

## 6.3 The Characteristic Polynomial

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Note that we ignore **boundary conditions** for the moment.

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### Approach:

- ▶ First determine all solutions that satisfy recurrence relation.
- ▶ Then pick the right one by analyzing boundary conditions.
- ▶ First consider the homogenous case.

# The Homogenous Case

The solution space

$$S = \{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \}$$

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**How do we find a non-trivial solution?**

We guess that the solution is of the form  $\lambda^n$ ,  $\lambda \neq 0$ , and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all  $n \geq k$ .

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Dividing by  $\lambda^{n-k}$  gives that all these constraints are identical to

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Let  $\lambda_1, \dots, \lambda_k$  be the  $k$  (complex) roots of  $P[\lambda]$ . Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \dots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values  $\alpha_i$ .

# The Homogenous Case

## Lemma 1

Assume that the characteristic polynomial has  $k$  *distinct* roots  $\lambda_1, \dots, \lambda_k$ . Then *all* solutions to the recurrence relation are of the form

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## Proof.

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We show that the above set of solutions contains one solution for every choice of boundary conditions.

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## Proof (cont.).

Suppose I am given boundary conditions  $T[i]$  and I want to see whether I can choose the  $\alpha'_i$ 's such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

# Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$= \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$



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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

# Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all  $\lambda_i$ 's are different, then the determinant is non-zero.

# The Homogeneous Case

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Suppose we have a root  $\lambda_i$  with multiplicity (**Vielfachheit**) at least 2. Then not only is  $\lambda_i^n$  a solution to the recurrence but also  $n\lambda_i^{n-1}$ .

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To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

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Since  $\lambda_i$  is a root we can write this as  $Q[\lambda] \cdot (\lambda - \lambda_i)^2$ . Calculating the derivative gives a polynomial that still has root  $\lambda_i$ .

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$



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Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

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Suppose  $\lambda_i$  has multiplicity  $j$ . We know that

$$c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with  $\lambda$ ; plugging in  $\lambda_i$ )

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

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Hence,  $n^\ell \lambda_i^n$  is a solution for  $\ell \in 0, \dots, j-1$ .

# The Homogeneous Case

## Lemma 2

Let  $P[\lambda]$  denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \dots + c_kT[n-k] = 0$$

Let  $\lambda_i, i = 1, \dots, m$  be the (complex) roots of  $P[\lambda]$  with multiplicities  $\ell_i$ . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of  $\alpha_{ij}$ 's is a solution to the recurrence.



## Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

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Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

## Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

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$$\alpha \left( \frac{1 + \sqrt{5}}{2} \right) + \beta \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

## Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$



# The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n - 1) + c_2T(n - 2) + \cdots + c_kT(n - k) = f(n)$$

with  $f(n) \neq 0$ .

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

# The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where  $T_h$  is **any** solution to the homogeneous equation, and  $T_p$  is **one** particular solution to the inhomogeneous equation.

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There is no general method to find a particular solution.

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I get a completely determined recurrence if I add  $T[0] = 1$  and  $T[1] = 2$ .



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$T[1] = 2$  gives  $1 + \beta = 2 \Rightarrow \beta = 1$ .

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Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3 \end{aligned}$$

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and so on...