Flows

Definition 3

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)

$$0 \leq f_{xy} \leq c_{xy} .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} .$$

   (flow conservation constraints)

Flows

Definition 4

The value of an (s,t)-flow f is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

LP-Formulation of Maxflow

$$
\begin{array}{rlrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \leq & c_{zw} & \ell_{zw} \\
& \forall w \neq s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
& & f_{zw} & \geq & 0 &
\end{array}
$$

$$
\begin{array}{rlrcl}
\min & & \sum_{(xy)} c_{xy} \ell_{xy} & & \\
\text{s.t.} & f_{xy} \ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& f_{sy} \ (y \neq s,t): & 1\ell_{sy} \qquad\; + 1p_y & \geq & 1 \\
& f_{xs} \ (x \neq s,t): & 1\ell_{xs} - 1p_x & \geq & -1 \\
& f_{ty} \ (y \neq s,t): & 1\ell_{ty} \qquad\; + 1p_y & \geq & 0 \\
& f_{xt} \ (x \neq s,t): & 1\ell_{xt} - 1p_x & \geq & 0 \\
& f_{st}: & 1\ell_{st} & \geq & 1 \\
& f_{ts}: & 1\ell_{ts} & \geq & -1 \\
& & \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{rlrcl}
\min & & \sum_{(xy)} c_{xy} \ell_{xy} & & \\
\text{s.t.} & f_{xy} \ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& f_{sy} \ (y \neq s,t): & 1\ell_{sy} - \quad 1 + 1p_y & \geq & 0 \\
& f_{xs} \ (x \neq s,t): & 1\ell_{xs} - 1p_x + \quad 1 & \geq & 0 \\
& f_{ty} \ (y \neq s,t): & 1\ell_{ty} - \quad 0 + 1p_y & \geq & 0 \\
& f_{xt} \ (x \neq s,t): & 1\ell_{xt} - 1p_x + \quad 0 & \geq & 0 \\
& f_{st}: & 1\ell_{st} - \quad 1 + \quad 0 & \geq & 0 \\
& f_{ts}: & 1\ell_{ts} - \quad 0 + \quad 1 & \geq & 0 \\
& & \ell_{xy} & \geq & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{lrcl}
\min & \multicolumn{3}{r}{\sum_{(xy)} c_{xy}\ell_{xy}} \\
\text{s.t.} \quad f_{xy}\,(x,y \neq s,t): & 1\ell_{xy}-1p_x+1p_y & \geq & 0 \\
\quad f_{sy}\,(y \neq s,t): & 1\ell_{sy}-\ \ p_s+1p_y & \geq & 0 \\
\quad f_{xs}\,(x \neq s,t): & 1\ell_{xs}-1p_x+\ \ p_s & \geq & 0 \\
\quad f_{ty}\,(y \neq s,t): & 1\ell_{ty}-\ \ p_t+1p_y & \geq & 0 \\
\quad f_{xt}\,(x \neq s,t): & 1\ell_{xt}-1p_x+\ \ p_t & \geq & 0 \\
\quad f_{st}: & 1\ell_{st}-\ \ p_s+\ \ p_t & \geq & 0 \\
\quad f_{ts}: & 1\ell_{ts}-\ \ p_t+\ \ p_s & \geq & 0 \\
& \ell_{xy} & \geq & 0
\end{array}
$$

with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

$$
\begin{array}{lrcl}
\min & \multicolumn{3}{r}{\sum_{(xy)} c_{xy}\ell_{xy}} \\
\text{s.t.} \quad f_{xy}: & 1\ell_{xy}-1p_x+1p_y & \geq & 0 \\
& \ell_{xy} & \geq & 0 \\
& p_s & = & 1 \\
& p_t & = & 0
\end{array}
$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.