Iterative Rounding

- Combinatorial optimization → Integer program
- LP Representation [extreme point optimal solution].
  - ↳ integral
    - → total unimodularity of constraint matrix (TUM)
    - → total dual integrality [TDI] of whole linear system including RHS.
  - ↳ fractional [rounding].

- Rand / Det. rounding: solve relaxation once & do the rounding based on this solution.
- Does not use full power of rounding.
- After part of the solution is rounded, the remaining fractional solution may not be the best to continue.
- Iterated rounding: (1) Round few.
  (2) Recompute fractional values for remaining variables.
  [Jain: Survivable network design: Gen Steiner Network FOCS 98]

- TUM matrix: Every square non-singular submatrix is unimodular. [only $0, \pm 1$ entries]
  ($Ax \le b$) A is TU, b integral.
  - incidence matrix of bipartite (matching) graph is TUM.
  - max flow.

  [~~mata~~ unimodular: square integer matrix w $|\det(A)| = \pm 1$].

- TDI: $Ax \le b$; $A, b \in \mathbb{Q}$ (rational) is TDI if $\forall c \in \mathbb{Z}^n$ if there is a bounded feasible solution to LP $\{\max c^T x : Ax \le b\}$ there is an integer opt dual solution. [$\min \{yb : y \geq 0, yA = c\}$, y is integer]
  (weaker sufficient condition than TUM)

  - If A is TUM then $Ax \le b$ is TDI for all b.
  - $Ax \le b$ is TDI & b integral ⇒ $Ax \le b$ is int polyhedron.

(1) Exponential sized LP.

LP Solvability (using separation oracle).

(2) Characterization of extreme point solution.

(3) Iterative Algo.

$\rightarrow x_v = 1$: include in integral.

$x_v = 0$: remove corrs. element

$\Big\}$ Resolve smaller

(4) Correctness : progress. [rank lemma]

Optimality : inductive argument

NP-Hard

Rounding : Pick x_v if $x_v > \frac{1}{\alpha}$. $\Rightarrow$ Mult. Appx

Relaxation : If $\sum_i a_i \leq b + \beta$ for some threshold β.

remove $\sum_i a_i x_i \leq b$ $\Rightarrow$ Additive Appx

$P = \{x : Ax = b, x \geq 0\}$

x is extreme pt soln if $\nexists y$ (nonzero) s.t. $x+y, x-y \in P$.

$\Rightarrow$ vertex soln / Basic feasible soln

- $\exists$ an optimal extreme pt soln.

- Rank lemma: x is an extreme point solution where $x_i > 0$ for each i. Then # variables = rank(A) i.e. # of lin independent constraint.

Note: lin independence : (nontrivial lin comb doesnot give 0)

linear programs

- integral polytope : every extreme point is integral.
- OPTIMIZATION = SEPARATION . (Ellipsoid method)
- Lemma 2.1.2. $P = \{x : Ax \geq b, x \geq 0\}$, $\min\{c^T x : x \in P\} < \infty$, then $\forall x \in P$, $\exists$ an extreme point optimal solution.

$\Rightarrow$ Assume x is not extreme; $\exists y$, s.t. $x+y, x-y \in P$.

$\therefore$ $A(x+y) \geq b$, $x+y \geq 0$ | $c^T x \leq c^T(x+y)$
$A(x-y) \geq b$, $x-y \geq 0$ | $\leq c^T(x-y)$ $\Big\}$ $\boxed{c^T y = 0}$

Let, $A^=$: A restricted to rows at equality at x i.e. $A^= x = b^=$.
$\therefore$ $A^= y \geq 0$, $A^=(-y) \geq 0 \Rightarrow \boxed{A^= y = 0}$

Since, $y \neq 0$, w.l.o.g. consider some $y_j < 0$. [otherwise take $-y_j$]
Consider $x + \lambda y$ for $\lambda > 0$ and increase λ untill
(i) $x + \lambda y \lneq 0$ [not feasible on nonneg constraint]
or (ii) $A(x+\lambda y) \lneq b$ [at " " other constraint] $\Big\}$ gets tight

claim : $x + \lambda^* y$ is optimal with one more tight constraint.
i) $x + y \gneq 0 \Rightarrow (x_i = 0 \Rightarrow y_i = 0)$.
(ii) $A^=(x+y) = A^= x = b^=$ (since, $A^= y = 0$) $\Big\}$ tight constraints remain tight
(iii) $c^T(x + \lambda^* y) = c^T x$ (so remains optimal)

- Lemma 2.1.3 : $A_x^=$ denote submatrix of $A^=$ restricted to columns corrs to nonzeros in x.

(1) x is extreme point $\Leftrightarrow$ (1) $A_x^=$ has full column rank.

(1) $\Rightarrow$ x is not extreme point, $\exists y$ s.t. $A^= y = 0$., $x_j = 0 \Rightarrow y_j = 0$.
$A_y^=$ is submatrix of $A_x^=$ & $A_y^=$ has lin dependent columns.

$(\overset{2}{\Leftarrow})$ $A_x^=$ has lin dependent columns.

$A_x^= y = 0$. set remaining coordinates 0. $\Rightarrow A^= y = 0$.

$x_j = 0 \Rightarrow y_j = 0$. [By construction].

$\therefore \exists \epsilon > 0$, $x + \epsilon y \geq 0$, $x - \epsilon y \geq 0$.

$A(x \pm \epsilon y) = Ax \pm \epsilon Ay \geq b$ $\Big\}$ x is not extreme pt.

<u>Lemma 2.1.4</u> $P = \{x : Ax \geq b, x \geq 0\}$,

<u>Rank Lemma</u>: x is an extreme point with $x_i > 0$ for all i.
Then # variables
= Any maximal # of lin indep tight constraints $A_i x = b_i$ for some row i of A

<u>Proof</u>: As $x_i > 0$ for all i, $A_x^= = A^=$.

- x is extreme point $\Rightarrow$ $A^=$ has full column rank = row rank.
- # columns = # nonzero variables.

Thus any maximal # of lin ind tight constraints
= maximal # of lin indep rows of $A^=$ = rowrank$(A^=)$.

- <u>Basis</u>: A subset of columns $\mathcal{B}$ of constraint matrix A is basis is A_B is invertible
- <u>Basic solution</u>: x is basic $\Leftrightarrow$ $\exists$ a basis B s.t. $x_j = 0$ if $j \notin B$ and $x_B = A_B^{-1} b$.
- Basic feasible solution $\Leftrightarrow$ extreme point solution.

Matching LP:

LP(G):

$$\max \sum_{e \in E} w_e x_e$$

$$\text{s.t.} \sum_{e \in \delta(v)} x_e \le 1 \quad \forall v \in V_1 \cup V_2$$

$$x_e \geq 0 \quad \forall e \in E.$$

Algo: (i) $F \leftarrow \phi$

(ii) while $E(G) \neq \phi$ do

(a) Find optimal x^*, remove e when $x_e = 0$

(b) If $x_e = 1$, $F \leftarrow F \cup \{e\}$, $G \leftarrow G \setminus \{u, v\}$.

(iii) Return F.

Lemma 3.1.3 (Conseq. of Rank lemma)

Given extreme point solution s.t. $x_e > 0 \ \forall e \in E$, $\exists W \subseteq V_1 \cup V_2$ such that

(i) $x(\delta(v)) = 1 \quad \forall v \in W$ (tight)

(ii) $|W| = |E|$ variable

(iii) The vectors in $\{X(\delta(v)) : v \in W\}$ are lin indep. (lin indep)

Correctness:

Claim 3.1.4: If the algo, in each iteration finds an e s.t. $x_e = 0$ or $x_e = 1$, then returns a matching of cost $\geq LP(G)$

→ Use induction. Base case: trivial

- **If $x_e = 0$:** $G' = G \setminus e$ is residual problem.

x_{res}, $x|_{G \setminus e}$ is a feasible soln for residual problem

By induction we get $F' \subseteq E(G')$

so that $w(F') \geq w \cdot x_{res} = w \cdot x$. ($\because x_e = 0$)

If $x_e = 1$: $\Rightarrow$ Residual problem $G' = G \setminus \{u, v\}$.

x_{res}, $x|_{G'}$ is again a feasible solution.

$\therefore w(F') \geq w \cdot x_{res}$.

$w(F) = w(F') + w_e \geq w \cdot x_{res} + w_e = w \cdot x$.

☒ Lemma 3.1.5: Always we get a $x_e = 0$ or $x_e = 1$.

$\Rightarrow$ Let $0 < x_e < 1$ for all edge.

From rank lemma, $\exists\, W \subseteq V_1 \cup V_2$ that are tight.

Now, $d_E(v) \geq 2$ for each $v \in W$. $\left[\begin{array}{l} \text{since } x(\delta(v)) = 1 \\ \& \ 0 < x_e < 1 \end{array}\right]$

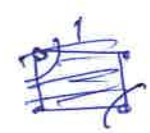
But, $\underset{\text{R.L.}}{2|W| = 2|E|} = \sum_{v \in V} d_E(v) \overset{(\text{subset})}{\geq} \sum_{v \in W} d_E(v) \geq 2|W|$.

To hold as equality, $d_E(v) = 0$ for $v \notin W$

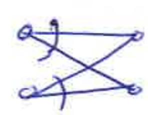
$d_E(v) = 2$ for $v \in W$.

So, $E|_W$ is union of even cycles. [$\because$ G is bipartite].

$\therefore \sum_{v \in C \cap V_1} x(\delta(v)) = \sum_{v \in C \cap V_2} x(\delta(v))$

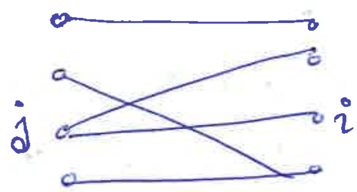


which contradicts independence of constraints.



Generalized Assignment Problem:

jobs J



machines M

p_{ij} = processing time
c_{ij} = cost
T_i = availability of machine i.

<u>Goal</u>: Assign each job to some machine s.t. total cost is min & no machine is scheduled more than T_i

[Shmoys + Tardos : machine is used $\leq 2T_i$].

<u>LP(GAP)</u> :

$$\min \sum_i c_{ij} x_{ij}$$

s.t. $\sum_{e \in \delta(j)} x_e = 1 \quad \forall j \in J$ [each job is assigned]

$\sum p_e x_e \leq T_i \quad \forall i \in M'$ [each machine is not overload]

$x_e \geq 0 \quad \forall e \in E$

[$M' \subseteq M$, $M' = M$ in the beginning] [Disallow $p_{ij} > T_i$].

◎ <u>Property of extreme point</u> : x be extreme point with $0 < x_e < 1$ for all e. Then $\exists$ $J' \subseteq J$ & $M'' \subseteq M$ such that

(i) <u>tight</u>: $\sum_{e \in \delta(j)} x_e = 1 \quad \forall j \in J'$, $\sum_{e \in \delta(i)} p_e x_e = T_i \quad \forall i \in M''$.

(ii) <u>lin indep</u>: constraints corrs. to J', M' are lin indep

(iii) $|J'| + |M''|$ = no of variables = $|E(G)|$.

• **Iterative GAP:**

(i) Initialize $F \leftarrow \phi$, $M' \leftarrow M$.

(ii) While $J \neq \phi$ [all jobs not assigned] do

a find extreme point opt x, remove all $x_{ij} = 0$

b if $x_{ij} = 1$, then update $F \leftarrow F \cup \{ij\}$, $J \leftarrow J \setminus \{j\}$, $T_i \leftarrow T_i - P_{ij}$.

c (Relax) if machine i s.t. $d(i) = 1$ or $d(i) = 2$ and $\sum_{j \in J} x_{ij} \geq 1$ then $M' \leftarrow M' \setminus i$

(iii) Return F.

⊙ Always progress: In extreme point always $\exists e$ satisfying a, b or c.

$\Rightarrow$ For contradiction assume, $0 < x_e < 1 \Rightarrow d(j) \geq 2$ or $(\sum x_e = 1)$

$d(i) \geq 2$ (for M' from c).

Rank lemma, $|J'| + |M''| = |E| \geq \frac{\sum d(i) + \sum d(j)}{2} \geq |J| + |M'| \geq |J'| + |M''|$.

All holds by equality, $d(i) = 0$ for $i \in M \setminus M'$;

$*$: all machines have deg exactly 2.

$\#$: $J = J'$, $M' = M''$.

- So, G is union of cycles with vertices in J', M''; #jobs = #machines

As each job has $\sum_{i \in M'} x_{ij} = 1$, $\exists$ machine with $\sum_{j \in J'} x_{ij} \geq 1$. ↯

⊙ **2 Approximation:**

• At any iteration $cost(F) + LP_{remaining} \leq LP_{original}$. Use induction.

step b: $cost(F) \uparrow = LP value \downarrow$.

step c: $cost(F)$ same, LP can only decrease.

thus finally when F is feasible assignment, $cost(F) \leq LP_{orig}$.

• For $i \in M'$, T_i' (residual time left) $+ T_i(F) \leq T_i$. (Use induction)

• Problem when machine i is removed

- $d(i) = 1 \Rightarrow T_i + P_{ij} \leq 2T_i$.

- $d(i) = 2 \Rightarrow T_i(F) + P_{ij_1} + P_{ij_2} \leq T_i - x_{ij_1}P_{ij_1} - x_{ij_2}P_{ij_2} + P_{ij_1} + P_{ij_2}$

$\leq T_i + (1 - x_{ij_1})P_{ij_1} + (1 - x_{ij_2})P_{ij_2} \leq T_i + (2 - x_{ij_1} - x_{ij_2})T_i \leq 2T_i$.