5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\max\{c^T x \mid Ax = b, x \geq 0\}$$

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$$\begin{aligned}&\max\{c^T x \mid Ax = b, x \geq 0\}\\&\quad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\end{aligned}$$

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$$\begin{aligned}
&\max\{c^T x \mid Ax = b, x \geq 0\} \\
&\quad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\
&\quad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}
\end{aligned}$$

Proof

Primal:

$$\begin{aligned}\max\{c^Tx \mid Ax = b, x \geq 0\} \\ &= \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ &= \max\{c^Tx \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}\end{aligned}$$

Dual:

$$\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \geq c, y \geq 0\}$$

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Dual:

$$\begin{aligned}&\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \geq c, y \geq 0\}\\ &\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}\end{aligned}$$

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$$\begin{aligned}&\max\{c^Tx \mid Ax = b, x \geq 0\}\\&\quad = \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\\&\quad = \max\{c^Tx \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}\end{aligned}$$

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$$\begin{aligned}&\min\{[b^T\ -b^T]y \mid [A^T\ -A^T]y \geq c, y \geq 0\}\\&\quad = \min\left\{[b^T\ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T\ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}\\&\quad = \min\left\{b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\}\end{aligned}$$

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&\quad = \min\left\{b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\} \\
&\quad = \min\left\{b^T y' \,\middle|\, A^T y' \geq c\right\}
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Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

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$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \geq c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.

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