Complementary Slackness

Lemma 3

Assume a linear program $P = \max\{c^T x \mid Ax \leq b; x \geq 0\}$ has solution x^* and its dual $D = \min\{b^T y \mid A^T y \geq c; y \geq 0\}$ has solution y^*.

1. *If $x_j^* > 0$ then the j-th constraint in D is tight.*
2. *If the j-th constraint in D is not tight than $x_j^* = 0$.*
3. *If $y_i^* > 0$ then the i-th constraint in P is tight.*
4. *If the i-th constraint in P is not tight than $y_i^* = 0$.*

If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \leq y^{*T} A x^* \leq b^T y^*$$

Because of strong duality we then get

$$c^T x^* = y^{*T} A x^* = b^T y^*$$

This gives e.g.

$$\sum_j (y^T A - c^T)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^T A \geq c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

▶ Brewer: find mix of ale and beer that maximizes profits

$$
\begin{aligned}
\max \quad & 13a + 23b \\
\text{s.t.} \quad & 5a + 15b \leq 480 \\
& 4a + 4b \leq 160 \\
& 35a + 20b \leq 1190 \\
& a, b \geq 0
\end{aligned}
$$

▶ Entrepeneur: buy resources from brewer at minimum cost C, H, M: unit price for corn, hops and malt.

$$
\begin{aligned}
\min \quad & 480C + 160H + 1190M \\
\text{s.t.} \quad & 5C + 4H + 35M \geq 13 \\
& 15C + 4H + 20M \geq 23 \\
& C, H, M \geq 0
\end{aligned}
$$

Note that brewer won't sell (at least not all) if e.g.
$5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

Interpretation of Dual Variables

Marginal Price:

▶ How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?

▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by $\varepsilon_C, \varepsilon_H$, and ε_M, respectively.

The profit increases to $\max\{c^T x \mid Ax \leq b + \varepsilon; x \geq 0\}$. Because of strong duality this is equal to

$$
\begin{aligned}
\min \quad & (b^T + \epsilon^T) y \\
\text{s.t.} \quad & A^T y \geq c \\
& y \geq 0
\end{aligned}
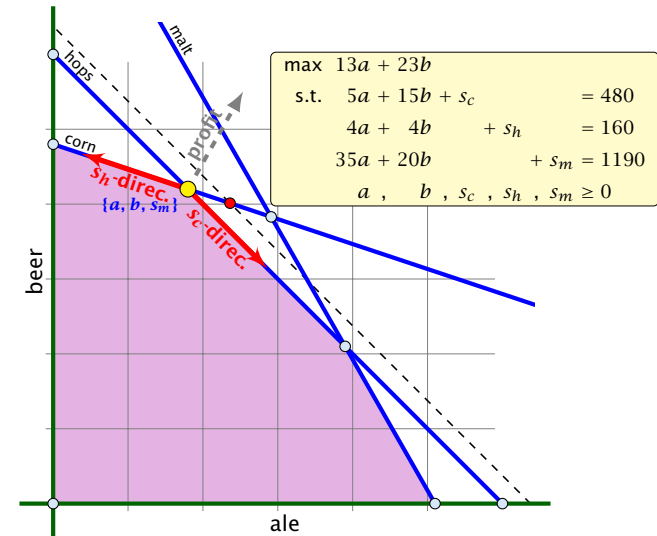$$

Interpretation of Dual Variables

If ϵ is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \epsilon_i y_i^*$.

Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

- ▶ If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).

- ▶ If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.

Example

$$\begin{aligned}
\max \;\; & 13a + 23b \\
\text{s.t.} \;\; & 5a + 15b + s_c && = 480 \\
& 4a + 4b && + s_h && = 160 \\
& 35a + 20b && && + s_m = 1190 \\
& a \;,\;\; b \;,\; s_c \;,\; s_h \;,\; s_m \geq 0
\end{aligned}$$

The change in profit when increasing hops by one unit is

$$= \underbrace{c_B^T A_B^{-1}}_{y^*} e_h.$$

Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.