

## 4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

**Simplex Algorithm** [George Dantzig 1947]

Move from BFS to **adjacent** BFS, without decreasing objective function.

Two BFSs are called **adjacent** if the bases just differ in one variable.

## 4 Simplex Algorithm

$$\begin{array}{rcl} \max & 13a + 23b & \\ \text{s.t.} & 5a + 15b + s_c & = 480 \\ & 4a + 4b + s_h & = 160 \\ & 35a + 20b + s_m & = 1190 \\ & a, b, s_c, s_h, s_m & \geq 0 \end{array}$$

$$\begin{array}{rcl} \max Z & & - Z = 0 \\ 13a + 23b & & - Z = 0 \\ 5a + 15b + s_c & & = 480 \\ 4a + 4b + s_h & & = 160 \\ 35a + 20b + s_m & & = 1190 \\ a, b, s_c, s_h, s_m & & \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

## Pivoting Step

$$\begin{array}{rcl} \max Z & & - Z = 0 \\ 13a + 23b & & - Z = 0 \\ 5a + 15b + s_c & & = 480 \\ 4a + 4b + s_h & & = 160 \\ 35a + 20b + s_m & & = 1190 \\ a, b, s_c, s_h, s_m & & \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- ▶ choose variable to bring into the basis
- ▶ chosen variable should have positive coefficient in objective function
- ▶ apply **min-ratio** test to find out by how much the variable can be increased
- ▶ pivot on row found by min-ratio test
- ▶ the existing basis variable in this row leaves the basis

$$\begin{array}{rcl} \max Z & & - Z = 0 \\ 13a + 23b & & - Z = 0 \\ 5a + 15b + s_c & & = 480 \\ 4a + 4b + s_h & & = 160 \\ 35a + 20b + s_m & & = 1190 \\ a, b, s_c, s_h, s_m & & \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{s_c, s_h, s_m\} \\ a = b = 0 \\ Z = 0 \\ s_c = 480 \\ s_h = 160 \\ s_m = 1190 \end{array}$$

- ▶ Choose variable with coefficient  $> 0$  as **entering variable**.
- ▶ If we keep  $a = 0$  and increase  $b$  from 0 to  $\theta > 0$  s.t. all constraints ( $Ax = b, x \geq 0$ ) are still fulfilled the objective value  $Z$  will strictly increase.
- ▶ For maintaining  $Ax = b$  we need e.g. to set  $s_c = 480 - 15\theta$ .
- ▶ Choosing  $\theta = \min\{480/15, 160/4, 1190/20\}$  ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- ▶ The basic variable in the row that gives  $\min\{480/15, 160/4, 1190/20\}$  becomes the **leaving variable**.

$$\begin{array}{rcl}
 \max Z & & \\
 13a + 23b & - Z = & 0 \\
 5a + 15b + s_c & = & 480 \\
 4a + 4b + s_h & = & 160 \\
 35a + 20b + s_m & = & 1190 \\
 a, b, s_c, s_h, s_m & \geq & 0
 \end{array}$$

$$\begin{array}{l}
 \text{basis} = \{s_c, s_h, s_m\} \\
 a = b = 0 \\
 Z = 0 \\
 s_c = 480 \\
 s_h = 160 \\
 s_m = 1190
 \end{array}$$

Substitute  $b = \frac{1}{15}(480 - 5a - s_c)$ .

$$\begin{array}{rcl}
 \max Z & & \\
 \frac{16}{3}a - \frac{23}{15}s_c & - Z = & -736 \\
 \frac{1}{3}a + b + \frac{1}{15}s_c & = & 32 \\
 \frac{8}{3}a - \frac{4}{15}s_c + s_h & = & 32 \\
 \frac{85}{3}a - \frac{4}{3}s_c + s_m & = & 550 \\
 a, b, s_c, s_h, s_m & \geq & 0
 \end{array}$$

$$\begin{array}{l}
 \text{basis} = \{b, s_h, s_m\} \\
 a = s_c = 0 \\
 Z = 736 \\
 b = 32 \\
 s_h = 32 \\
 s_m = 550
 \end{array}$$

$$\begin{array}{rcl}
 \max Z & & \\
 \frac{16}{3}a - \frac{23}{15}s_c & - Z = & -736 \\
 \frac{1}{3}a + b + \frac{1}{15}s_c & = & 32 \\
 \frac{8}{3}a - \frac{4}{15}s_c + s_h & = & 32 \\
 \frac{85}{3}a - \frac{4}{3}s_c + s_m & = & 550 \\
 a, b, s_c, s_h, s_m & \geq & 0
 \end{array}$$

$$\begin{array}{l}
 \text{basis} = \{b, s_h, s_m\} \\
 a = s_c = 0 \\
 Z = 736 \\
 b = 32 \\
 s_h = 32 \\
 s_m = 550
 \end{array}$$

Choose variable  $a$  to bring into basis.

Computing  $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$  means pivot on line 2.

Substitute  $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$ .

$$\begin{array}{rcl}
 \max Z & & \\
 -s_c - 2s_h & - Z = & -800 \\
 b + \frac{1}{10}s_c - \frac{1}{8}s_h & = & 28 \\
 a - \frac{1}{10}s_c + \frac{3}{8}s_h & = & 12 \\
 \frac{3}{2}s_c - \frac{85}{8}s_h + s_m & = & 210 \\
 a, b, s_c, s_h, s_m & \geq & 0
 \end{array}$$

$$\begin{array}{l}
 \text{basis} = \{a, b, s_m\} \\
 s_c = s_h = 0 \\
 Z = 800 \\
 b = 28 \\
 a = 12 \\
 s_m = 210
 \end{array}$$

## 4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

**Solution is optimal:**

- ▶ any feasible solution satisfies all equations in the tableaux
- ▶ in particular:  $Z = 800 - s_c - 2s_h, s_c \geq 0, s_h \geq 0$
- ▶ hence optimum solution value is at most 800
- ▶ the current solution has value 800

## Matrix View

Let our linear program be

$$\begin{array}{l}
 c_B^T x_B + c_N^T x_N = Z \\
 A_B x_B + A_N x_N = b \\
 x_B, x_N \geq 0
 \end{array}$$

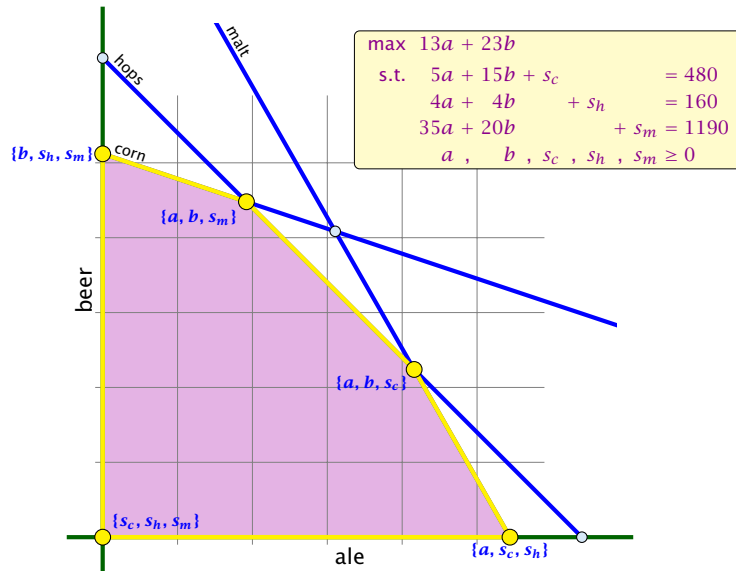
The simplex tableaux for basis  $B$  is

$$\begin{array}{l}
 (c_N^T - c_B^T A_B^{-1} A_N) x_N = Z - c_B^T A_B^{-1} b \\
 I x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\
 x_B, x_N \geq 0
 \end{array}$$

The BFS is given by  $x_N = 0, x_B = A_B^{-1} b$ .

If  $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$  we know that we have an optimum solution.

## Geometric View of Pivoting



## Algebraic Definition of Pivoting

- ▶ Given basis  $B$  with BFS  $x^*$ .
- ▶ Choose index  $j \notin B$  in order to increase  $x_j^*$  from 0 to  $\theta > 0$ .
  - ▶ Other non-basis variables should stay at 0.
  - ▶ Basis variables change to maintain feasibility.
- ▶ Go from  $x^*$  to  $x^* + \theta \cdot d$ .

### Requirements for $d$ :

- ▶  $d_j = 1$  (normalization)
- ▶  $d_\ell = 0, \ell \notin B, \ell \neq j$
- ▶  $A(x^* + \theta d) = b$  must hold. Hence  $Ad = 0$ .
- ▶ Altogether:  $A_B d_B + A_{*j} = Ad = 0$ , which gives  $d_B = -A_B^{-1} A_{*j}$ .

## Algebraic Definition of Pivoting

### Definition 2 ( $j$ -th basis direction)

Let  $B$  be a basis, and let  $j \notin B$ . The vector  $d$  with  $d_j = 1$  and  $d_\ell = 0, \ell \notin B, \ell \neq j$  and  $d_B = -A_B^{-1} A_{*j}$  is called the  $j$ -th basis direction for  $B$ .

Going from  $x^*$  to  $x^* + \theta \cdot d$  the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

## Algebraic Definition of Pivoting

### Definition 3 (Reduced Cost)

For a basis  $B$  the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the **reduced cost** for variable  $x_j$ .

Note that this is defined for every  $j$ . If  $j \in B$  then the above term is 0.

## Algebraic Definition of Pivoting

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\A_B x_B + A_N x_N &= b \\x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis  $B$  is

$$\begin{aligned}(c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\I x_B + A_B^{-1} A_N x_N &= A_B^{-1} b \\x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by  $x_N = 0, x_B = A_B^{-1} b$ .

If  $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$  we know that we have an optimum solution.

## 4 Simplex Algorithm

Questions:

- ▶ What happens if the min ratio test fails to give us a value  $\theta$  by which we can safely increase the entering variable?
- ▶ How do we find the initial basic feasible solution?
- ▶ Is there always a basis  $B$  such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 ?$$

Then we can terminate because we know that the solution is optimal.

- ▶ If yes how do we make sure that we reach such a basis?

## Min Ratio Test

The min ratio test computes a value  $\theta \geq 0$  such that after setting the entering variable to  $\theta$  the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes  $b_i/A_{ie}$  for all constraints  $i$  and calculates the minimum positive value.

What does it mean that the ratio  $b_i/A_{ie}$  (and hence  $A_{ie}$ ) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase  $b$ . Hence, there is no danger of this basic variable becoming negative

What happens if **all**  $b_i/A_{ie}$  are negative? Then we do not have a leaving variable. **Then the LP is unbounded!**

## Termination

The objective function does not decrease during one iteration of the simplex-algorithm.

Does it always increase?

## Termination

The objective function may not increase!

Because a variable  $x_\ell$  with  $\ell \in B$  is already 0.

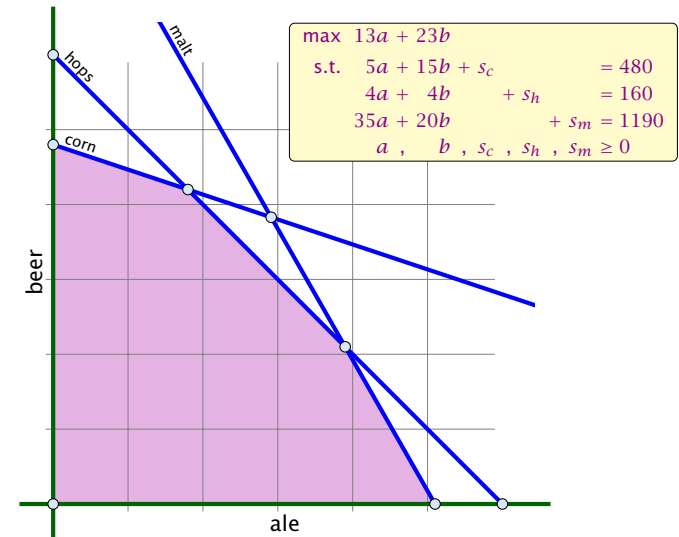
The set of inequalities is **degenerate** (also the basis is degenerate).

### Definition 4 (Degeneracy)

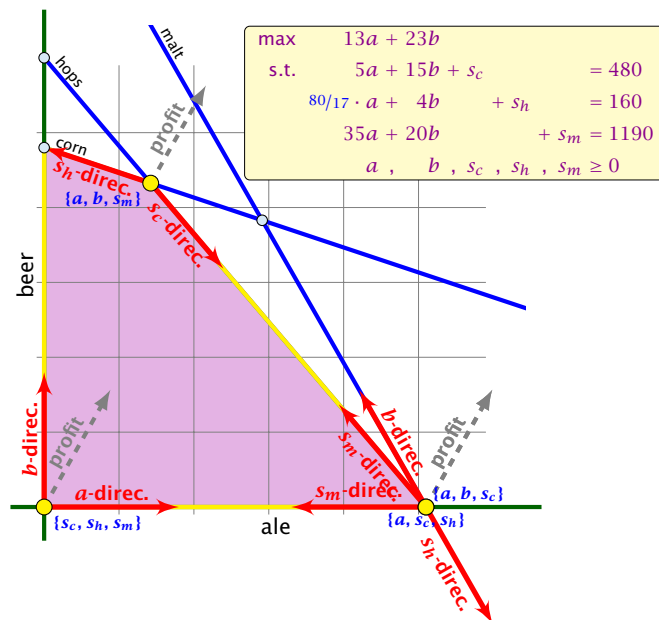
A BFS  $x^*$  is called **degenerate** if the set  $J = \{j \mid x_j^* > 0\}$  fulfills  $|J| < m$ .

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

## Non Degenerate Example



## Degenerate Example



## Summary: How to choose pivot-elements

- ▶ We can choose a column  $e$  as an entering variable if  $\tilde{c}_e > 0$  ( $\tilde{c}_e$  is reduced cost for  $x_e$ ).
- ▶ The standard choice is the column that maximizes  $\tilde{c}_e$ .
- ▶ If  $A_{ie} \leq 0$  for all  $i \in \{1, \dots, m\}$  then the maximum is not bounded.
- ▶ Otw. choose a leaving variable  $\ell$  such that  $b_\ell / A_{\ell e}$  is minimal among all variables  $i$  with  $A_{ie} > 0$ .
- ▶ If several variables have minimum  $b_\ell / A_{\ell e}$  you reach a **degenerate** basis.
- ▶ Depending on the choice of  $\ell$  it may happen that the algorithm runs into a cycle where it does not escape from a degenerate vertex.

## Termination

### What do we have so far?

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is **unbounded**, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an **optimum solution**.

### How do we come up with an initial solution?

- ▶  $Ax \leq b, x \geq 0$ , and  $b \geq 0$ .
- ▶ The standard slack form for this problem is  $Ax + Is = b, x \geq 0, s \geq 0$ , where  $s$  denotes the vector of slack variables.
- ▶ Then  $s = b, x = 0$  is a basic feasible solution (how?).
- ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

## Two phase algorithm

Suppose we want to maximize  $c^T x$  s.t.  $Ax = b, x \geq 0$ .

1. Multiply all rows with  $b_i < 0$  by  $-1$ .
2. maximize  $-\sum_i v_i$  s.t.  $Ax + Iv = b, x \geq 0, v \geq 0$  using Simplex.  $x = 0, v = b$  is initial feasible.
3. If  $\sum_i v_i > 0$  then the original problem is **infeasible**.
4. Otw. you have  $x \geq 0$  with  $Ax = b$ .
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

## Optimality

### Lemma 5

Let  $B$  be a basis and  $x^*$  a BFS corresponding to basis  $B$ .  $\tilde{c} \leq 0$  implies that  $x^*$  is an optimum solution to the LP.