# Repetition: Primal Dual for Set Cover

**Primal Relaxation:**

$$\begin{array}{lrrcl} \min & & \sum_{i=1}^{k} w_i x_i & & \\ \text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq & 1 \\ & \forall i \in \{1,\dots,k\} & x_i & \geq & 0 \end{array}$$

**Dual Formulation:**

$$\begin{array}{lrrcl} \max & & \sum_{u\in U} y_u & & \\ \text{s.t.} & \forall i \in \{1,\dots,k\} & \sum_{u:u\in S_i} y_u & \leq & w_i \\ & & y_u & \geq & 0 \end{array}$$

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# Repetition: Primal Dual for Set Cover

**Algorithm:**

- Start with $y = 0$ (feasible dual solution).
  Start with $x = 0$ (integral primal solution that may be infeasible).
- While $x$ not feasible
  - Identify an element $e$ that is not covered in current primal integral solution.
  - Increase dual variable $y_e$ until a dual constraint becomes tight (maybe increase by $0$!).
  - If this is the constraint for set $S_j$ set $x_j = 1$ (add this set to your solution).

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# Repetition: Primal Dual for Set Cover

**Analysis:**

- For every set $S_j$ with $x_j = 1$ we have

$$\sum_{e\in S_j} y_e = w_j$$

- Hence our cost is

$$\begin{aligned} \sum_j w_j x_j &= \sum_j \sum_{e\in S_j} y_e = \sum_e |\{j : e \in S_j\}| \cdot y_e \\ &\leq f \cdot \sum_e y_e \leq f \cdot \mathrm{OPT} \end{aligned}$$

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Note that the constructed pair of primal and dual solution fulfills primal slackness conditions.

This means

$$x_j > 0 \Rightarrow \sum_{e\in S_j} y_e = w_j$$

If we would also fulfill dual slackness conditions

$$y_e > 0 \Rightarrow \sum_{j:e\in S_j} x_j = 1$$

then the solution would be optimal!!!

We don't fulfill these constraint but we fulfill an approximate version:

$$y_e > 0 \Rightarrow 1 \le \sum_{j:e\in S_j} x_j \le f$$

This is sufficient to show that the solution is an $f$-approximation.

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Suppose we have a primal/dual pair

$$\begin{array}{|llrcl|}\hline \min & & \sum_j c_j x_j & & \\ \text{s.t.} & \forall i & \sum_{j:} a_{ij}x_j & \ge & b_i \\ & \forall j & x_j & \ge & 0 \\ \hline \end{array} \quad \begin{array}{|llrcl|}\hline \max & & \sum_i b_i y_i & & \\ \text{s.t.} & \forall j & \sum_i a_{ij}y_i & \le & c_j \\ & \forall i & y_i & \ge & 0 \\ \hline \end{array}$$

and solutions that fulfill approximate slackness conditions:

$$x_j > 0 \Rightarrow \sum_i a_{ij} y_i \ge \frac{1}{\alpha} c_j$$

$$y_i > 0 \Rightarrow \sum_j a_{ij} x_j \le \beta b_i$$

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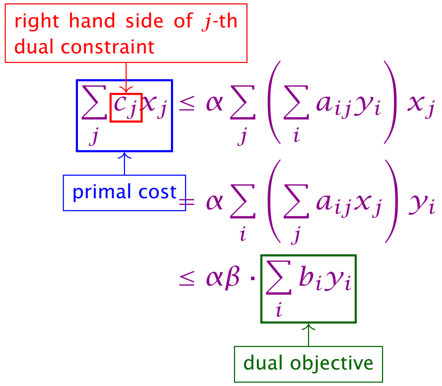
Then



right hand side of $j$-th dual constraint

$$\begin{aligned} \boxed{\sum_j \boxed{c_j} x_j} &\le \alpha \sum_j \left(\sum_i a_{ij} y_i\right) x_j \\ &= \alpha \sum_i \left(\sum_j a_{ij} x_j\right) y_i \\ &\le \alpha\beta \cdot \boxed{\sum_i b_i y_i} \end{aligned}$$

primal cost

dual objective

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# Feedback Vertex Set for Undirected Graphs

- Given a graph $G = (V, E)$ and non-negative weights $w_v \ge 0$ for vertex $v \in V$.
- Choose a minimum cost subset of vertices s.t. every cycle contains at least one vertex.

We can encode this as an instance of Set Cover

- Each vertex can be viewed as a set that contains some cycles.
- However, this encoding gives a Set Cover instance of non-polynomial size.
- The $O(\log n)$-approximation for Set Cover does not help us to get a good solution.

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Let $\mathbb{C}$ denote the set of all cycles (where a cycle is identified by its set of vertices)

**Primal Relaxation:**

$$\begin{array}{llrcl} \min & & \sum_v w_v x_v & & \\ \text{s.t.} & \forall C \in \mathbb{C} & \sum_{v \in C} x_v & \geq & 1 \\ & \forall v & x_v & \geq & 0 \end{array}$$

**Dual Formulation:**

$$\begin{array}{llrcl} \max & & \sum_{C \in \mathbb{C}} y_C & & \\ \text{s.t.} & \forall v \in V & \sum_{C : v \in C} y_C & \leq & w_v \\ & \forall C & y_C & \geq & 0 \end{array}$$

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If we perform the previous dual technique for Set Cover we get the following:

- Start with $x = 0$ and $y = 0$
- While there is a cycle $C$ that is not covered (does not contain a chosen vertex).
  - Increase $y_C$ until dual constraint for some vertex $v$ becomes tight.
  - set $x_v = 1$.

---

Then

$$\begin{aligned} \sum_v w_v x_v &= \sum_v \sum_{C : v \in C} y_C x_v \\ &= \sum_{v \in S} \sum_{C : v \in C} y_C \\ &= \sum_C |S \cap C| \cdot y_C \end{aligned}$$

where $S$ is the set of vertices we choose.

If every cycle is short we get a good approximation ratio, but this is unrealistic.

**Algorithm 1** FeedbackVertexSet

```
$y \leftarrow 0$
$x \leftarrow 0$
while exists cycle $C$ in $G$ do
    increase $y_C$ until there is $v \in C$ s.t. $\sum_{C: v \in C} y_C = w_v$
    $x_v = 1$
    remove $v$ from $G$
    repeatedly remove vertices of degree 1 from $G$
```

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**Idea:**
Always choose a short cycle that is not covered. If we always find a cycle of length at most $\alpha$ we get an $\alpha$-approximation.

**Observation:**
For any path $P$ of vertices of degree $2$ in $G$ the algorithm chooses at most one vertex from $P$.

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**Observation:**
If we always choose a cycle for which the number of vertices of degree at least 3 is at most $\alpha$ we get a $2\alpha$-approximation.

**Theorem 3**
*In any graph with no vertices of degree 1, there always exists a cycle that has at most $\mathcal{O}(\log n)$ vertices of degree 3 or more. We can find such a cycle in linear time.*

This means we have

$$y_C > 0 \Rightarrow |S \cap C| \le \mathcal{O}(\log n) \ .$$

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# Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \to \mathbb{R}^+$ find a shortest path between $s$ and $t$ w.r.t. edge-weights $c$.

$$\begin{array}{llrcl} \min & & \sum_e c(e) x_e & & \\ \text{s.t.} & \forall S \in \mathcal{S} & \sum_{e:\delta(S)} x_e & \ge & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in $S$, and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.

## Primal Dual for Shortest Path

**The Dual:**

$$\begin{array}{llrcl} \max & & \sum_S y_S & & \\ \text{s.t.} & \forall e \in E & \sum_{S:e\in\delta(S)} y_S & \le & c(e) \\ & \forall S \in \mathcal{S} & y_S & \ge & 0 \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in $S$, and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.

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## Primal Dual for Shortest Path

We can interpret the value $y_S$ as the width of a moat surounding the set $S$.

Each set can have its own moat but all moats must be disjoint.

An edge cannot be shorter than all the moats that it has to cross.

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**Algorithm 1** PrimalDualShortestPath

```
$y \leftarrow 0$
$F \leftarrow \emptyset$
while there is no $s$-$t$ path in $(V,F)$ do
    Let $C$ be the connected component of $(V,F)$ containing $s$
    Increase $y_C$ until there is an edge $e' \in \delta(C)$ such that $\sum_{S:e'\in\delta(S)} y_S = c(e')$.
    $F \leftarrow F \cup \{e'\}$
Let $P$ be an $s$-$t$ path in $(V,F)$
return $P$
```

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**Lemma 4**
*At each point in time the set $F$ forms a tree.*

**Proof:**

- In each iteration we take the current connected component from $(V,F)$ that contains $s$ (call this component $C$) and add some edge from $\delta(C)$ to $F$.
- Since, at most one end-point of the new edge is in $C$ the edge cannot close a cycle.

$$\begin{aligned}\sum_{e\in P} c(e) &= \sum_{e\in P}\sum_{S:e\in\delta(S)} y_S\\ &= \sum_{S:s\in S,t\notin S} |P\cap\delta(S)|\cdot y_S\ .\end{aligned}$$

If we can show that $y_S > 0$ implies $|P\cap\delta(S)| = 1$ gives

$$\sum_{e\in P} c(e) = \sum_S y_S \le \mathrm{OPT}$$

by weak duality.

Hence, we find a shortest path.

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If $\delta(S)$ contains two edges from $P$ then there must exist a subpath $P'$ of $P$ that starts and ends with a vertex from $S$ (and all interior vertices are not in $S$).

When we increased $y_S$, $S$ was a connected component of the set of edges $F'$ that we had chosen till this point.

$F'\cup P'$ contains a cycle. Hence, also the final set of edges contains a cycle.

This is a contradiction.

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**Steiner Forest Problem:**
Given a graph $G = (V, E)$, together with source-target pairs $s_i, t_i$, $i = 1, \dots, k$, and a cost function $c : E \to \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that for every $i \in \{1, \dots, k\}$ there is a path between $s_i$ and $t_i$ only using edges in $F$.

$$\begin{array}{llrcl}\min & & \sum_e c(e)x_e & & \\ \text{s.t.} & \forall S\subseteq V: S\in\mathcal{S}_i \text{ for some } i & \sum_{e\in\delta(S)} x_e & \ge & 1\\ & \forall e\in E & x_e & \in & \{0,1\}\end{array}$$

Here $\mathcal{S}_i$ contains all sets $S$ such that $s_i \in S$ and $t_i \notin S$.

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$$\begin{array}{llrcl}\max & & \sum_{S\,:\,\exists i \text{ s.t. } S\in\mathcal{S}_i} y_S & & \\ \text{s.t.} & \forall e\in E & \sum_{S:e\in\delta(S)} y_S & \le & c(e)\\ & & y_S & \ge & 0\end{array}$$

The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).

**Algorithm 1** FirstTry

```
y ← 0
F ← ∅
while not all $s_i$-$t_i$ pairs connected in $F$ do
    Let $C$ be some connected component of $(V,F)$ such that $|C \cap \{s_i,t_i\}| = 1$ for some $i$.
    Increase $y_C$ until there is an edge $e' \in \delta(C)$ s.t. $\sum_{S \in S_i : e' \in \delta(S)} y_S = c_{e'}$
    $F \leftarrow F \cup \{e'\}$
return $\bigcup_i P_i$
```

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$$\sum_{e \in F} c(e) = \sum_{e \in F} \sum_{S : e \in \delta(S)} y_S = \sum_S |\delta(S) \cap F| \cdot y_S \ .$$

If we show that $y_S > 0$ implies that $|\delta(S) \cap F| \leq \alpha$ we are in good shape.

However, this is not true:

- Take a complete graph on $k+1$ vertices $v_0, v_1, \dots, v_k$.
- The $i$-th pair is $v_0$-$v_i$.
- The first component $C$ could be $\{v_0\}$.
- We only set $y_{\{v_0\}} = 1$. All other dual variables stay $0$.
- The final set $F$ contains all edges $\{v_0, v_i\}$, $i = 1, \dots, k$.
- $y_{\{v_0\}} > 0$ but $|\delta(\{v_0\}) \cap F| = k$.

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**Algorithm 1** SecondTry

```
$y \leftarrow 0$; $F \leftarrow \emptyset$; $\ell \leftarrow 0$
while not all $s_i$-$t_i$ pairs connected in $F$ do
    $\ell \leftarrow \ell + 1$
    Let $\mathfrak{C}$ be set of all connected components $C$ of $(V,F)$ such that $|C \cap \{s_i,t_i\}| = 1$ for some $i$.
    Increase $y_C$ for all $C \in \mathfrak{C}$ uniformly until for some edge $e_\ell \in \delta(C')$, $C' \in \mathfrak{C}$ s.t. $\sum_{S : e_\ell \in \delta(S)} y_S = c_{e_\ell}$
    $F \leftarrow F \cup \{e_\ell\}$
$F' \leftarrow F$
for $k \leftarrow \ell$ downto 1 do // reverse deletion
    if $F' - e_k$ is feasible solution then
        remove $e_k$ from $F'$
return $F'$
```

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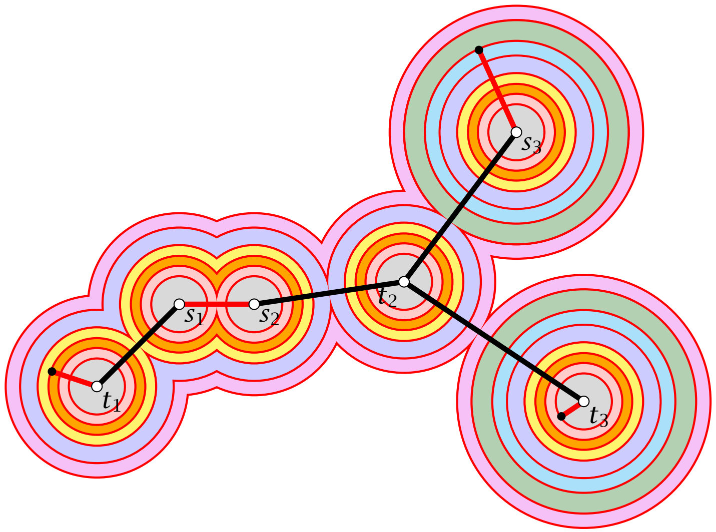
The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

## Example



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**Lemma 5**

*For any $\mathbb{C}$ in any iteration of the algorithm*

$$\sum_{C\in\mathbb{C}} |\delta(C)\cap F'| \le 2|\mathbb{C}|$$

This means that the number of times a moat from $\mathbb{C}$ is crossed in the final solution is at most twice the number of moats.

**Proof:** later...

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$$\sum_{e\in F'} c_e = \sum_{e\in F'} \sum_{S:e\in\delta(S)} y_S = \sum_S |F'\cap\delta(S)|\cdot y_S \ .$$

We want to show that

$$\sum_S |F'\cap\delta(S)|\cdot y_S \le 2\sum_S y_S$$

- In the $i$-th iteration the increase of the left-hand side is

  $$\epsilon\sum_{C\in\mathbb{C}} |F'\cap\delta(C)|$$

  and the increase of the right hand side is $2\epsilon|\mathbb{C}|$.
- Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

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**Lemma 6**

*For any set of connected components $\mathbb{C}$ in any iteration of the algorithm*

$$\sum_{C\in\mathbb{C}} |\delta(C)\cap F'| \le 2|\mathbb{C}|$$

**Proof:**

- At any point during the algorithm the set of edges forms a forest (why?).
- Fix iteration $i$. Let $F_i$ be the set of edges in $F$ at the beginning of the iteration.
- Let $H = F' - F_i$.
- All edges in $H$ are necessary for the solution.

- Contract all edges in $F_i$ into single vertices $V'$.
- We can consider the forest $H$ on the set of vertices $V'$.
- Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- Color a vertex $v \in V'$ red if it corresponds to a component from $\mathbb{C}$ (an active component). Otw. color it blue. (Let $B$ the set of blue vertices (with non-zero degree) and $R$ the set of red vertices)
- We have

$$\sum_{v \in R} \deg(v) \geq \sum_{C \in \mathbb{C}} |\delta(C) \cap F'| \overset{?}{\leq} 2|\mathbb{C}| = 2|R|$$

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- Suppose that no node in $B$ has degree one.
- Then

$$\begin{aligned}\sum_{v \in R} \deg(v) &= \sum_{v \in R \cup B} \deg(v) - \sum_{v \in B} \deg(v) \\ &\leq 2(|R| + |B|) - 2|B| = 2|R|\end{aligned}$$

- Every blue vertex with non-zero degree must have degree at least two.
  - Suppose not. The single edge connecting $b \in B$ comes from $H$, and, hence, is necessary.
  - But this means that the cluster corresponding to $b$ must separate a source-target pair.
  - But then it must be a red node.