14 Rounding Data + Dynamic Programming

Knapsack:
Given a set of items $\{1,\ldots,n\}$, where the i-th item has weight $w_i \in \mathbb{N}$ and profit $p_i \in \mathbb{N}$, and given a threshold W. Find a subset $I \subseteq \{1,\ldots,n\}$ of items of total weight at most W such that the profit is maximized (we can assume each $w_i \le W$).

$$\begin{array}{llrcl} \max & & \sum_{i=1}^{n} p_i x_i & & \\ \text{s.t.} & & \sum_{i=1}^{n} w_i x_i & \le & W \\ & \forall i \in \{1,\ldots,n\} & x_i & \in & \{0,1\} \end{array}$$

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Algorithm 1 Knapsack

$A(1) \leftarrow [(0,0),(p_1,w_1)]$
for $j \leftarrow 2$ to n **do**
    $A(j) \leftarrow A(j-1)$
    for each $(p,w) \in A(j-1)$ **do**
        if $w + w_j \leq W$ **then**
            add $(p+p_j, w+w_j)$ to $A(j)$
        remove dominated pairs from $A(j)$
return $\max_{(p,w)\in A(n)} p$

The running time is $\mathcal{O}(n \cdot \min\{W, P\})$, where $P = \sum_i p_i$ is the total profit of all items. This is only pseudo-polynomial.

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Definition 3
An algorithm is said to have pseudo-polynomial running time if the running time is polynomial when the numerical part of the input is encoded in unary.

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Let S be the set of items returned by the algorithm, and let O be an optimum set of items.

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Scheduling Revisited

The previous analysis of the scheduling algorithm gave a makespan of

$$\frac{1}{m}\sum_{j\neq\ell} p_j + p_\ell$$

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Together with the obervation that if each $p_i \geq \frac{1}{3}C^*_{\max}$ then LPT is optimal this gave a $4/3$-approximation.

14.2 Scheduling Revisited

Partition the input into long jobs and short jobs.

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Idea:

1. Find the optimum Makespan for the long jobs by brute force.

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Idea:

1. Find the optimum Makespan for the long jobs by brute force.
2. Then use the list scheduling algorithm for the short jobs, always assigning the next job to the least loaded machine.

We still have a cost of

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If ℓ is a long job, then the schedule must be optimal, as it consists of an optimal schedule of long jobs plus a schedule for short jobs.

If ℓ is a short job its length is at most

$$p_\ell \leq \sum_j p_j/(mk)$$

which is at most $C^*_{\max}/k$.

Hence we get a schedule of length at most

$$\left(1 + \frac{1}{k}\right) C^*_{\max}$$

There are at most km long jobs. Hence, the number of possibilities of scheduling these jobs on m machines is at most m^{km}, which is constant if m is constant. Hence, it is easy to implement the algorithm in polynomial time.

Theorem 4

The above algorithm gives a polynomial time approximation scheme (PTAS) for the problem of scheduling n jobs on m identical machines if m is constant.

We choose $k = \lceil \frac{1}{\epsilon} \rceil$.

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How to get rid of the requirement that m is constant?

We first design an algorithm that works as follows:
On input of T it either finds a schedule of length $(1+\frac{1}{k})T$ or certifies that no schedule of length at most T exists (assume $T \geq \frac{1}{m}\sum_j p_j$).

We partition the jobs into long jobs and short jobs:

- A job is long if its size is larger than T/k.
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- We round all long jobs down to multiples of T/k^2.
- For these rounded sizes we first find an optimal schedule.
- If this schedule does not have length at most T we conclude that also the original sizes don't allow such a schedule.
- If we have a good schedule we extend it by adding the short jobs according to the LPT rule.

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After the first phase the rounded sizes of the long jobs assigned to a machine add up to at most T.

There can be at most k (long) jobs assigned to a machine as otw. their rounded sizes would add up to more than T (note that the rounded size of a long job is at least T/k).

Since, jobs had been rounded to multiples of T/k^2 going from rounded sizes to original sizes gives that the Makespan is at most

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During the second phase there always must exist a machine with load at most T, since T is larger than the average load.

Assigning the current (short) job to such a machine gives that the new load is at most

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Running Time for scheduling large jobs: There should not be a job with rounded size more than T as otw. the problem becomes trivial.

Hence, any large job has rounded size of $\frac{i}{k^2}T$ for $i \in \{k, \ldots, k^2\}$. Therefore the number of different inputs is at most n^{k^2} (described by a vector of length k^2 where, the i-th entry describes the number of jobs of size $\frac{i}{k^2}T$). This is polynomial.

The schedule/configuration of a particular machine x can be described by a vector of length k^2 where the i-th entry describes the number of jobs of rounded size $\frac{i}{k^2}T$ assigned to x. There are only $(k+1)^{k^2}$ different vectors.

This means there are a constant number of different machine configurations.

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Let $\mathrm{OPT}(n_1,\ldots,n_{k^2})$ be the number of machines that are required to schedule input vector $(n_1,\ldots,n_{k^2})$ with Makespan at most T.

If $\mathrm{OPT}(n_1,\ldots,n_{k^2}) \leq m$ we can schedule the input.

We have

$$\mathrm{OPT}(n_1,\ldots,n_{k^2}) = \begin{cases} 0 & (n_1,\ldots,n_{k^2}) = 0 \\ 1 + \min\limits_{(s_1,\ldots,s_{k^2})\in C} \mathrm{OPT}(n_1 - s_1,\ldots,n_{k^2} - s_{k^2}) & (n_1,\ldots,n_{k^2}) \ngeq 0 \\ \infty & \text{otw.} \end{cases}$$

where C is the set of all configurations.

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We can turn this into a PTAS by choosing $k = \lceil 1/\epsilon \rceil$ and using binary search. This gives a running time that is exponential in $1/\epsilon$.

Can we do better?

Scheduling on identical machines with the goal of minimizing Makespan is a strongly NP-complete problem.

Theorem 5

There is no FPTAS for problems that are strongly NP-hard.

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There is no FPTAS for problems that are strongly NP-hard.

- Suppose we have an instance with polynomially bounded processing times $p_i \leq q(n)$
- We set $k := \lceil 2nq(n) \rceil \geq 2\,\mathrm{OPT}$
- Then

$$\mathrm{ALG} \leq \left(1 + \frac{1}{k}\right)\mathrm{OPT} \leq \mathrm{OPT} + \frac{1}{2}$$

- But this means that the algorithm computes the optimal solution as the optimum is integral.
- This means we can solve problem instances if processing times are polynomially bounded
- Running time is $\mathcal{O}(\mathrm{poly}(n,k)) = \mathcal{O}(\mathrm{poly}(n))$
- For strongly NP-complete problems this is not possible unless P=NP

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More General

Let $\mathrm{OPT}(n_1,\dots,n_A)$ be the number of machines that are required to schedule input vector $(n_1,\dots,n_A)$ with Makespan at most T (A: number of different sizes).

If $\mathrm{OPT}(n_1,\dots,n_A) \le m$ we can schedule the input.

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The running time is then $O((B+1)^A n^A)$ because the dynamic programming table has just n^A entries.

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Bin Packing

Given n items with sizes $s_1, \ldots, s_n$ where

$$1 > s_1 \geq \cdots \geq s_n > 0 \ .$$

Pack items into a minimum number of bins where each bin can hold items of total size at most 1.

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Proof

- In the partition problem we are given positive integers $b_1, \dots, b_n$ with $B = \sum_i b_i$ even. Can we partition the integers into two sets S and T s.t.

$$\sum_{i \in S} b_i = \sum_{i \in T} b_i \quad ?$$

- We can solve this problem by setting $s_i := 2b_i/B$ and asking whether we can pack the resulting items into 2 bins or not.
- A ρ-approximation algorithm with $\rho < 3/2$ cannot output 3 or more bins when 2 are optimal.
- Hence, such an algorithm can solve Partition.

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Definition 7

An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithms $\{A_\epsilon\}$ along with a constant c such that A_ϵ returns a solution of value at most $(1+\epsilon)\mathrm{OPT} + c$ for minimization problems.

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Bin Packing

Again we can differentiate between small and large items.

Lemma 8

Any packing of items into ℓ bins can be extended with items of size at most γ s.t. we use only $\max\{\ell, \frac{1}{1-\gamma}\mathrm{SIZE}(I) + 1\}$ bins, where $\mathrm{SIZE}(I) = \sum_i s_i$ is the sum of all item sizes.

- If after Greedy we use more than ℓ bins, all bins (apart from the last) must be full to at least $1 - \gamma$.
- Hence, $r(1 - \gamma) \leq \mathrm{SIZE}(I)$ where r is the number of nearly-full bins.
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Choose $\gamma = \epsilon/2$. Then we either use ℓ bins or at most

$$\frac{1}{1-\epsilon/2} \cdot \mathrm{OPT} + 1 \leq (1+\epsilon) \cdot \mathrm{OPT} + 1$$

bins.

It remains to find an algorithm for the large items.

Bin Packing

Linear Grouping:

Generate an instance I' (for large items) as follows.

- Order large items according to size.
- Let the first k items belong to group 1; the following k items belong to group 2; etc.
- Delete items in the first group;
- Round items in the remaining groups to the size of the largest item in the group.

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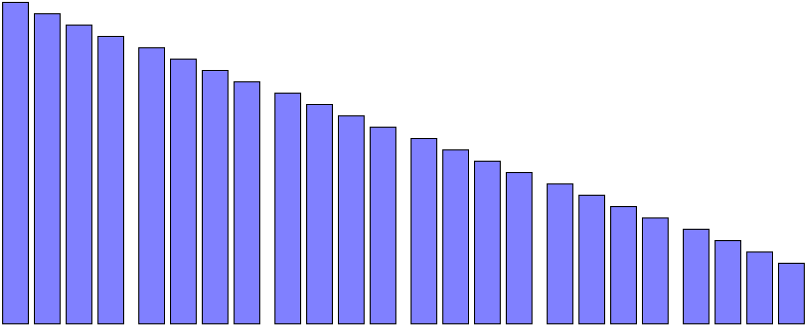

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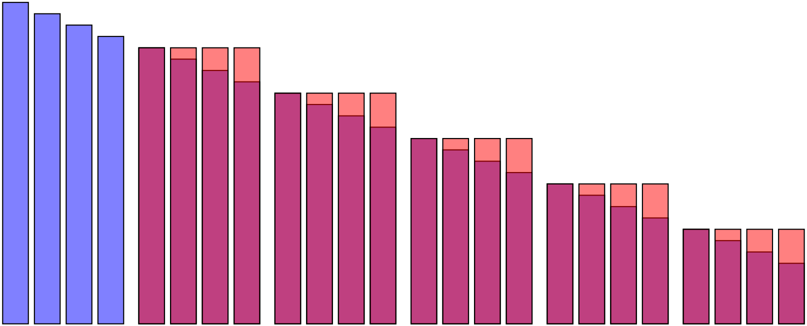

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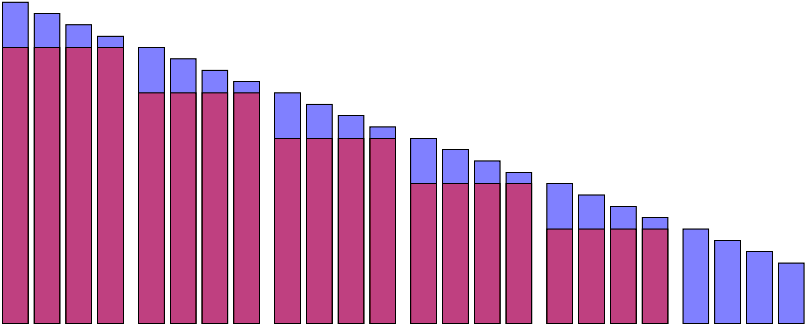

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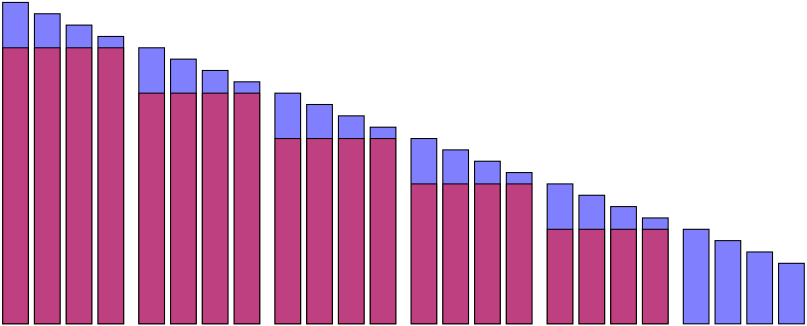

Lemma 9

$\mathrm{OPT}(I') \leq \mathrm{OPT}(I) \leq \mathrm{OPT}(I') + k$

Proof 1:

- Any bin packing for I gives a bin packing for I' as follows.
- Pack the items of group 2, where in the packing for I the items for group 1 have been packed;
- Pack the items of groups 3, where in the packing for I the items for group 2 have been packed;
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Assume that our instance does not contain pieces smaller than $\epsilon/2$. Then $\mathrm{SIZE}(I) \geq \epsilon n/2$.

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Hence, after grouping we have a constant number of piece sizes ($4/\epsilon^2$) and at most a constant number ($2/\epsilon$) can fit into any bin.

We can find an optimal packing for such instances by the previous Dynamic Programming approach.

- cost (for large items) at most

$$\mathrm{OPT}(I') + k \leq \mathrm{OPT}(I) + \epsilon \mathrm{SIZE}(I) \leq (1+\epsilon)\mathrm{OPT}(I)$$

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Change of Notation:

- Group pieces of identical size.
- Let s_1 denote the largest size, and let b_1 denote the number of pieces of size s_1.
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$$\sum_i t_i \cdot s_i \leq 1 \enspace .$$

We call a vector that fulfills the above constraint a configuration.

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Let $T_1, \dots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).

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\min & & \sum_{j=1}^{N} x_j & & \\
\text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\
 & \forall j \in \{1, \dots, N\} & x_j & \geq & 0 \\
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How to solve this LP?

later...

We can assume that each item has size at least $1/\mathrm{SIZE}(I)$.

Harmonic Grouping

- Sort items according to size (monotonically decreasing).
- Process items in this order; close the current group if size of items in the group is at least 2 (or larger). Then open new group.
- I.e., G_1 is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for $G_2, \ldots, G_{r-1}$.
- Only the size of items in the last group G_r may sum up to less than 2.

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- Process items in this order; close the current group if size of items in the group is at least 2 (or larger). Then open new group.
- I.e., G_1 is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for $G_2, \dots, G_{r-1}$.
- Only the size of items in the last group G_r may sum up to less than 2.

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Harmonic Grouping

From the grouping we obtain instance I' as follows:

- Round all items in a group to the size of the largest group member.
- Delete all items from group G_1 and G_r.
- For groups $G_2, \ldots, G_{r-1}$ delete $n_i - n_{i-1}$ items.
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Lemma 11

The number of different sizes in I' is at most $\mathrm{SIZE}(I)/2$.

- Each group that survives (recall that G_1 and G_k are deleted) has total size at least 2.
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Lemma 12

The total size of deleted items is at most $\mathcal{O}(\log(\mathrm{SIZE}(I)))$.

- The total size of items in G_1 and G_r is at most 6 as a group has total size at most 3.
- Consider a group G_i that has strictly more items than G_{i-1}.
- It discards $n_i - n_{i-1}$ pieces of total size at most

$$3\frac{n_i - n_{i-1}}{n_i} \le \sum_{j=n_{i-1}+1}^{n_i} \frac{3}{j}$$

since the average piece size is only $3/n_i$.

- Summing over all i that have $n_i > n_{i-1}$ gives a bound of at most

$$\sum_{j=1}^{n_{r-1}} \frac{3}{j} \le \mathcal{O}(\log(\mathrm{SIZE}(I))) .$$

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Algorithm 1 BinPack

if $\text{SIZE}(I) < 10$ **then**
    pack remaining items greedily
Apply harmonic grouping to create instance I'; pack discarded items in at most $\mathcal{O}(\log(\text{SIZE}(I)))$ bins.
Let x be optimal solution to configuration LP
Pack $\lfloor x_j \rfloor$ bins in configuration T_j for all j; call the packed instance I_1.
Let I_2 be remaining pieces from I'
Pack I_2 via BinPack(I_2)

Analysis

$$\mathrm{OPT}_{\mathrm{LP}}(I_1) + \mathrm{OPT}_{\mathrm{LP}}(I_2) \le \mathrm{OPT}_{\mathrm{LP}}(I') \le \mathrm{OPT}_{\mathrm{LP}}(I)$$

Proof:

- Each piece surviving in I' can be mapped to a piece in I of no lesser size. Hence, $\mathrm{OPT}_{\mathrm{LP}}(I') \le \mathrm{OPT}_{\mathrm{LP}}(I)$
- $\lfloor x_j \rfloor$ is feasible solution for I_1 (even integral).
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Analysis

Each level of the recursion partitions pieces into three types

1. Pieces discarded at this level.
2. Pieces scheduled because they are in I_1.
3. Pieces in I_2 are handed down to the next level.

Pieces of type 2 summed over all recursion levels are packed into at most $\mathrm{OPT}_{\mathrm{LP}}$ many bins.

Pieces of type 1 are packed into at most

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How to solve the LP?

Let $T_1, \dots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).

In total we have b_i pieces of size s_i.

Primal

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\begin{array}{llrcl}
\min & & \sum_{j=1}^{N} x_j & & \\
\text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\
 & \forall j \in \{1, \dots, N\} & x_j & \geq & 0
\end{array}
$$

Dual

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\begin{array}{llrcl}
\max & & \sum_{i=1}^{m} y_i b_i & & \\
\text{s.t.} & \forall j \in \{1, \dots, N\} & \sum_{i=1}^{m} T_{ji} y_i & \leq & 1 \\
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Separation Oracle

Suppose that I am given variable assignment y for the dual.

How do I find a violated constraint?

I have to find a configuration $T_j = (T_{j1}, \dots, T_{jm})$ that

- is feasible, i.e.,

$$\sum_{i=1}^{m} T_{ji} \cdot s_i \le 1 ,$$

- and has a large profit

$$\sum_{i=1}^{m} T_{ji} y_i > 1$$

But this is the Knapsack problem.

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We have FPTAS for Knapsack. This means if a constraint is violated with $1+\epsilon' = 1+\frac{\epsilon}{1-\epsilon}$ we find it, since we can obtain at least $(1-\epsilon)$ of the optimal profit.

The solution we get is feasible for:

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$$\begin{array}{llrcl}
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If the value of the computed dual solution (which may be infeasible) is z then

$$\mathrm{OPT} \le z \le (1 + \epsilon')\mathrm{OPT}$$

How do we get good primal solution (not just the value)?

- The constraints used when computing z certify that the solution is feasible for DUAL′.
- Suppose that we drop all unused constraints in DUAL. We will compute the same solution feasible for DUAL′.
- Let DUAL″ be DUAL without unused constraints.
- The dual to DUAL″ is PRIMAL where we ignore variables for which the corresponding dual constraint has not been used.
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This gives that overall we need at most

$$(1+\epsilon')\mathrm{OPT}_{\mathrm{LP}}(I)+\mathcal{O}(\log^2(\mathrm{SIZE}(I)))$$

bins.

We can choose $\epsilon' = \frac{1}{\mathrm{OPT}}$ as $\mathrm{OPT} \leq \#\mathrm{items}$ and since we have a fully polynomial time approximation scheme (FPTAS) for knapsack.

This gives that overall we need at most

$$(1+\epsilon')\mathrm{OPT}_{\mathrm{LP}}(I) + \mathcal{O}(\log^2(\mathrm{SIZE}(I)))$$

bins.

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