

## 6.2 Master Theorem

Note that the cases do not cover all possibilities.

### Lemma 1

Let  $a \geq 1$ ,  $b > 1$  and  $\epsilon > 0$  denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

#### Case 1.

If  $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$  then  $T(n) = \Theta(n^{\log_b a})$ .

#### Case 2.

If  $f(n) = \Theta(n^{\log_b(a)} \log^k n)$  then  $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$ ,  
 $k \geq 0$ .

#### Case 3.

If  $f(n) = \Omega(n^{\log_b(a)+\epsilon})$  and for sufficiently large  $n$   
 $af\left(\frac{n}{b}\right) \leq cf(n)$  for some constant  $c < 1$  then  $T(n) = \Theta(f(n))$ .

## 6.2 Master Theorem

We prove the Master Theorem for the case that  $n$  is of the form  $b^\ell$ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

# The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

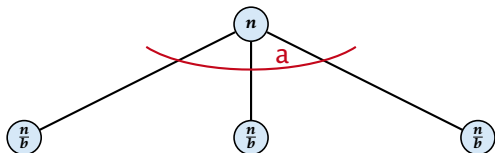
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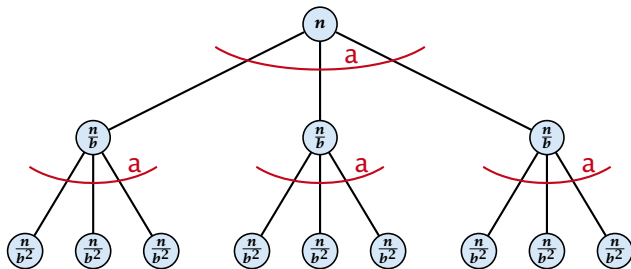
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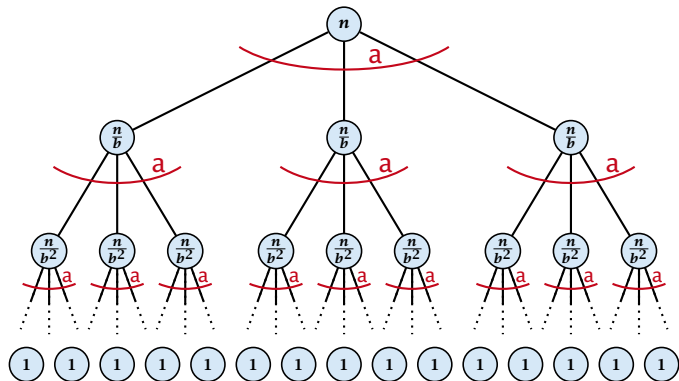
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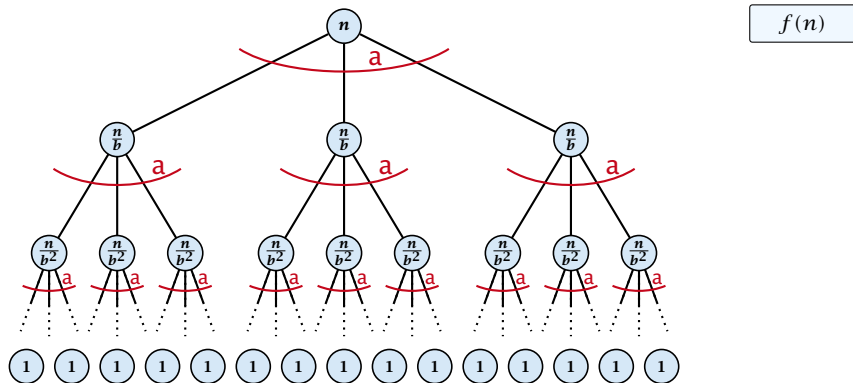
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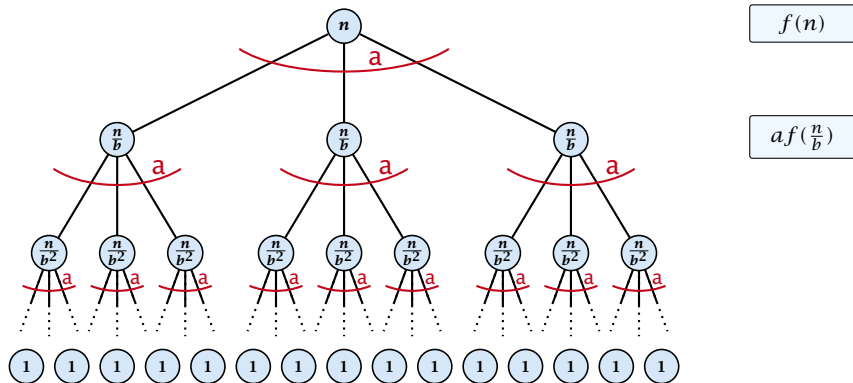
The running time of a recursive algorithm can be visualized by a recursion tree:





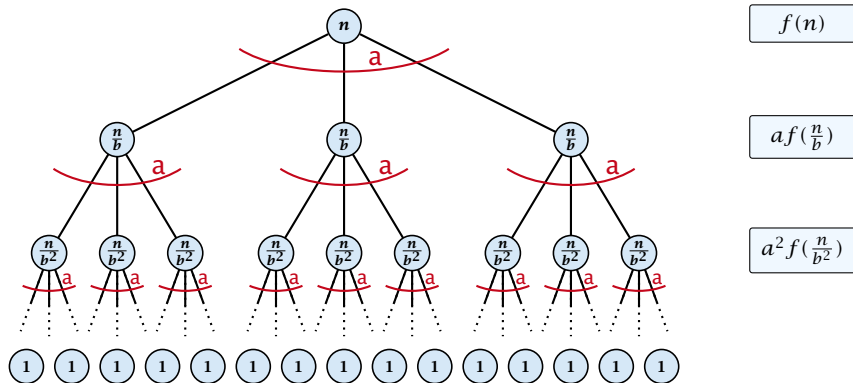
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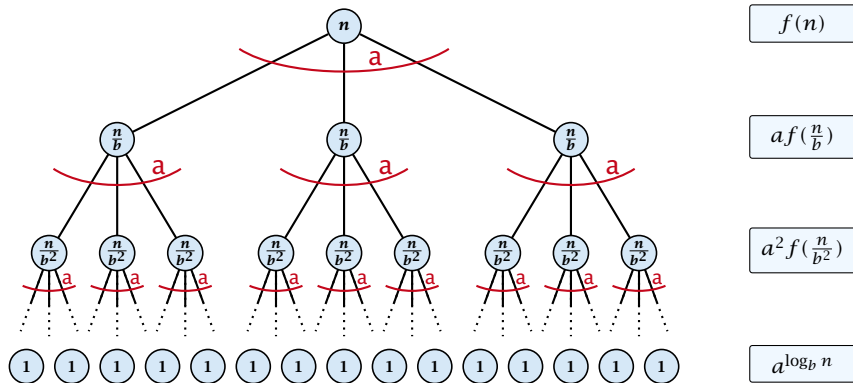
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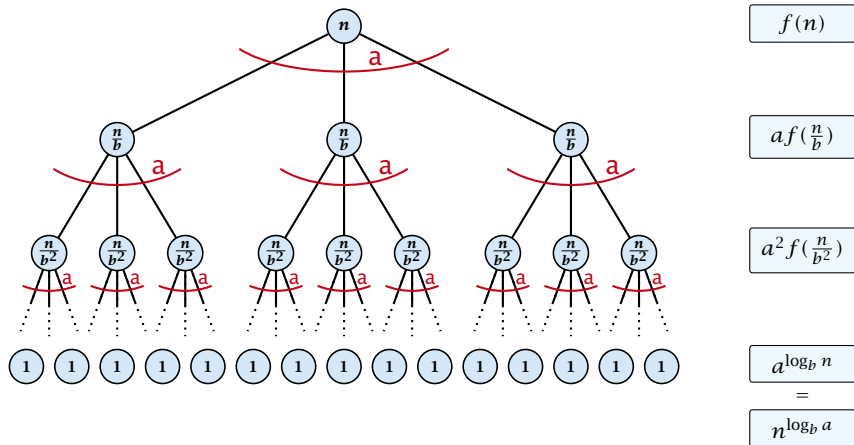
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## 6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$



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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$



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Hence,

$$T(n) \leq \left( \frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

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$n = b^\ell \Rightarrow \ell = \log_b n$	$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$
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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \end{aligned}$$



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$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

Case 2. Now suppose that  $f(n) \leq cn^{\log_b a} (\log_b(n))^k$ .

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$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \\ &\approx \frac{c}{k} n^{\log_b a} \ell^{k+1} \end{aligned}$$

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$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k$$

$$\approx \frac{c}{k} n^{\log_b a} \ell^{k+1}$$

$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

Where did we use  $f(n) \geq \Omega(n^{\log_b a + \epsilon})$ ?

**Case 3.** Now suppose that  $f(n) \geq dn^{\log_b a + \epsilon}$ , and that for sufficiently large  $n$ :  $af(n/b) \leq cf(n)$ , for  $c < 1$ .

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

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Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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From this we get  $a^i f(n/b^i) \leq c^i f(n)$ , where we assume that  $n/b^{i-1} \geq n_0$  is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Where did we use  $f(n) \geq \Omega(n^{\log_b a + \epsilon})$ ?

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ \hline \end{array}$$

The diagram shows two 9-bit integers,  $A$  and  $B$ , aligned for addition. Integer  $A$  is represented by the red bits 1 1 0 1 1 0 1 0 1, and integer  $B$  is represented by the blue bits 1 0 0 0 1 0 0 1 1. A horizontal line is drawn under the bits of  $B$ . A vertical light blue box highlights the rightmost bit of  $A$  (the least significant bit) and the rightmost bit of  $B$  (the least significant bit), which are the bits that would be added together.



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1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
								0	



## Example: Multiplying Two Integers

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 0\ 0 \end{array}$$

The diagram shows the addition of two 10-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the 8th bit of B. A vertical box highlights the 8th and 9th bits of both numbers, which are 0 and 1 in A, and 1 and 1 in B. Below the line, the result of the addition for these two bits is shown as 0 and 0.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
						1	1		
							0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	1	1	A
1	0	0	0	1	0	0	1	1	1	1	B
						0	0	0			

(Note: In the original image, a vertical box highlights the 7th, 8th, and 9th bits of the result row, which are 0, 0, and 0. Small '1' characters are placed below the 6th, 7th, and 8th bits of the B row, indicating carry bits.)

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
					0	0	0		

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical line separates the bits into two groups of 5 bits each. A light blue vertical box highlights the 5-bit segment of the first group (bits 4 to 8) in both A and B. The result of the addition is shown below the horizontal line, with carry bits '1' indicated under the 5th, 6th, and 7th bits of the second group.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
					0	1	1	1	
					1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$



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For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
				0	1	0	0	0	

Carry bits: 1, 0, 1, 1, 1

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Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
<hr/>									
			1	0	1	1	1		
				0	1	0	0	0	

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Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, to produce a 9-bit result. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. The result is 0, 0, 1, 0, 0, 0, 0. A vertical box highlights the 4th bit of A (1) and the 4th bit of B (0), which are being added to produce the 4th bit of the result (0). Small subscripts are present below the 4th and 5th bits of B: 1, 0, 1, 1, 1.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & 1 & 1 & 0 & 1 & 1 & 1 & & \\ & & & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & B \\ \hline & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \\ \hline & & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B, using a ripple carry adder. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 0, 1. The carry propagation is shown by a vertical bar highlighting the carry chain starting from the second bit from the right (index 7) and moving left through the third, fourth, fifth, sixth, seventh, and eighth bits. The final carry-out is 1, which is placed at the ninth bit position of the result.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

1	1	0	1	1	0	1	0	1	$A$
1	0	0	0	1	0	0	1	1	$B$
	0	1	1	0	1	1	1		
<hr/>									
		1	0	0	1	0	0	0	

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	<hr/>									
	1	1	0	0	1	0	0	0		

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. The result of the addition is 1, 1, 0, 0, 1, 0, 0, 0. A vertical box highlights the first two bits of the result, 1 and 1, which correspond to the carry bits from the addition of the first two bits of A and B.

## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

	1	1	0	1	1	0	1	0	1	$A$
	1	0	0	0	1	0	0	1	1	$B$
	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	



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	1	0	0	1	1	0	1	1	1	
	0	1	1	0	0	1	0	0	0	





## Example: Multiplying Two Integers

Suppose we want to multiply two  $n$ -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers  $A$  and  $B$ :

$$\begin{array}{r} 110110101 \quad A \\ 1000010011 \quad B \\ \hline 1011001000 \end{array}$$

This gives that two  $n$ -bit integers can be added in time  $\mathcal{O}(n)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most  $m + n \leq 2n$  bits.

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \\ \times 1011 \\ \hline \end{array}$$

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$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 00000 \\ 00000 \\ 10001 \\ \hline 0 \end{array}$$

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Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ \phantom{1\ 0\ 0\ 0\ 1\ 0} 0\ 0 \end{array}$$

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Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00000000 \end{array}$$

- This is also known as the “school method” for multiplying integers.
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## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00001000 \\ \hline 100011001 \end{array}$$

- This is also known as the “school method” for multiplying integers.
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## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline \end{array}$$

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$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 10001000 \\ \hline 10111011 \end{array}$$

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$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00001000 \\ \hline 10111011 \end{array}$$

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**Time requirement:**

## Example: Multiplying Two Integers

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- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most  $m + n \leq 2n$  bits.

**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .

## Example: Multiplying Two Integers

Suppose that we want to multiply an  $n$ -bit integer  $A$  and an  $m$ -bit integer  $B$  ( $m \leq n$ ).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 10001000 \\ \hline 10111011 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most  $m + n \leq 2n$  bits.

**Time requirement:**

- ▶ Computing intermediate results:  $\mathcal{O}(nm)$ .
- ▶ Adding  $m$  numbers of length  $\leq 2n$ :  $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$ .

## Example: Multiplying Two Integers

**A recursive approach:**

Suppose that integers  $A$  and  $B$  are of length  $n = 2^k$ , for some  $k$ .

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Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

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### Algorithm 3 $\text{mult}(A, B)$

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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

# Example: Multiplying Two Integers

**Master Theorem:** Recurrence:  $T[n] = aT(\frac{n}{b}) + f(n)$ .

- ▶ Case 1:  $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$        $T(n) = \Theta(n^{\log_b a})$
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In our case  $a = 4$ ,  $b = 2$ , and  $f(n) = \Theta(n)$ . Hence, we are in Case 1, since  $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$ .

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We get a running time of  $\mathcal{O}(n^2)$  for our algorithm.

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⇒ Not better than the “school method”.

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We can use the following identity to compute  $Z_1$ :

A more precise  
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A more precise (correct) analysis would say that computing  $Z_1$  needs time  $T(\frac{n}{2} + 1) + \mathcal{O}(n)$ .

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We can use the following identity to compute  $Z_1$ :

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Hence,

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A huge improvement over the “school method”.