

# Part II

## Foundations

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- ▶ Learn how to analyze and judge the efficiency of algorithms.
- ▶ Learn how to design efficient algorithms.

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- ▶ Theoretical analysis in a specific **model of computation**.
  - ▶ Gives **asymptotic bounds** like “this algorithm always runs in time  $\mathcal{O}(n^2)$ ”.
  - ▶ Typically focuses on the **worst case**.
  - ▶ Can give lower bounds like “any comparison-based sorting algorithm needs at least  $\Omega(n \log n)$  comparisons in the worst case”.

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### Example 1

Suppose  $n$  numbers from the interval  $\{1, \dots, N\}$  have to be sorted. In this case we usually say that the input length is  $n$  instead of e.g.  $n \log N$ , which would be the number of bits required to encode the input.

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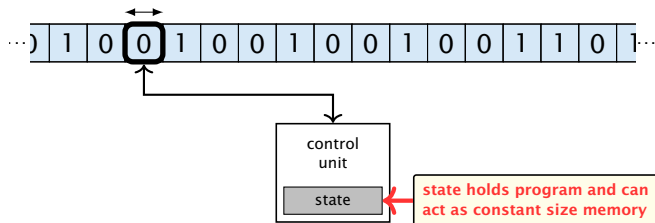
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Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.



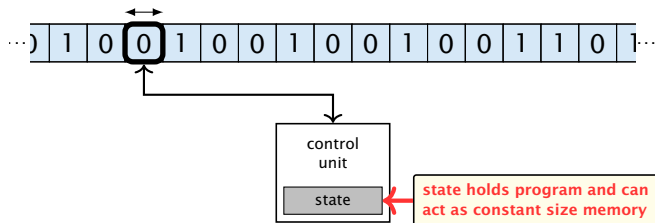
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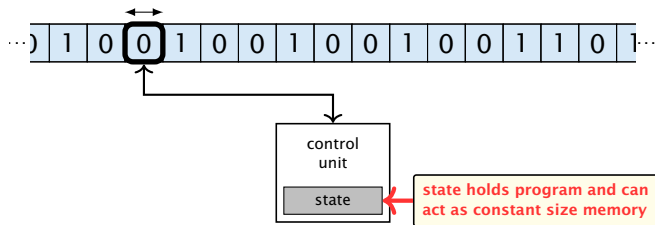
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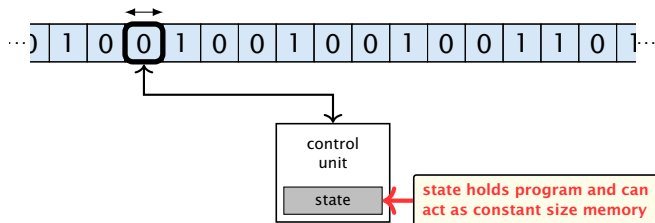
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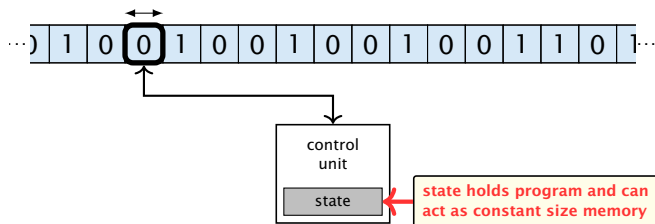
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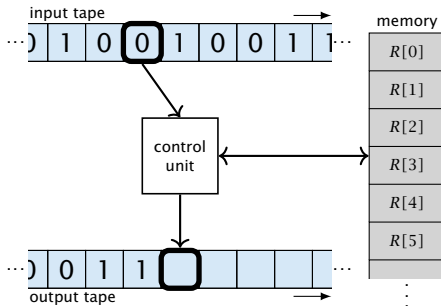
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⇒ **Not a good model for developing efficient algorithms.**



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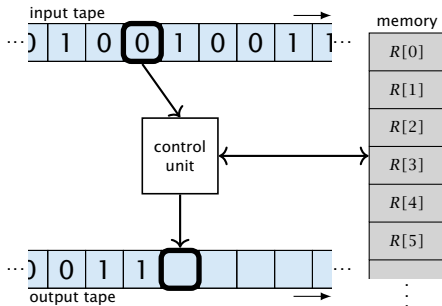
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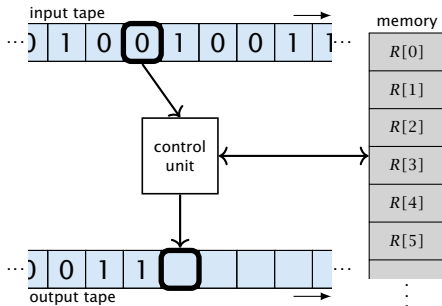
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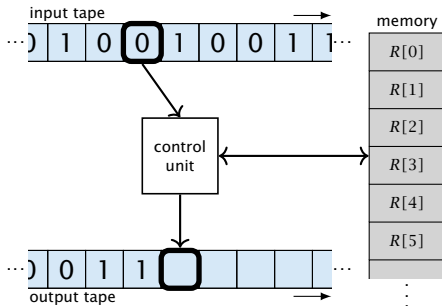


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- ▶ **uniform** cost model  
Every operation takes time 1.

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**Bounded word RAM model:** cost is uniform but the largest value stored in a register may not exceed  $2^w$ , where usually  $w = \log_2 n$ .

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## Example 2

### Algorithm 1 RepeatedSquaring( $n$ )

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There are **different types of complexity bounds**:

- ▶ **best-case** complexity:

$$C_{bc}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

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more general: probability measure  $\mu$

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▶ **randomized** complexity:

The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input  $x$ . Then take the worst-case over all  $x$  with  $|x| = n$ .

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- ▶ A linear speed-up (i.e., by a constant factor) is always possible by e.g. implementing the algorithm on a faster machine.

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- ▶ We are usually interested in the running times for large values of  $n$ . Then constant additive terms do not play an important role.
- ▶ An exact analysis (e.g. *exactly* counting the number of operations in a RAM) may be hard, but wouldn't lead to more precise results as the computational model is already quite a distance from reality.
- ▶ A linear speed-up (i.e., by a constant factor) is always possible by e.g. implementing the algorithm on a faster machine.
- ▶ Running time should be expressed by simple functions.



# Asymptotic Notation

## Formal Definition

Let  $f, g$  denote functions from  $\mathbb{N}$  to  $\mathbb{R}^+$ .

- ▶  $\mathcal{O}(f) = \{g \mid \exists c > 0 \exists n_0 \in \mathbb{N}_0 \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$   
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There is an equivalent definition using limes notation (**assuming that the respective limes exists**).  $f$  and  $g$  are functions from  $\mathbb{N}_0$  to  $\mathbb{R}_0^+$ .

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## Abuse of notation

1. People write  $f = \mathcal{O}(g)$ , when they mean  $f \in \mathcal{O}(g)$ . This is **not** an equality (how could a function be equal to a set of functions).

2. In this context  $f(n)$  does **not** mean the function  $f$  evaluated at  $n$ , but instead it is a shorthand for the function itself (leaving out domain and codomain and only giving the rule of correspondence of the function).

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4. People write  $\mathcal{O}(f(n)) = \mathcal{O}(g(n))$ , when they mean  $\mathcal{O}(f(n)) \subseteq \mathcal{O}(g(n))$ . Again this is not an equality.

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How do we interpret an expression like:

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Note that  $\Theta(n)$  is on the right hand side, otw. this interpretation is wrong.

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How do we interpret an expression like:

$$2n^2 + \mathcal{O}(n) = \Theta(n^2)$$

Regardless of how we choose the anonymous function  $f(n) \in \mathcal{O}(n)$  there is an anonymous function  $g(n) \in \Theta(n^2)$  that makes the expression true.

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**Careful!**

# Asymptotic Notation in Equations

The  $\Theta(i)$ -symbol on the left represents **one** anonymous function  $f : \mathbb{N} \rightarrow \mathbb{R}^+$ , and then  $\sum_i f(i)$  is computed.

How do we interpret an expression like:

$$\sum_{i=1}^n \Theta(i) = \Theta(n^2)$$

**Careful!**

“It is understood” that every occurrence of an  $\Theta$ -symbol (or  $\Omega, o, \omega$ ) on the left represents **one anonymous function**.

Hence, the left side is **not** equal to

$$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$$

$\Theta(1) + \Theta(2) + \dots + \Theta(n-1) + \Theta(n)$  does not really have a reasonable interpretation.

# Asymptotic Notation in Equations

We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\{f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n)\}$$

with  $g(n) \in \mathcal{O}(n)$  and  $h(n) \in \mathcal{O}(\log n)$

Recall that according to the previous slide e.g. the expressions  $\sum_{i=1}^n \mathcal{O}(i)$  and  $\sum_{i=1}^{n/2} \mathcal{O}(i) + \sum_{i=n/2+1}^n \mathcal{O}(i)$  generate different sets.

# Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containment btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

Note that the equation does not hold.



# Asymptotic Notation

## Lemma 3

Let  $f, g$  be functions with the property

$\exists n_0 > 0 \forall n \geq n_0 : f(n) > 0$  (the same for  $g$ ). Then

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The expressions also hold for  $\Omega$ . Note that this means that  $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$ .

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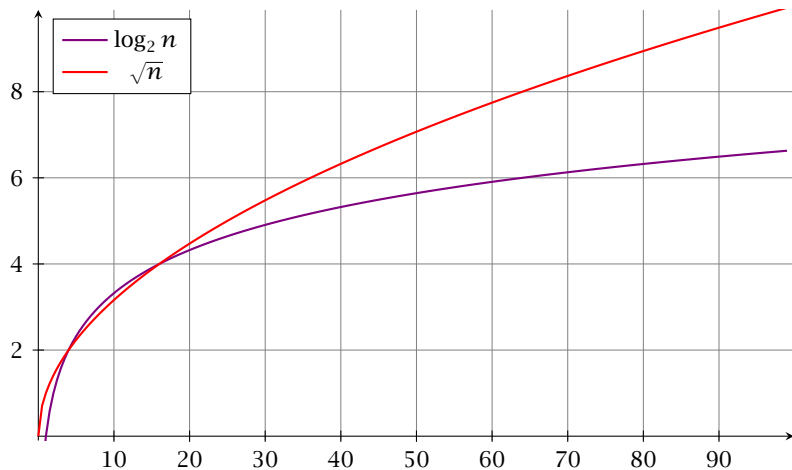
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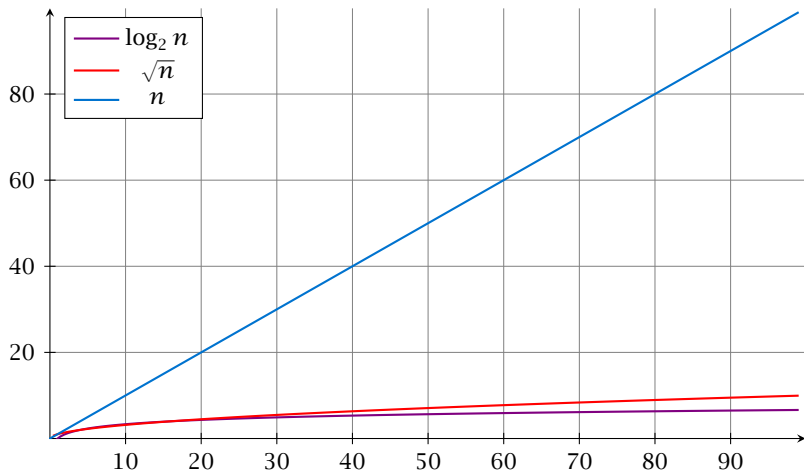
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- ▶ In general  $\log n = \log_2 n$ , i.e., we use 2 as the default base for the logarithm.



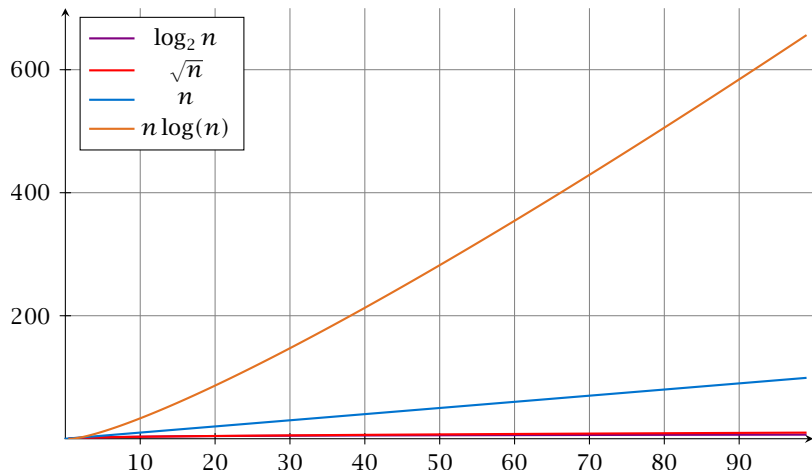
# Funktionen



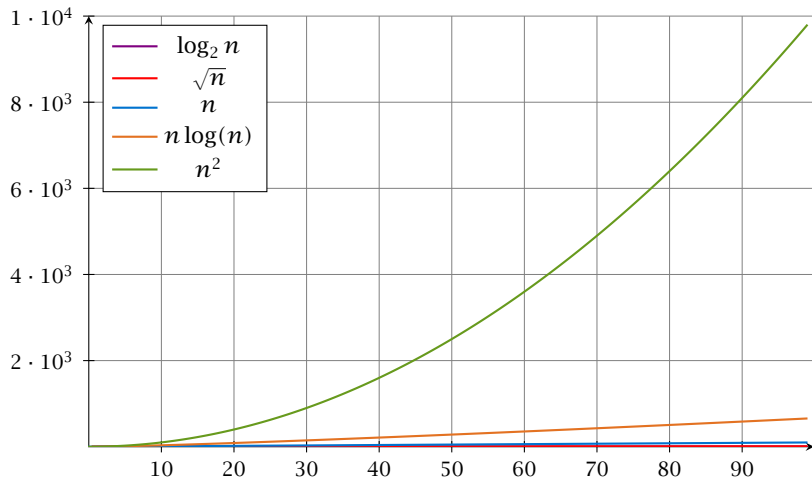
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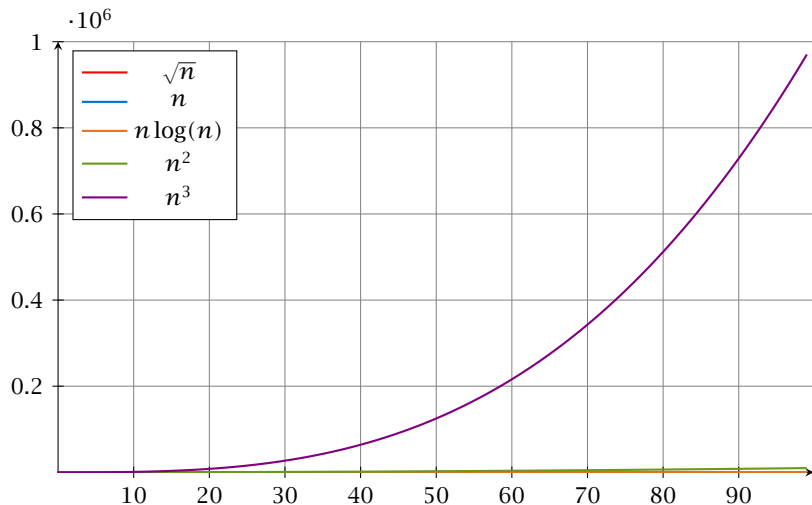
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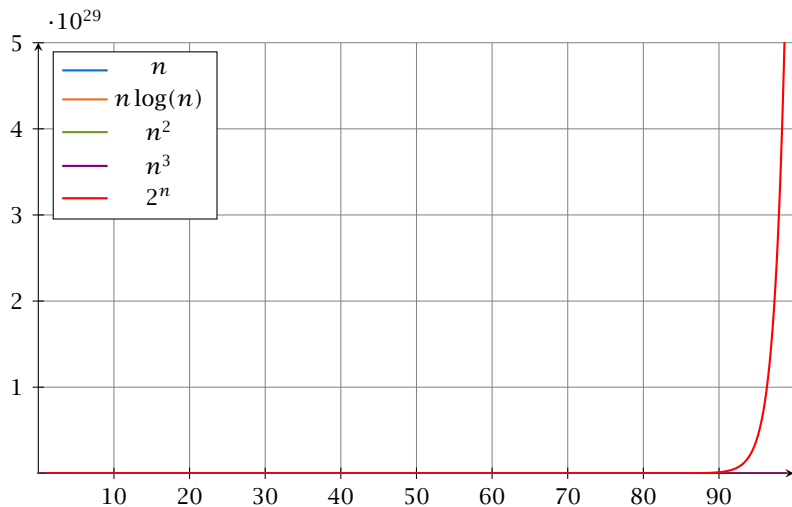
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# Laufzeiten

Funktion	Eingabelänge $n$							
	10	$10^2$	$10^3$	$10^4$	$10^5$	$10^6$	$10^7$	$10^8$
$\log n$	33ns	66ns	0.1 $\mu$ s	0.1 $\mu$ s	0.2 $\mu$ s	0.2 $\mu$ s	0.2 $\mu$ s	0.3 $\mu$ s
$\sqrt{n}$	32ns	0.1 $\mu$ s	0.3 $\mu$ s	1 $\mu$ s	3.1 $\mu$ s	10 $\mu$ s	31 $\mu$ s	0.1ms
$n$	100ns	1 $\mu$ s	10 $\mu$ s	0.1ms	1ms	10ms	0.1s	1s
$n \log n$	0.3 $\mu$ s	6.6 $\mu$ s	0.1ms	1.3ms	16ms	0.2s	2.3s	27s
$n^{3/2}$	0.3 $\mu$ s	10 $\mu$ s	0.3ms	10ms	0.3s	10s	5.2min	2.7h
$n^2$	1 $\mu$ s	0.1ms	10ms	1s	1.7min	2.8h	11d	3.2y
$n^3$	10 $\mu$ s	10ms	10s	2.8h	115d	317y	$3.2 \cdot 10^5$ y	
$1.1^n$	26ns	0.1ms	$7.8 \cdot 10^{25}$ y					
$2^n$	10 $\mu$ s	$4 \cdot 10^{14}$ y						
$n!$	36ms	$3 \cdot 10^{142}$ y						

1 Operation = 10ns; 100MHz

Alter des Universums: ca.  $13.8 \cdot 10^9$ y

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In general asymptotic classification of running times is a good measure for comparing algorithms:

- ▶ If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of  $n$ .



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Clearly  $f = o(g)$ . However, as long as  $\log n \leq 1000$  Algorithm B will be more efficient.

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## Formal Definition

Let  $f, g$  denote functions from  $\mathbb{N}^d$  to  $\mathbb{R}_0^+$ .

$$\blacktriangleright \mathcal{O}(f) = \{g \mid \exists c > 0 \exists N \in \mathbb{N}_0 \forall \vec{n} \text{ with } n_i \geq N \text{ for some } i : [g(\vec{n}) \leq c \cdot f(\vec{n})]\}$$

(set of functions that asymptotically grow **not faster** than  $f$ )

# Multiple Variables in Asymptotic Notation

## Example 4

►  $f : \mathbb{N} \rightarrow \mathbb{R}_0^+$ ,  $f(n, m) = 1$  und  $g : \mathbb{N} \rightarrow \mathbb{R}_0^+$ ,  $g(n, m) = n - 1$



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## 6 Recurrences

### Algorithm 2 mergesort(list $L$ )

```
1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
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This algorithm requires

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n)$$

comparisons when  $n > 1$  and 0 comparisons when  $n \leq 1$ .

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For this we need to **solve** the recurrence.

# Methods for Solving Recurrences

## 1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

## 2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

## 3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

## 4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

## 5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

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First we need to get rid of the  $\mathcal{O}$ -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

**Informal way:**

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One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

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Formally, this is not correct if  $n$  is not a power of 2. Also even in this case one would need to do an induction proof.

## 6.1 Guessing+Induction

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

- Note that this proves the statement for  $n = 2^k$ ,  $k \in \mathbb{N}_{\geq 1}$ , as the statement is wrong for  $n = 1$ .
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Hence, statement is **true** if we choose  $d \geq c$ .

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Note that we can do this as for constant-sized inputs the running time is always some constant ( $b$  in the above case).

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$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

$$= dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\log n \leq \frac{n}{4}$$

## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$T(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn$$

$$\boxed{\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1} \leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn$$

$$\boxed{\frac{n}{2} + 1 \leq \frac{9}{16}n} \leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn$$

$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)} = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\boxed{\log n \leq \frac{n}{4}} \leq dn \log n + (\log 9 - 3.5)dn + cn$$



## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$\begin{aligned}T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn \\&\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn \\&= dn \log n + (\log 9 - 4)dn + 2d \log n + cn \\&\leq dn \log n + (\log 9 - 3.5)dn + cn \\&\leq dn \log n - 0.33dn + cn\end{aligned}$$

$$\left\lceil \frac{n}{2} \right\rceil \leq \frac{n}{2} + 1$$

$$\frac{n}{2} + 1 \leq \frac{9}{16}n$$

$$\log \frac{9}{16}n = \log n + (\log 9 - 4)$$

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## 6.1 Guessing+Induction

We also make a guess of  $T(n) \leq dn \log n$  and get

$$\begin{aligned}T(n) &\leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left\lceil \frac{n}{2} \right\rceil \log \left\lceil \frac{n}{2} \right\rceil\right) + cn \\&\leq 2\left(d\left(\frac{n}{2} + 1\right) \log\left(\frac{n}{2} + 1\right)\right) + cn \\&\leq dn \log\left(\frac{9}{16}n\right) + 2d \log n + cn \\&= dn \log n + (\log 9 - 4)dn + 2d \log n + cn \\&\leq dn \log n + (\log 9 - 3.5)dn + cn \\&\leq dn \log n - 0.33dn + cn \\&\leq dn \log n\end{aligned}$$

for a suitable choice of  $d$ .

## 6.2 Master Theorem

Note that the cases do not cover all possibilities.

### Lemma 5

Let  $a \geq 1$ ,  $b > 1$  and  $\epsilon > 0$  denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

#### Case 1.

If  $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$  then  $T(n) = \Theta(n^{\log_b a})$ .

#### Case 2.

If  $f(n) = \Theta(n^{\log_b(a)} \log^k n)$  then  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ ,  
 $k \geq 0$ .

#### Case 3.

If  $f(n) = \Omega(n^{\log_b(a)+\epsilon})$  and for sufficiently large  $n$   
 $af\left(\frac{n}{b}\right) \leq cf(n)$  for some constant  $c < 1$  then  $T(n) = \Theta(f(n))$ .

## 6.2 Master Theorem

We prove the Master Theorem for the case that  $n$  is of the form  $b^\ell$ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

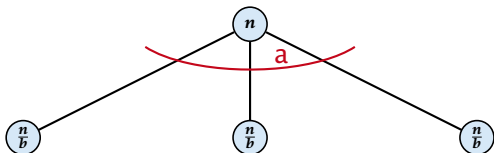
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



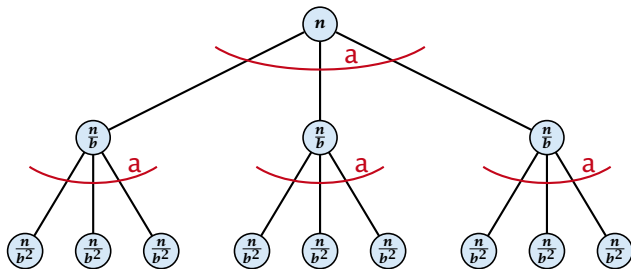
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# The Recursion Tree

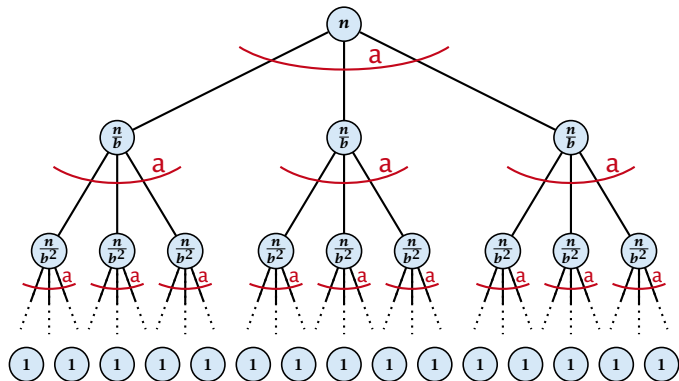
The running time of a recursive algorithm can be visualized by a recursion tree:





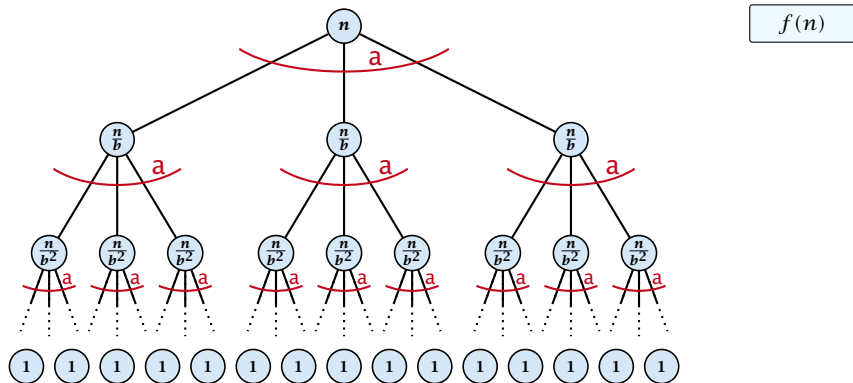
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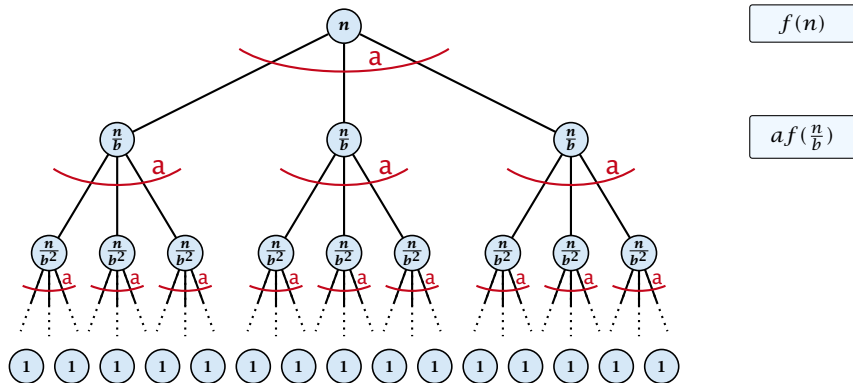
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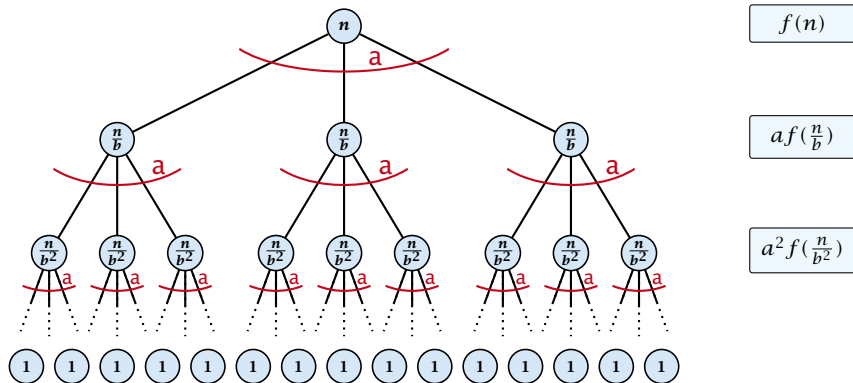
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The running time of a recursive algorithm can be visualized by a recursion tree:



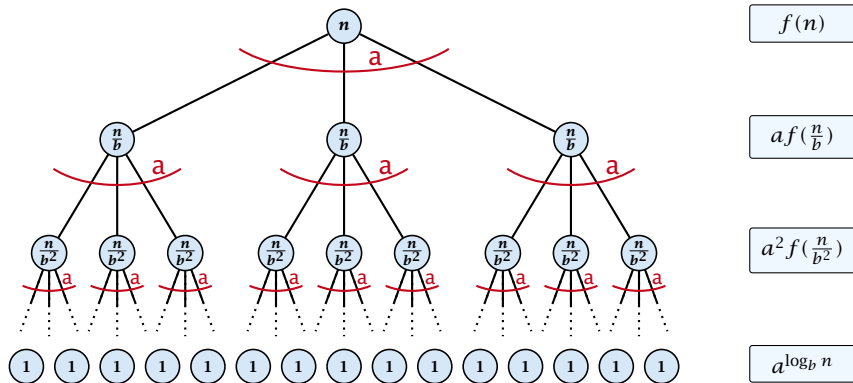
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



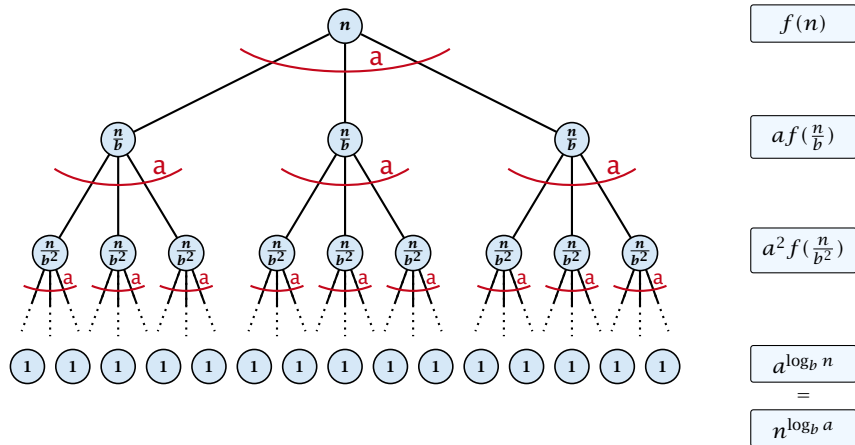
# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



## 6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .



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$$T(n) = n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

Case 1. Now suppose that  $f(n) \leq cn^{\log_b a - \epsilon}$ .

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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a}$ .

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \quad \Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

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Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

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$$n = b^\ell \Rightarrow \ell = \log_b n$$



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$n = b^\ell \Rightarrow \ell = \log_b n$
--

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

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$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\ &\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \end{aligned}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

Where did we use  $f(n) \geq \Omega(n^{\log_b a + \epsilon})$ ?

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Where did we use  $f(n) \geq \Omega(n^{\log_b a + \epsilon})$ ?

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

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Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

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1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$

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1	1	0	1	1	0	1	0	1	$A$
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<hr/>									
								0	

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For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \phantom{1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ } 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\end{array}$$

The diagram illustrates the addition of two 10-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A vertical box highlights the 8th bit position (from the right), where a carry of 1 is shown. The result of the addition is 1, 1, 0, 1, 1, 0, 1, 0, 1, 0.

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For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & & 1 & 1 & \\ & & & & & & & 0 & 0 & \end{array}$$

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For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
						1	1		
							0	0	

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Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two integers, A and B, using a register of constant size. The integers are represented as binary strings: A = 110110101 and B = 100010011. The addition is performed bit-by-bit from right to left. The result of the addition is shown below the horizontal line, with the carry bits (0, 0, 0) highlighted in a light blue box. The carry bits are shown as 0, 0, 0, indicating that the addition of the two integers results in a carry of 0 for each bit position.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
					1	1	1		
						0	0	0	



## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
					0	1	1	1	
					1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
					0	1	1	1	
						1	0	0	0

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 1 0 0 0. A vertical box highlights the 5th bit of the result, which is 0. This bit is the result of adding the 5th bits of A and B (1 + 1) and the carry-in from the 4th bit (1). The carry-out from the 5th bit is 1, which is the carry-in for the 6th bit.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
			1	0	1	1	1		
			0	1	0	0	0		

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
			0	0	1	0	0	0	

*Note: In the original image, a vertical box highlights the 4th bit (index 3) of both A and B, and the 4th bit of the result. Carry bits are shown below the B row: 1, 1, 0, 1, 1, 1.*

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
			0	0	1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The numbers are aligned by their least significant bits. A vertical blue box highlights the third bit position (index 2 from the right). Below the horizontal line, the carry bits are shown: a '1' is written below the third bit of A, and '1', '0', '1', '1', '1' are written below the fourth, fifth, sixth, seventh, and eighth bits of A, respectively. The result of the addition is shown in the bottom row, with a '0' in the third bit position and '0', '1', '0', '0', '0' in the fourth, fifth, sixth, seventh, and eighth bit positions, respectively.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & B \\ \hline & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, using a ripple carry method. The numbers are aligned by their least significant bits. A horizontal line is drawn under the second row. The result of the addition is shown in the third row. The carry bits are indicated by small subscripts below the digits in the second row: 0 under the first 0, 1 under the first 0, 1 under the first 1, 0 under the second 0, 1 under the second 0, 1 under the second 1, and 1 under the second 1. The carry bit 1 is shown in a light blue box in the third row, under the second column.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & 0 & 1 & 1 & 0 & 1 & 1 & 1 & & \\ & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$



## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	0	0	1	1	0	1	1	1		
	1	1	0	0	1	0	0	0		

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	<hr/>									
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

		1	1	0	1	1	0	1	0	1	$A$
		1	0	0	0	1	0	0	1	1	$B$
		<hr/>									
		1	0	1	1	0	0	1	0	0	0

The diagram shows the addition of two 10-bit integers, A and B. A light blue vertical box highlights the carry propagation from the least significant bit (LSB) to the most significant bit (MSB). The carry starts at 1 in the second column from the left and propagates through all columns, resulting in a final carry of 1 in the eleventh column.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

This gives that two  $n$ -bit integers can be added in time  $\mathcal{O}(n)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most  $m + n \leq 2n$  bits.

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \\ \times 1011 \\ \hline \end{array}$$

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$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ \phantom{10001}00001 \\ \phantom{10001}00000 \\ \phantom{10001}00000 \\ \phantom{10001}00000 \end{array}$$

- This is also known as the “school method” for multiplying integers.
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## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \end{array}$$

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## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 00000 \\ 00000 \\ 10001 \\ \hline 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

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- ▶ Adding  $m$  numbers of length  $\leq 2n$ :  $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$ .



## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

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2:   return  $a_0 \cdot b_0$   
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

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**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
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In our case  $a = 4$ ,  $b = 2$ , and  $f(n) = \Theta(n)$ . Hence, we are in Case 1, since  $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$ .

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⇒ Not better than the “school method”.

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We can use the following identity to compute  $Z_1$ :

A more precise  
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A huge improvement over the “school method”.

## 6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$



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Note that we ignore **boundary conditions** for the moment.

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- ▶ First consider the homogenous case.

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The solution space

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**How do we find a non-trivial solution?**

We guess that the solution is of the form  $\lambda^n$ ,  $\lambda \neq 0$ , and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all  $n \geq k$ .

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Dividing by  $\lambda^{n-k}$  gives that all these constraints are identical to

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Let  $\lambda_1, \dots, \lambda_k$  be the  $k$  (complex) roots of  $P[\lambda]$ . Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \dots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values  $\alpha_i$ .

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## Lemma 6

Assume that the characteristic polynomial has  $k$  *distinct* roots  $\lambda_1, \dots, \lambda_k$ . Then *all* solutions to the recurrence relation are of the form

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We show that the above set of solutions contains one solution for every choice of boundary conditions.

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## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ 's such that these conditions are met:

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## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ s such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & & \vdots & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.



# Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$= \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

# Computing the Determinant

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot \mathbf{1} & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot \mathbf{1} & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix} =$$



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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

# Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all  $\lambda_i$ 's are different, then the determinant is non-zero.

# The Homogeneous Case

**What happens if the roots are not all distinct?**

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Suppose we have a root  $\lambda_i$  with multiplicity (**Vielfachheit**) at least 2. Then not only is  $\lambda_i^n$  a solution to the recurrence but also  $n\lambda_i^{n-1}$ .

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To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

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$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

Since  $\lambda_i$  is a root we can write this as  $Q[\lambda] \cdot (\lambda - \lambda_i)^2$ . Calculating the derivative gives a polynomial that still has root  $\lambda_i$ .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

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Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n-1) \lambda_i^{n-1}}_{T[n-1]} + \dots + \underbrace{c_k (n-k) \lambda_i^{n-k}}_{T[n-k]} = 0$$

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Doing this again gives

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We can continue  $j-1$  times.

Hence,  $n^\ell \lambda_i^n$  is a solution for  $\ell \in 0, \dots, j-1$ .

# The Homogeneous Case

## Lemma 7

Let  $P[\lambda]$  denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let  $\lambda_i$ ,  $i = 1, \dots, m$  be the (complex) roots of  $P[\lambda]$  with multiplicities  $\ell_i$ . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of  $\alpha_{ij}$ 's is a solution to the recurrence.

## Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$



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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

## Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

## Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

# The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with  $f(n) \neq 0$ .

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.



# The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where  $T_h$  is **any** solution to the homogeneous equation, and  $T_p$  is **one** particular solution to the inhomogeneous equation.

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There is no general method to find a particular solution.

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$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

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I get a completely determined recurrence if I add  $T[0] = 1$  and  $T[1] = 2$ .

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Example: Characteristic polynomial:

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$T[0] = 1$  gives  $\alpha = 1$ .

$T[1] = 2$  gives  $1 + \beta = 2 \Rightarrow \beta = 1$ .

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Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3 \end{aligned}$$

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$$T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2$$

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$$T[n] = 3T[n - 1] - 3T[n - 2] + T[n - 3] + 2$$

and so on...



## 6.4 Generating Functions

### Definition 8 (Generating Function)

Let  $(a_n)_{n \geq 0}$  be a sequence. The corresponding

- ▶ **generating function** (**Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} a_n z^n ;$$

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- ▶ **exponential generating function** (**exponentielle Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n .$$

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### Example 9

1. The generating function of the sequence  $(1, 0, 0, \dots)$  is

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There are no convergence issues here.

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Then, it is important to think about convergence/convergence radius etc.

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This is well-defined.

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Hence, the generating function of the sequence  $a_n = n + 1$  is  $1/(1-z)^2$ .

Formally the derivative of a formal power series  $\sum_{n \geq 0} a_n z^n$  is defined as  $\sum_{n \geq 0} n a_n z^{n-1}$ .

The known rules for differentiation work for this definition. In particular, e.g. the derivative of  $\frac{1}{1-z}$  is  $\frac{1}{(1-z)^2}$ .

Note that this requires a proof if we consider power series as algebraic objects. However, we did not prove this in the lecture.

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Hence, the generating function of the sequence

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Hence:

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The generating function of the sequence  $a_n = \binom{n+k}{k}$  is  $\frac{1}{(1-z)^{k+1}}$ .

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Suppose we have the recurrence  $a_n = a_{n-1} + 1$  for  $n \geq 1$  and  $a_0 = 1$ .

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6. The coefficients of the resulting power series are the  $a_n$ .

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$$A(z) = 1 + \sum_{n \geq 1} (2a_{n-1})z^n$$

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means  $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$ .



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$$f_0 = 1$$

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$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

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Let  $n = 2^k$ :

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

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