Brewery Problem

Brewery brews ale and beer.

- Production limited by supply of corn, hops and barley malt
- Recipes for ale and beer require different amounts of resources

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

Brewery Problem

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

How can brewer maximize profits?

- only brew ale: 34 barrels of ale $\Longrightarrow$ 442 €
- only brew beer: 32 barrels of beer $\Longrightarrow$ 736 €
- 7.5 barrels ale, 29.5 barrels beer $\Longrightarrow$ 776 €
- 12 barrels ale, 28 barrels beer $\Longrightarrow$ 800 €

Brewery Problem

Linear Program

- Introduce variables a and b that define how much ale and beer to produce.
- Choose the variables in such a way that the objective function (profit) is maximized.
- Make sure that no constraints (due to limited supply) are violated.

$$
\begin{array}{llcll}
\max & 13a & + & 23b & \\
\text{s.t.} & 5a & + & 15b & \leq 480 \\
 & 4a & + & 4b & \leq 160 \\
 & 35a & + & 20b & \leq 1190 \\
 & & & a,b & \geq 0
\end{array}
$$

Standard Form LPs

LP in standard form:

- input: numbers a_{ij}, c_j, b_i
- output: numbers x_j
- n = #decision variables, m = #constraints
- maximize linear objective function subject to linear (in)equalities

$$
\begin{array}{llcll}
\max & \sum_{j=1}^{n} c_j x_j & & & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \leq i \leq m \\
 & x_j & \geq & 0 & 1 \leq j \leq n
\end{array}
$$

$$
\begin{array}{llcl}
\max & c^T x & & \\
\text{s.t.} & Ax & = & b \\
 & x & \geq & 0
\end{array}
$$

Standard Form LPs

Original LP

$$\begin{array}{lrcrl} \max & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a,b & \ge 0 \end{array}$$

Standard Form

Add a slack variable to every constraint.

$$\begin{array}{lrcrcrcrcrl} \max & 13a & + & 23b & & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & & = 480 \\ & 4a & + & 4b & & & + & s_h & & & = 160 \\ & 35a & + & 20b & & & & & + & s_m & = 1190 \\ & a & , & b & , & s_c & , & s_h & , & s_m & \ge 0 \end{array}$$

Standard Form LPs

There are different standard forms:

standard form

$$\begin{array}{lrcl} \max & c^Tx & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

$$\begin{array}{lrcl} \min & c^Tx & & \\ \text{s.t.} & Ax & = & b \\ & x & \ge & 0 \end{array}$$

standard maximization form

$$\begin{array}{lrcl} \max & c^Tx & & \\ \text{s.t.} & Ax & \le & b \\ & x & \ge & 0 \end{array}$$

standard minimization form

$$\begin{array}{lrcl} \min & c^Tx & & \\ \text{s.t.} & Ax & \ge & b \\ & x & \ge & 0 \end{array}$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **less or equal to equality:**

$$a - 3b + 5c \le 12 \implies \begin{array}{r} a - 3b + 5c + s = 12 \\ s \ge 0 \end{array}$$

- **greater or equal to equality:**

$$a - 3b + 5c \ge 12 \implies \begin{array}{r} a - 3b + 5c - s = 12 \\ s \ge 0 \end{array}$$

- **min to max:**

$$\min a - 3b + 5c \implies \max -a + 3b - 5c$$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- **equality to less or equal:**

$$a - 3b + 5c = 12 \implies \begin{array}{r} a - 3b + 5c \le 12 \\ -a + 3b - 5c \le -12 \end{array}$$

- **equality to greater or equal:**

$$a - 3b + 5c = 12 \implies \begin{array}{r} a - 3b + 5c \ge 12 \\ -a + 3b - 5c \ge -12 \end{array}$$

- **unrestricted to nonnegative:**

$$x \text{ unrestricted} \implies x = x^+ - x^-,\ x^+ \ge 0,\ x^- \ge 0$$

Standard Form LPs

Observations:

- a linear program does not contain x^2, $\cos(x)$, etc.
- transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

Fundamental Questions

Definition 1 (Linear Programming Problem (LP))
Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

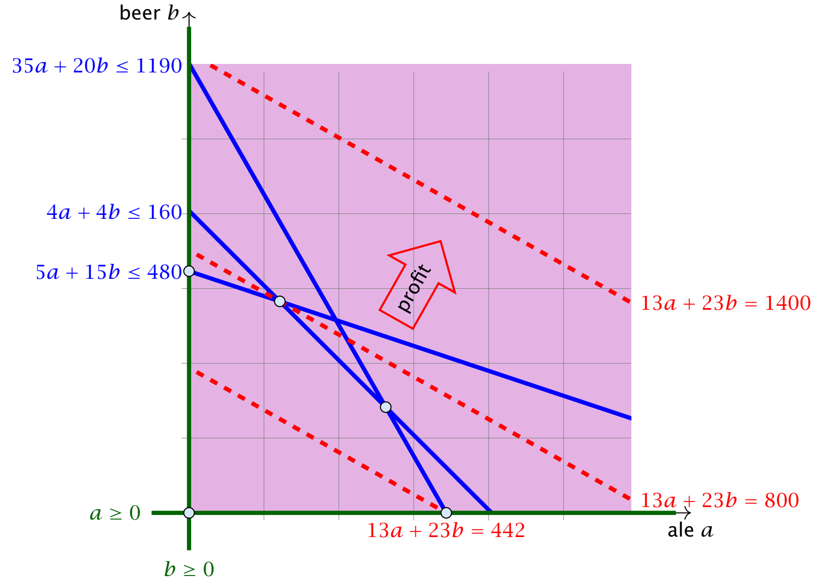
Questions:

- Is LP in NP?
- Is LP in co-NP?
- Is LP in P?

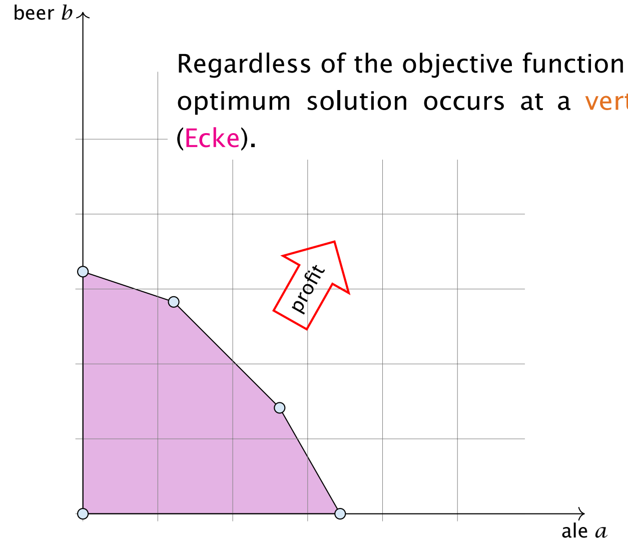
Input size:

- n number of variables, m constraints, L number of bits to encode the input

Geometry of Linear Programming



Geometry of Linear Programming



Regardless of the objective function an optimum solution occurs at a vertex (Ecke).

Definitions

Let for a Linear Program in standard form
$P = \{x \mid Ax = b, x \geq 0\}$.

- P is called the feasible region (Lösungsraum) of the LP.
- A point $x \in P$ is called a feasible point (gültige Lösung).
- If $P \neq \emptyset$ then the LP is called feasible (erfüllbar). Otherwise, it is called infeasible (unerfüllbar).
- An LP is bounded (beschränkt) if it is feasible and
  - $c^T x < \infty$ for all $x \in P$ (for maximization problems)
  - $c^T x > -\infty$ for all $x \in P$ (for minimization problems)

Definition 2
Given vectors/points $x_1, \dots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- linear combination if $\lambda_i \in \mathbb{R}$.
- affine combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- convex combination if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
- conic combination if $\lambda_i \in \mathbb{R}$ and $\lambda_i \geq 0$.

Note that a combination involves only finitely many vectors.

Definition 3
A set $X \subseteq \mathbb{R}^n$ is called

- a linear subspace if it is closed under linear combinations.
- an affine subspace if it is closed under affine combinations.
- convex if it is closed under convex combinations.
- a convex cone if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

Definition 4
Given a set $X \subseteq \mathbb{R}^n$.

- $\mathrm{span}(X)$ is the set of all linear combinations of X (linear hull, span)
- $\mathrm{aff}(X)$ is the set of all affine combinations of X (affine hull)
- $\mathrm{conv}(X)$ is the set of all convex combinations of X (convex hull)
- $\mathrm{cone}(X)$ is the set of all conic combinations of X (conic hull)

Definition 5
A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

Lemma 6
If $P \subseteq \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ convex then also

$$Q = \{x \in P \mid f(x) \le t\}$$

Dimensions

Definition 7
The dimension $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

Definition 8
The dimension $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension of its affine hull $\text{aff}(X)$.

Definition 9
A set $H \subseteq \mathbb{R}^n$ is a hyperplane if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

Definition 10
A set $H' \subseteq \mathbb{R}^n$ is a (closed) halfspace if $H = \{x \mid a^T x \le b\}$, for $a \neq 0$.

Definitions

Definition 11
A polytop is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a finite set of points, i.e., $P = \text{conv}(X)$ where $|X| = c$.

Definitions

Definition 12
A polyhedron is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of finitely many half-spaces $\{H(a_1, b_1), \ldots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} \ .$$

Definition 13
A polyhedron P is bounded if there exists B s.t. $\|x\|_2 \leq B$ for all $x \in P$.

Definitions

Theorem 14
P is a bounded polyhedron iff P is a polytop.

Definition 15
Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a, b) = \{x \in \mathbb{R}^n \mid a^T x = b\}$$

is a supporting hyperplane of P if $\max\{a^T x \mid x \in P\} = b$.

Definition 16
Let $P \subseteq \mathbb{R}^n$. F is a face of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H.

Definition 17
Let $P \subseteq \mathbb{R}^n$.

- a face v is a vertex of P if $\{v\}$ is a face of P.
- a face e is an edge of P if e is a face and $\dim(e) = 1$.
- a face F is a facet of P if F is a face and $\dim(F) = \dim(P) - 1$.

Equivalent definition for vertex:

Definition 18
Given polyhedron P. A point $x \in P$ is a vertex if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

Definition 19
Given polyhedron P. A point $x \in P$ is an extreme point if $\nexists a, b \neq x$, $a, b \in P$, with $\lambda a + (1 - \lambda) b = x$ for $\lambda \in [0, 1]$.

Lemma 20
A vertex is also an extreme point.

Observation
The feasible region of an LP is a Polyhedron.

Convex Sets

Theorem 21
If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

- suppose x is optimal solution that is not extreme point
- there exists direction $d \neq 0$ such that $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- Wlog. assume $c^T d \geq 0$ (by taking either d or $-d$)
- Consider $x + \lambda d$, $\lambda > 0$

Convex Sets

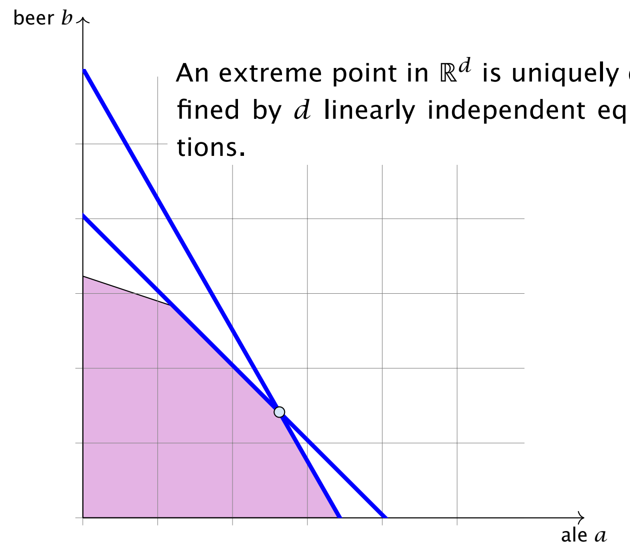
Case 1. [$\exists j$ s.t. $d_j < 0$]

- increase λ to λ' until first component of $x + \lambda d$ hits 0
- $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- $c^T x' = c^T(x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

- $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$
- as $\lambda \to \infty$, $c^T(x + \lambda d) \to \infty$ as $c^T d > 0$

Algebraic View



Notation

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

Theorem 22

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.*

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Proof (⇐)

- assume x is not extreme point
- there exists direction d s.t. $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- define $B' = \{j \mid d_j \neq 0\}$
- $A_{B'}$ has linearly dependent columns as $Ad = 0$
- $d_j = 0$ for all j with $x_j = 0$ as $x \pm d \geq 0$
- Hence, $B' \subseteq B$, $A_{B'}$ is sub-matrix of A_B

Theorem 22

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.*

Proof (⇒)

- assume A_B has linearly dependent columns
- there exists $d \neq 0$ such that $A_B d = 0$
- extend d to $\mathbb{R}^n$ by adding 0-components
- now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$
- for sufficiently small λ we have $x \pm \lambda d \in P$
- hence, x is not extreme point

Theorem 23

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P.

- define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
- then $c^T x = 0$ and $c^T y \leq 0$ for $y \in P$
- assume $c^T y = 0$; then $y_j = 0$ for all $j \notin B$
- $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
- this means that $x_B = y_B$ since A_B has linearly independent columns
- we get $y = x$
- hence, x is a vertex of P

Observation

For an LP we can assume wlog. that the matrix A has full row-rank. This means $\operatorname{rank}(A) = m$.

- assume that $\operatorname{rank}(A) < m$
- assume wlog. that the first row A_1 lies in the span of the other rows $A_2, \dots, A_m$; this means

$$A_1 = \sum_{i=2}^{m} \lambda_i \cdot A_i, \text{ for suitable } \lambda_i$$

C1 if now $b_1 = \sum_{i=2}^{m} \lambda_i \cdot b_i$ then for all x with $A_i x = b_i$ we also have $A_1 x = b_1$; hence the first constraint is superfluous

C2 if $b_1 \neq \sum_{i=2}^{m} \lambda_i \cdot b_i$ then the LP is infeasible, since for all x that fulfill constraints $A_2, \dots, A_m$ we have

$$A_1 x = \sum_{i=2}^{m} \lambda_i \cdot A_i x = \sum_{i=2}^{m} \lambda_i \cdot b_i \neq b_1$$

From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

Theorem 24

Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- A_B is non-singular
- $x_B = A_B^{-1} b \geq 0$
- $x_N = 0$

where $N = \{1, \dots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\operatorname{rank}(A) = m$.

Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and $\operatorname{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a basic **feasible** solution (gültige Basislösung) if in addition $x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \dots, n\}$ with $\operatorname{rank}(A_B) = m$ and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x_B = b$ and $x_j = 0$ for all $j \notin B$ is the basic solution associated to basis B (die zu B assoziierte Basislösung)

Basic Feasible Solutions

A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i's are zero. The corresponding non-negativity constraint is fulfilled with equality.

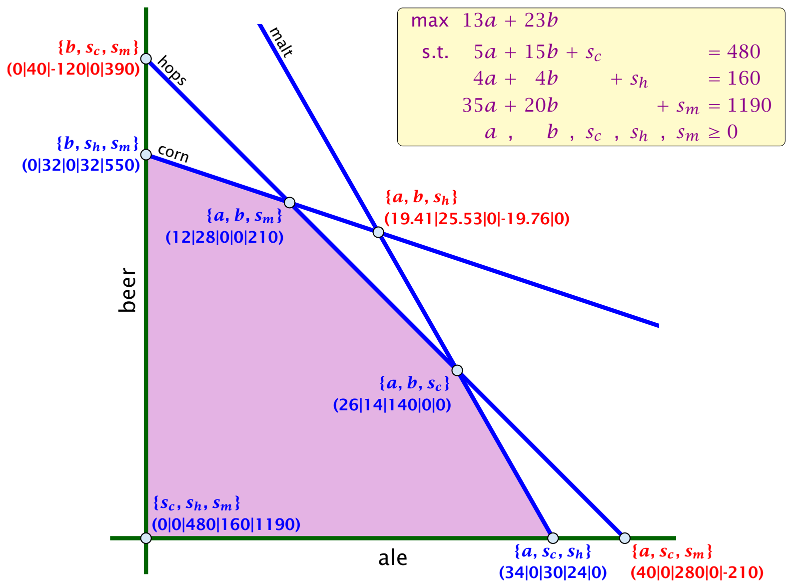
Fact:
In a BFS at least n constraints are fulfilled with equality.

Basic Feasible Solutions

Definition 25
For a general LP ($\max\{c^Tx \mid Ax \le b\}$) with n variables a point x is a basic feasible solution if x is feasible and there exist n (linearly independent) constraints that are tight.

Algebraic View

$$\begin{array}{llcl} \max & 13a + 23b & & \\ \text{s.t.} & 5a + 15b + s_c & = & 480 \\ & 4a + 4b + s_h & = & 160 \\ & 35a + 20b + s_m & = & 1190 \\ & a, b, s_c, s_h, s_m & \ge & 0 \end{array}$$



Fundamental Questions

Linear Programming Problem (LP)
Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \ge 0$, $c^Tx \ge \alpha$?

Questions:

- Is LP in NP? yes!
- Is LP in co-NP?
- Is LP in P?

Proof:

- Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n, m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

What happens if LP is unbounded?