P$  is infeasible. Contradiction to the assumption of the lemma.

## Proof of Strong Duality

Hence, there exists a solution  $y, v$  with  $v > 0$ .

We can rescale this solution (scaling both  $y$  and  $v$ ) s.t.  $v = 1$ .

Then  $y$  is feasible for the dual but  $b^T y < \alpha$ . This means that  $w < \alpha$ .

## Fundamental Questions

### Definition 12 (Linear Programming Problem (LP))

Let  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ ,  $c \in \mathbb{Q}^n$ ,  $\alpha \in \mathbb{Q}$ . Does there exist  $x \in \mathbb{Q}^n$  s.t.  $Ax = b$ ,  $x \geq 0$ ,  $c^T x \geq \alpha$ ?

#### Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP? yes!
- ▶ Is LP in P?

#### Proof:

- ▶ Given a primal maximization problem  $P$  and a parameter  $\alpha$ . Suppose that  $\alpha > \text{opt}(P)$ .
- ▶ We can prove this by providing an optimal basis for the dual.
- ▶ A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost  $< \alpha$ .

## Complementary Slackness

### Lemma 13

Assume a linear program  $P = \max\{c^T x \mid Ax \leq b; x \geq 0\}$  has solution  $x^*$  and its dual  $D = \min\{b^T y \mid A^T y \geq c; y \geq 0\}$  has solution  $y^*$ .

1. If  $x_j^* > 0$  then the  $j$ -th constraint in  $D$  is tight.
2. If the  $j$ -th constraint in  $D$  is not tight than  $x_j^* = 0$ .
3. If  $y_i^* > 0$  then the  $i$ -th constraint in  $P$  is tight.
4. If the  $i$ -th constraint in  $P$  is not tight than  $y_i^* = 0$ .

If we say that a variable  $x_j^*$  ( $y_i^*$ ) has slack if  $x_j^* > 0$  ( $y_i^* > 0$ ), (i.e., the corresponding variable restriction is not tight) and a constraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

## Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \leq y^{*T} A x^* \leq b^T y^*$$

Because of strong duality we then get

$$c^T x^* = y^{*T} A x^* = b^T y^*$$

This gives e.g.

$$\sum_j (y^T A - c^T)_j x_j^* = 0$$

From the constraint of the dual it follows that  $y^T A \geq c^T$ . Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g.  $(y^T A - c^T)_j > 0$  (the  $j$ -th constraint in the dual is not tight) then  $x_j = 0$  (2.). The result for (1./3./4.) follows similarly.

## Interpretation of Dual Variables

- ▶ Brewer: find mix of ale and beer that maximizes profits

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

- ▶ Entrepreneur: buy resources from brewer at minimum cost  
 $C, H, M$ : unit price for corn, hops and malt.

$$\begin{aligned} \min \quad & 480C + 160H + 1190M \\ \text{s.t.} \quad & 5C + 4H + 35M \geq 13 \\ & 15C + 4H + 20M \geq 23 \\ & C, H, M \geq 0 \end{aligned}$$

Note that brewer won't sell (at least not all) if e.g.  
 $5C + 4H + 35M < 13$  as then brewing ale would be advantageous.

## Interpretation of Dual Variables

### Marginal Price:

- ▶ How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- ▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by  $\epsilon_C, \epsilon_H$ , and  $\epsilon_M$ , respectively.

The profit increases to  $\max\{c^T x \mid Ax \leq b + \epsilon; x \geq 0\}$ . Because of strong duality this is equal to

$$\begin{aligned} \min \quad & (b^T + \epsilon^T)y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

## Interpretation of Dual Variables

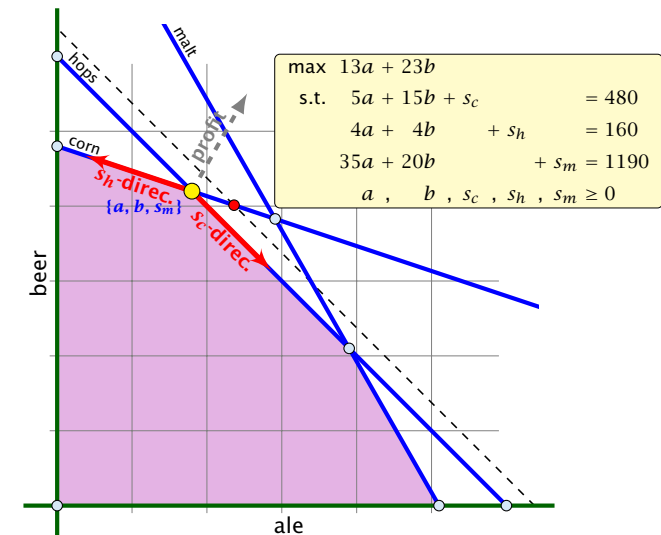
If  $\epsilon$  is "small" enough then the optimum dual solution  $y^*$  might not change. Therefore the profit increases by  $\sum_i \epsilon_i y_i^*$ .

Therefore we can interpret the dual variables as **marginal prices**.

Note that with this interpretation, complementary slackness becomes obvious.

- ▶ If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
- ▶ If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.

## Example



The change in profit when increasing hops by one unit is

$$= \underbrace{c_B^T A_B^{-1}}_{y^*} e_h.$$



Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

## Flows

### Definition 14

An  $(s, t)$ -flow in a (complete) directed graph  $G = (V, V \times V, c)$  is a function  $f : V \times V \rightarrow \mathbb{R}_0^+$  that satisfies

1. For each edge  $(x, y)$

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} .$$

(flow conservation constraints)

## Flows

### Definition 15

The value of an  $(s, t)$ -flow  $f$  is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

### Maximum Flow Problem:

Find an  $(s, t)$ -flow with maximum value.

## LP-Formulation of Maxflow

$$\begin{array}{ll} \max & \sum_z f_{sz} - \sum_z f_{zs} \\ \text{s.t.} & \forall (z, w) \in V \times V \quad f_{zw} \leq c_{zw} \quad \ell_{zw} \\ & \forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} = 0 \quad p_w \\ & f_{zw} \geq 0 \end{array}$$

$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} (x, y \neq s, t) : 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} (y \neq s, t) : 1\ell_{sy} \quad +1p_y \geq 1 \\ & f_{xs} (x \neq s, t) : 1\ell_{xs} - 1p_x \geq -1 \\ & f_{ty} (y \neq s, t) : 1\ell_{ty} \quad +1p_y \geq 0 \\ & f_{xt} (x \neq s, t) : 1\ell_{xt} - 1p_x \geq 0 \\ & f_{st} : 1\ell_{st} \geq 1 \\ & f_{ts} : 1\ell_{ts} \geq -1 \\ & \ell_{xy} \geq 0 \end{array}$$

## LP-Formulation of Maxflow

$$\begin{array}{ll}
 \min & \sum_{(x,y)} c_{xy} l_{xy} \\
 \text{s.t.} & f_{xy} (x, y \neq s, t): 1l_{xy} - 1p_x + 1p_y \geq 0 \\
 & f_{sy} (y \neq s, t): 1l_{sy} - 1 + 1p_y \geq 0 \\
 & f_{xs} (x \neq s, t): 1l_{xs} - 1p_x + 1 \geq 0 \\
 & f_{ty} (y \neq s, t): 1l_{ty} - 0 + 1p_y \geq 0 \\
 & f_{xt} (x \neq s, t): 1l_{xt} - 1p_x + 0 \geq 0 \\
 & f_{st}: 1l_{st} - 1 + 0 \geq 0 \\
 & f_{ts}: 1l_{ts} - 0 + 1 \geq 0 \\
 & l_{xy} \geq 0
 \end{array}$$

## LP-Formulation of Maxflow

$$\begin{array}{ll}
 \min & \sum_{(x,y)} c_{xy} l_{xy} \\
 \text{s.t.} & f_{xy} (x, y \neq s, t): 1l_{xy} - 1p_x + 1p_y \geq 0 \\
 & f_{sy} (y \neq s, t): 1l_{sy} - p_s + 1p_y \geq 0 \\
 & f_{xs} (x \neq s, t): 1l_{xs} - 1p_x + p_s \geq 0 \\
 & f_{ty} (y \neq s, t): 1l_{ty} - p_t + 1p_y \geq 0 \\
 & f_{xt} (x \neq s, t): 1l_{xt} - 1p_x + p_t \geq 0 \\
 & f_{st}: 1l_{st} - p_s + p_t \geq 0 \\
 & f_{ts}: 1l_{ts} - p_t + p_s \geq 0 \\
 & l_{xy} \geq 0
 \end{array}$$

with  $p_t = 0$  and  $p_s = 1$ .

## LP-Formulation of Maxflow

$$\begin{array}{ll}
 \min & \sum_{(x,y)} c_{xy} l_{xy} \\
 \text{s.t.} & f_{xy}: 1l_{xy} - 1p_x + 1p_y \geq 0 \\
 & l_{xy} \geq 0 \\
 & p_s = 1 \\
 & p_t = 0
 \end{array}$$

We can interpret the  $l_{xy}$  value as assigning a length to every edge.

The value  $p_x$  for a variable, then can be seen as the distance of  $x$  to  $t$  (where the distance from  $s$  to  $t$  is required to be 1 since  $p_s = 1$ ).

The constraint  $p_x \leq l_{xy} + p_y$  then simply follows from triangle inequality ( $d(x, t) \leq d(x, y) + d(y, t) \Rightarrow d(x, t) \leq l_{xy} + d(y, t)$ ).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means  $p_x = 1$  or  $p_x = 0$  for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.