

4 Modelling Issues

What do you measure?

- ▶ Memory requirement
- ▶ Running time
- ▶ Number of comparisons
- ▶ Number of multiplications
- ▶ Number of hard-disc accesses
- ▶ Program size
- ▶ Power consumption
- ▶ ...

4 Modelling Issues

How do you measure?

- ▶ Implementing and testing on representative inputs
 - ▶ How do you choose your inputs?
 - ▶ May be very time-consuming.
 - ▶ Very reliable results if done correctly.
 - ▶ Results only hold for a specific machine and for a specific set of inputs.
- ▶ Theoretical analysis in a specific **model of computation**.
 - ▶ Gives **asymptotic bounds** like “this algorithm always runs in time $\mathcal{O}(n^2)$ ”.
 - ▶ Typically focuses on the **worst case**.
 - ▶ Can give lower bounds like “any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case”.

4 Modelling Issues

Input length

The theoretical bounds are usually given by a function $f: \mathbb{N} \rightarrow \mathbb{N}$ that maps the **input length** to the running time (or storage space, comparisons, multiplications, program size etc.).

The **input length** may e.g. be

- ▶ the size of the input (number of bits)
- ▶ the number of arguments

Example 1

Suppose n numbers from the interval $\{1, \dots, N\}$ have to be sorted. In this case we usually say that the input length is n instead of e.g. $n \log N$, which would be the number of bits required to encode the input.

Model of Computation

How to measure performance

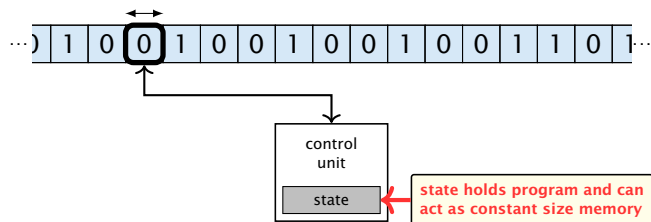
1. Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), ...
2. Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, ...

Version 2. is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

Turing Machine

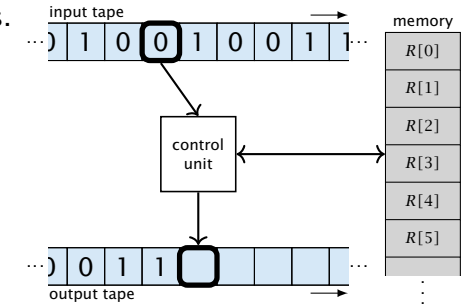
- ▶ Very simple model of computation.
- ▶ Only the “current” memory location can be altered.
- ▶ Very good model for discussing computability, or polynomial vs. exponential time.
- ▶ Some simple problems like recognizing whether input is of the form xx , where x is a string, have quadratic lower bound.

⇒ **Not a good model for developing efficient algorithms.**



Random Access Machine (RAM)

- ▶ Input tape and output tape (sequences of zeros and ones; unbounded length).
- ▶ Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \dots$
- ▶ Registers hold integers.
- ▶ Indirect addressing.



Note that in the picture on the right the tapes are one-directional, and that a READ- or WRITE-operation always advances its tape.

Random Access Machine (RAM)

Operations

- ▶ input operations (input tape $\rightarrow R[i]$)
 - ▶ READ i
- ▶ output operations ($R[i] \rightarrow$ output tape)
 - ▶ WRITE i
- ▶ register-register transfers
 - ▶ $R[j] := R[i]$
 - ▶ $R[j] := 4$
- ▶ indirect addressing
 - ▶ $R[j] := R[R[i]]$
loads the content of the $R[i]$ -th register into the j -th register
 - ▶ $R[R[i]] := R[j]$
loads the content of the j -th into the $R[i]$ -th register

Random Access Machine (RAM)

Operations

- ▶ branching (including loops) based on comparisons
 - ▶ jump x
jumps to position x in the program;
sets instruction counter to x ;
 - ▶ jumpz $x R[i]$
jump to x if $R[i] = 0$
if not the instruction counter is increased by 1;
 - ▶ jumpi i
jump to $R[i]$ (indirect jump);
- ▶ arithmetic instructions: $+$, $-$, \times , $/$
 - ▶ $R[i] := R[j] + R[k]$;
 - ▶ $R[i] := -R[k]$;

The jump-directives are very close to the jump-instructions contained in the assembler language of real machines.

Model of Computation

- ▶ **uniform** cost model
Every operation takes time 1.
- ▶ **logarithmic** cost model
The cost depends on the content of memory cells:
 - ▶ The time for a step is equal to the largest operand involved;
 - ▶ The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w , where usually $w = \log_2 n$.

The latter model is quite realistic as the word-size of a standard computer that handles a problem of size n must be at least $\log_2 n$ as otherwise the computer could either not store the problem instance or not address all its memory.

4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

```

1:  $r \leftarrow 2$ ;
2: for  $i = 1 \rightarrow n$  do
3:    $r \leftarrow r^2$ 
4: return  $r$ 
    
```

- ▶ running time (for Line 3):
 - ▶ uniform model: n steps
 - ▶ logarithmic model:
 $2 + 3 + 5 + \dots + (1 + 2^n) = 2^{n+1} - 1 + n = \Theta(2^n)$
- ▶ space requirement:
 - ▶ uniform model: $\mathcal{O}(1)$
 - ▶ logarithmic model: $\mathcal{O}(2^n)$

There are **different types of complexity bounds**:

- ▶ **best-case** complexity:

$$C_{bc}(n) := \min\{C(x) \mid |x| = n\}$$

Usually easy to analyze, but not very meaningful.

- ▶ **worst-case** complexity:

$$C_{wc}(n) := \max\{C(x) \mid |x| = n\}$$

Usually moderately easy to analyze; sometimes too pessimistic.

- ▶ **average case** complexity:

$$C_{avg}(n) := \frac{1}{|I_n|} \sum_{|x|=n} C(x)$$

more general: probability measure μ

$$C_{avg}(n) := \sum_{x \in I_n} \mu(x) \cdot C(x)$$

μ is a probability distribution over inputs of length n .

$C(x)$	cost of instance x
$ x $	input length of instance x
I_n	set of instances of length n

There are **different types of complexity bounds**:

- ▶ **amortized** complexity:

The average cost of data structure operations over a worst case sequence of operations.

- ▶ **randomized** complexity:

The algorithm may use random bits. Expected running time (over all possible choices of random bits) for a fixed input x . Then take the worst-case over all x with $|x| = n$.

μ is a probability distribution over inputs of length n .

$C(x)$	cost of instance x
$ x $	input length of instance x
I_n	set of instances of length n

4 Modelling Issues

Bibliography

[MS08] Kurt Mehlhorn, Peter Sanders:
Algorithms and Data Structures — The Basic Toolbox,
Springer, 2008

[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
Introduction to algorithms (3rd ed.),
McGraw-Hill, 2009

Chapter 2.1 and 2.2 of [MS08] and Chapter 2 of [CLRS90] are relevant for this section.