

There are many practically important optimization problems that are NP-hard.

### **What can we do?**

- ▶ Heuristics.
- ▶ Exploit special structure of instances occurring in practise.
- ▶ Consider algorithms that do not compute the optimal solution but provide solutions that are close to optimum.

## Definition 1

An  $\alpha$ -approximation for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of  $\alpha$  of the value of an optimal solution.

## Why approximation algorithms?

- ▶ We need algorithms for hard problems.
- ▶ It gives a rigorous mathematical base for studying heuristics.
- ▶ It provides a metric to compare the difficulty of various optimization problems.
- ▶ Proving theorems may give a deeper theoretical understanding which in turn leads to new algorithmic approaches.

## Why not?

- ▶ Sometimes the results are very pessimistic due to the fact that an algorithm has to provide a close-to-optimum solution on every instance.

## Definition 2

An optimization problem  $P = (\mathcal{I}, \text{sol}, m, \text{goal})$  is in **NPO** if

- ▶  $x \in \mathcal{I}$  can be **decided** in polynomial time
- ▶  $y \in \text{sol}(\mathcal{I})$  can be **verified** in polynomial time
- ▶  $m$  can be computed in polynomial time
- ▶  $\text{goal} \in \{\text{min}, \text{max}\}$

In other words: the decision problem **is there a solution  $y$  with  $m(x, y)$  at most/at least  $z$**  is in NP.

- ▶  $x$  is problem instance
- ▶  $y$  is candidate solution
- ▶  $m^*(x)$  cost/profit of an optimal solution

### Definition 3 (Performance Ratio)

$$R(x, y) := \max \left\{ \frac{m(x, y)}{m^*(x)}, \frac{m^*(x)}{m(x, y)} \right\}$$

### Definition 4 ( $r$ -approximation)

An algorithm  $A$  is an  $r$ -approximation algorithm iff

$$\forall x \in \mathcal{I} : R(x, A(x)) \leq r ,$$

and  $A$  runs in polynomial time.

### Definition 5 (PTAS)

A PTAS for a problem  $P$  from NPO is an algorithm that takes as input  $x \in \mathcal{I}$  and  $\epsilon > 0$  and produces a solution  $y$  for  $x$  with

$$R(x, y) \leq 1 + \epsilon .$$

The running time is polynomial in  $|x|$ .

approximation with arbitrary good factor... fast?

## Problems that have a PTAS

**Scheduling.** Given  $m$  jobs with known processing times; schedule the jobs on  $n$  machines such that the MAKESPAN is minimized.



## Definition 6 (FPTAS)

An FPTAS for a problem  $P$  from NPO is an algorithm that takes as input  $x \in \mathcal{I}$  and  $\epsilon > 0$  and produces a solution  $y$  for  $x$  with

$$R(x, y) \leq 1 + \epsilon .$$

The running time is polynomial in  $|x|$  and  $1/\epsilon$ .

approximation with arbitrary good factor... fast!

## Problems that have an FPTAS

**KNAPSACK.** Given a set of items with profits and weights choose a subset of total weight at most  $W$  s.t. the profit is maximized.

### Definition 7 (APX – approximable)

A problem  $P$  from NPO is in APX if there exist a constant  $r \geq 1$  and an  $r$ -approximation algorithm for  $P$ .

constant factor approximation...

## Problems that are in APX

**MAXCUT.** Given a graph  $G = (V, E)$ ; partition  $V$  into two disjoint pieces  $A$  and  $B$  s. t. the number of edges between both pieces is maximized.

**MAX-3SAT.** Given a 3CNF-formula. Find an assignment to the variables that satisfies the maximum number of clauses.

## Problems with polylogarithmic approximation guarantees

- ▶ Set Cover
- ▶ Minimum Multicut
- ▶ Sparsest Cut
- ▶ Minimum Bisection

There is an  $r$ -approximation with  $r \leq \mathcal{O}(\log^c(|x|))$  for some constant  $c$ .

Note that only for some of the above problem a matching lower bound is known.

**There are really difficult problems!**

### Theorem 8

*For any constant  $\epsilon > 0$  there does not exist an  $\Omega(n^{1-\epsilon})$ -approximation algorithm for the maximum clique problem on a given graph  $G$  with  $n$  nodes unless  $P = NP$ .*

Note that an  $n$ -approximation is trivial.

## There are weird problems!

Asymmetric  $k$ -Center admits an  $\mathcal{O}(\log^* n)$ -approximation.

There is no  $o(\log^* n)$ -approximation to Asymmetric  $k$ -Center unless  $NP \subseteq DTIME(n^{\log \log \log n})$ .

Class APX not important in practise.

Instead of saying **problem  $P$  is in APX** one says **problem  $P$  admits a 4-approximation**.

One only says that a problem is **APX-hard**.