

6 Recurrences

Algorithm 2 mergesort(list L)

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1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
7:  $L \leftarrow \text{merge}(L_1, L_2)$ 
8: return  $L$ 
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This algorithm requires

$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

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For this we need to **solve** the recurrence.

Methods for Solving Recurrences

1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

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First we need to get rid of the \mathcal{O} -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Informal way:

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One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

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Formally, this is not correct if n is not a power of 2. Also even in this case one would need to do an induction proof.

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$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 16 \\ b & \text{otw.} \end{cases}$$

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Hence, statement is **true** if we choose $d \geq c$.

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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$$\boxed{\log \frac{9}{16}n = \log n + (\log 9 - 4)} = dn \log n + (\log 9 - 4)dn + 2d \log n + cn$$

$$\boxed{\log n \leq \frac{n}{4}} \leq dn \log n + (\log 9 - 3.5)dn + cn$$

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$$\leq dn \log n$$

for a suitable choice of d .

6.2 Master Theorem

Lemma 1

Let $a \geq 1$, $b > 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
 $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
 $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

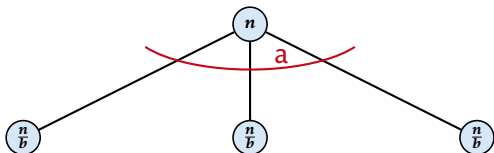
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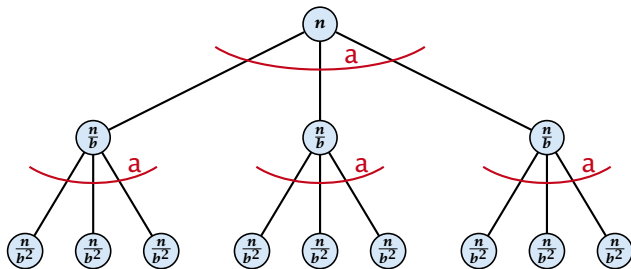
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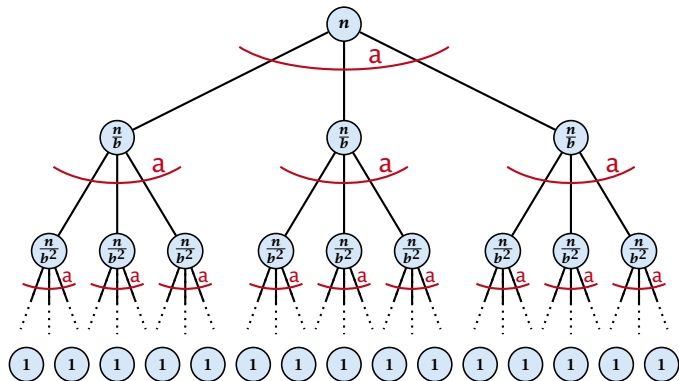
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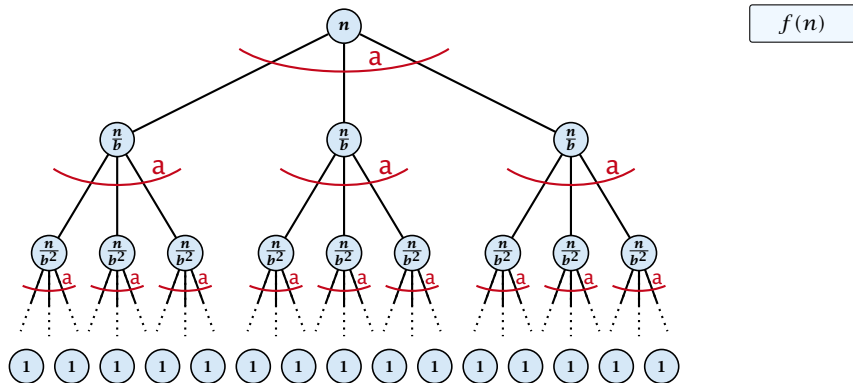
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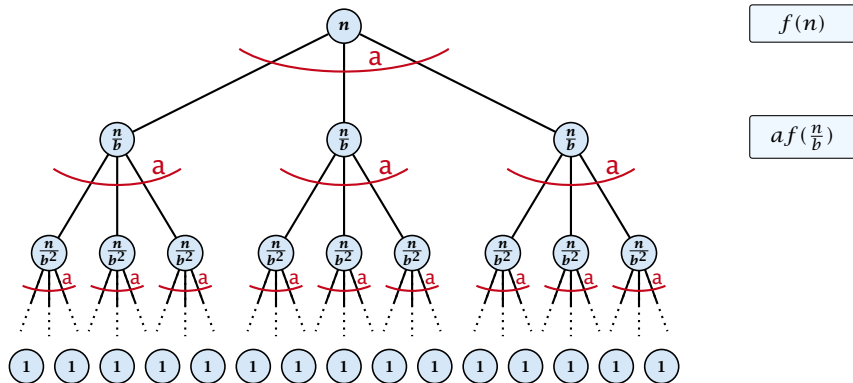
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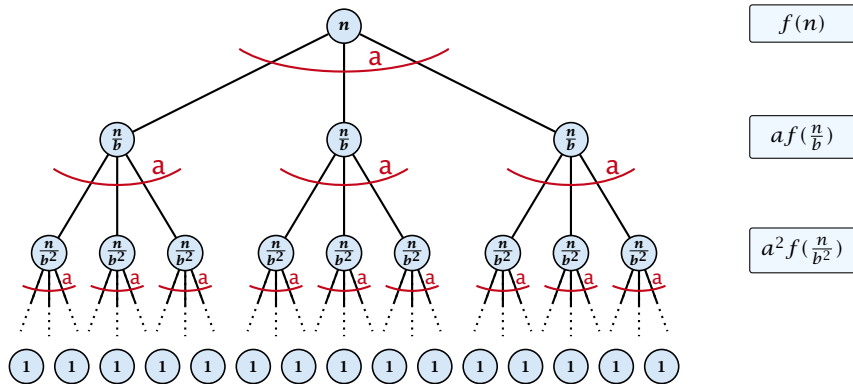
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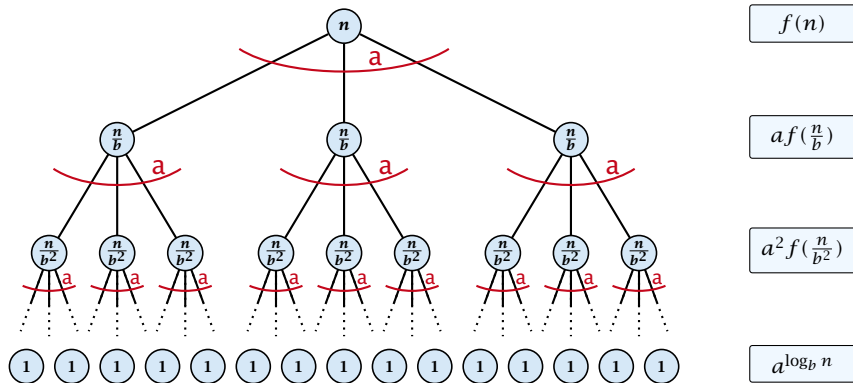
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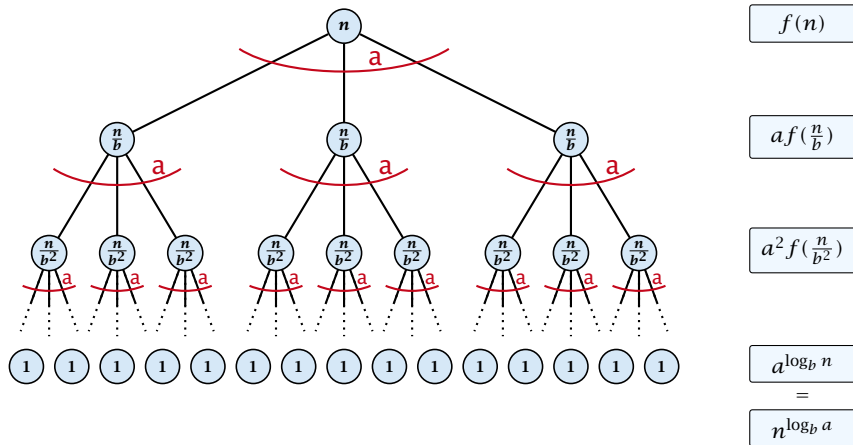
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6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

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$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$$

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$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \end{aligned}$$

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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

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$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

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$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

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Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

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For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline \end{array}$$

The diagram shows two 9-bit integers, A and B, aligned for addition. Integer A is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer B is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. A vertical light blue box highlights the rightmost bit of A (the least significant bit) and the rightmost bit of B, which are both 1. To the right of the bits, the labels A and B are written in red and blue respectively.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
								0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>								1	
									0

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & & B \\ \hline & & & & & & & 1 & 1 & & \\ & & & & & & & 0 & 0 & & \end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
						0	1	1	
							0	0	

The diagram illustrates the addition of two integers A and B . The numbers are represented as binary strings: $A = 110110101$ and $B = 100010011$. A horizontal line is drawn under the numbers. A vertical blue box highlights the 7th bit (from the left) of both numbers, which are both 1. Below the line, the 7th bit of the result is 0, and a carry of 1 is shown below it. The 8th bit of the result is 1, and the 9th bit is 1. The final result is 00.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>						0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line as 0 0 0. A vertical box highlights the 7th, 8th, and 9th bits of the result, which are 0, 0, and 0 respectively. Small '1' characters are placed below the 6th, 7th, and 8th bits of the result, indicating carry bits.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
					0	0	0		

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical line is drawn between the 5th and 6th bits. A light blue vertical box highlights the 5th bit of A (0) and the 5th bit of B (0). Below the 5th bit of B, there is a small '1' indicating a carry. Below the 6th bit of B, there is a small '1' indicating a carry. Below the 7th bit of B, there is a small '1' indicating a carry. The result of the addition is shown as 000 in the bottom row, starting from the 6th bit position.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
					0	1	1	1	
					1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
				1	0	1	1	1	
-----				0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			1	0	1	1	1		
			0	1	0	0	0		

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, to produce a 9-bit result. The numbers are aligned by their least significant bits. A vertical box highlights the 4th bit position (index 3 from the right), where a carry-in of 1 from the 3rd bit position is added to the bits of A and B. The carry-in is shown as a '1' below the 3rd bit of B. The result of the addition is shown below a horizontal line, with the 4th bit being 0. The carry-out from the 4th bit position is 1, which is shown as a '1' below the 4th bit of B.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			1	1	0	1	1	1	
									0 0 1 0 0 0

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

		1	1	0	1	1	0	1	0	1	A
		1	0	0	0	1	0	0	1	1	B
			0	1	1	0	1	1	1		
				1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 110110101 \quad A \\ 100010011 \quad B \\ \hline 1001000 \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, to produce a 7-bit result. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. The result is 1, 0, 0, 1, 0, 0, 0. A light blue vertical box highlights the first two bits of the input (1, 1) and the first bit of the result (1). Small subscripts are placed below the bits of B: 0, 1, 1, 0, 1, 1, 1.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1		
	1	1	0	0	1	0	0	0		

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

The diagram illustrates the addition of two integers A and B in binary. The numbers are aligned to the right, and a horizontal line is drawn under the bottom number B . The result of the addition is shown below the line. A carry-out of 1 is shown in a light blue box on the left, aligned with the 10th bit position. The bits of A and B are color-coded: A is red and B is blue. The bits of the result are black.

1	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
<hr/>										
1	0	1	1	0	0	1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 110110101 \quad A \\ 1000010011 \quad B \\ \hline 1011001000 \end{array}$$

This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \\ \times 1011 \\ \hline \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 0 \\ 0 \\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0 \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

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Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \quad b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \quad a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
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7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

$\mathcal{O}(1)$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split** A into A_0 and A_1

4: **split** B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split** A into A_0 and A_1

$\mathcal{O}(n)$

4: **split** B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: if $ A = B = 1$ then	$\mathcal{O}(1)$
2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
4: split B into B_0 and B_1	$\mathcal{O}(n)$
5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$	
6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$	
7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$	
8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: split A into A_0 and A_1

$\mathcal{O}(n)$

4: split B into B_0 and B_1

$\mathcal{O}(n)$

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

$T(\frac{n}{2})$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: **split** A into A_0 and A_1

$\mathcal{O}(n)$

4: **split** B into B_0 and B_1

$\mathcal{O}(n)$

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

$T(\frac{n}{2})$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

$2T(\frac{n}{2}) + \mathcal{O}(n)$

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Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

$\mathcal{O}(1)$

2: **return** $a_0 \cdot b_0$

$\mathcal{O}(1)$

3: split A into A_0 and A_1

$\mathcal{O}(n)$

4: split B into B_0 and B_1

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5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

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Algorithm 3 $\text{mult}(A, B)$

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2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
3: split A into A_0 and A_1	$\mathcal{O}(n)$
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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⇒ Not better than the “school method”.

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1: if  $|A| = |B| = 1$  then  
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4: split  $B$  into  $B_0$  and  $B_1$   
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A huge improvement over the “school method”.

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$

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Note that we ignore **boundary conditions** for the moment.

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- ▶ First determine all solutions that satisfy recurrence relation.
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- ▶ First consider the homogenous case.

The Homogenous Case

The solution space

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is a **vector space**. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants α, β .

The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

is a **vector space**. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants α, β .

How do we find a non-trivial solution?

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How do we find a non-trivial solution?

We guess that the solution is of the form λ^n , $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

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This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \dots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i .

The Homogenous Case

Lemma 2

Assume that the characteristic polynomial has k *distinct* roots $\lambda_1, \dots, \lambda_k$. Then *all* solutions to the recurrence relation are of the form

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the α'_i 's such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i 's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

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Suppose we have a root λ_i with multiplicity (**Vielfachheit**) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^{n-1}$.

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Suppose we have a root λ_i with multiplicity (**Vielfachheit**) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^{n-1}$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

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Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

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Hence,

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Hence,

$$\underbrace{c_0 n \lambda_i^n}_{T[n]} + \underbrace{c_1 (n-1) \lambda_i^{n-1}}_{T[n-1]} + \dots + \underbrace{c_k (n-k) \lambda_i^{n-k}}_{T[n-k]} = 0$$

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(after taking the derivative; multiplying with λ ; plugging in λ_i)

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Doing this again gives

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We can continue $j-1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \dots, j-1$.

The Homogeneous Case

Lemma 3

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let $\lambda_i, i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij} 's is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

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Hence, the solution is of the form

$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right)^n + \beta \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

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$T[0] = 0$ gives $\alpha + \beta = 0$.

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$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

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where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

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$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

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Then,

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Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

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I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

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Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

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$T[1] = 2$ gives $1 + \beta = 2 \Rightarrow \beta = 1$.

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and so on...

6.4 Generating Functions

Definition 4 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

- ▶ **generating function** (**Erzeugendenfunktion**) is

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- ▶ **exponential generating function** (**exponentielle Erzeugendenfunktion**) is

$$F(z) := \sum_{n \geq 0} \frac{a_n}{n!} z^n .$$

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There are no convergence issues here.

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Then, it is important to think about convergence/convergence radius etc.

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It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are inverses, i.e.,

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This is well-defined.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence

$$a_n = (n+1)(n+2) \text{ is } \frac{2}{(1-z)^3} .$$

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Computing the k -th derivative of $\sum z^n$.

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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The generating function of the sequence $f_n = a^n$ is $\frac{1}{1-az}$.

Example: $a_n = a_{n-1} + 1$, $a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

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Hence, $a_n = n + 1$.

Some Generating Functions

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$\frac{1}{n!}$	e^z

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f_{n-k} ($n \geq k$); 0 otw.	$z^k F$
$\sum_{i=0}^n f_i$	$\frac{F(z)}{1-z}$

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Solving Recursions with Generating Functions

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3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.

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Techniques:

Solving Recursions with Generating Functions

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This leads to the following conditions:

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 6

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

6.5 Transformation of the Recurrence

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$$f_n = 2^{F_n}$$

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We get

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6 Recurrences

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

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Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

$$\begin{aligned} f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log_3 3})^k - 2 \cdot 2^k \\ &= 3(2^k)^{\log_3 3} - 2 \cdot 2^k \\ &= 3n^{\log_3 3} - 2n . \end{aligned}$$