Spanning Trees

Subtour LP : LP_{ST}

$$\min \sum_{e \in E} c_e x_e$$

set of edges having both endpoints in S

$$\text{subject to} \quad x(E(S)) \leq |S| - 1 \quad \forall \phi \neq S \subset V$$
$$x(E(V)) = |V| - 1$$
$$x_e \geq 0 \quad \forall e \in E$$

(1) LP solvability : polytime separation oracle.

- Is there a set S s.t. $x(E(S)) > |S| - 1$. (Mincut doesnot help)

$\iff \min_{S \subset V} \{|S| - 1 - x(E(S))\} < 0$

$\iff \min_S \{|S| - 1 + x(E(V)) - x(E(S)) < |V| - 1\}. \Rightarrow$ Use $2|V|-2$ min cut problems

Fix a root vertex $r \in V$.

For each $k \in V \setminus \{r\}$, construct two min-cuts

(i) check inequality all ~~all~~ subset containing r but not k

(ii) " " " k but not r.

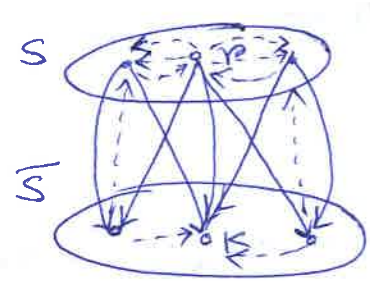
Construction for (1) :

Directed graph $\hat{G}$ with vertex set V and arcs (i,j) and (j,i) $\forall$ $(i,j) \in E(G)$

a $w(i,j) = w(j,i) = \frac{x_{ij}}{2}$.

b Arcs from $v \in V \setminus \{r,k\} \to k$, weight 1.

c Arcs from r to $v \in V \setminus r$, weight $\frac{x(\delta(v))}{2}$.



$[S, \bar{S}]$ separates r & k.

$$\text{Total weight of cut} = \underbrace{|S| - 1}_{\text{type b}} + \underbrace{\frac{x(\delta(S))}{2}}_{\text{type a}} + \underbrace{\sum_{v \notin S} \frac{x(\delta(v))}{2}}_{\text{type c}}$$
$$= |S| - 1 + x(E(V)) - x(E(S)).$$

Hence, checking min cut separating r from $k < |V| - 1$ suffices to check whether $\exists$ a violating set S containing r but not k.

- characterization of extreme point solution
  - finding a good set of tight inequalities.
    as there are exponential equalities, many are satisfied by equalities.

4.1.3 : Uncrossing Techniques.

- Induced edge function is supermodular.

  $x(E(X)) + x(E(Y)) \le x(E(X \cup Y)) + x(E(X \cap Y))$

  and equality holds iff $E(X \setminus Y, Y \setminus X) = \phi$.

- Let x be an extreme point optimal solution.

  $\mathcal{F} = \{ S \subseteq V \mid x(E(S)) = |S| - 1 \}$: family of tight ineq for x

- Lemma 4.1.4 : $\mathcal{F}$ is closed under $\cup$ & $\cap$.

  Furthermore, $x(E(S)) + x(E(T)) = x(E(S \cap T)) + x(E(S \cup T))$.

$$
\begin{aligned}
|S| - 1 + |T| - 1 &= x(E(S)) + x(E(T)) && \text{(tight sets)} \\
&\le x(E(S \cap T)) + x(E(S \cup T)) && \text{(supermodular)} \\
&\le |S \cap T| - 1 + |S \cup T| - 1 && \text{(subtour LP constraint)} \\
&\le |S| - 1 + |T| - 1 && \text{(|.| is modular)}
\end{aligned}
$$

Hence, = everywhere, $\left.\begin{aligned} |S \cap T| - 1 &= x(E(S \cap T)) \\ |S \cup T| - 1 &= x(E(S \cup T)) \end{aligned}\right\}$ closed

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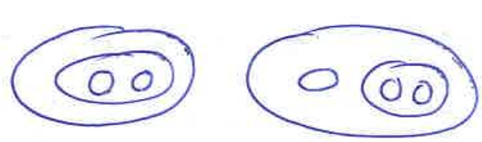
- span $(\mathcal{F})$: vector space generated by $\{ \chi(E(S)) \mid S \in \mathcal{F} \}$
- X, Y are intersecting:  $X \cap Y,\ X \setminus Y,\ Y \setminus X$ are nonempty
- laminar : no two sets are intersecting.



~~Bipartite matching / Assignment problem:~~

- **Lemma 4.1.5**: If $\mathcal{L}$ is a maximal laminar subfamily of $\mathcal{F}$, then $\boxed{\text{span}(\mathcal{L}) = \text{span}(\mathcal{F})}$

$\Rightarrow$ let $\mathcal{L}$ is max lam. subfamily and $\text{span}(\mathcal{L}) \subset \text{span}(\mathcal{F})$

For any $S \notin \mathcal{L}$, define $\text{intersect}(S, \mathcal{L}) = |\{T \in \mathcal{L} \mid T, S \text{ intersects}\}|$

As, $\text{span}(\mathcal{L}) \subset \text{span}(\mathcal{F})$; $\exists S$ s.t. $\chi(E(S)) \notin \text{span}(\mathcal{L})$.

choose S with minimum $\text{intersect}(S, \mathcal{L})$.

Clearly, $\text{intersect}(S, \mathcal{L}) \geq 1$ ($\because$ max lam subfamily)

let, T intersects S & $T \in \mathcal{L}$.

By Lemma 4.1.4, $S \cap T, S \cup T \in \mathcal{F}$.

Prop 4.1.6 $\Rightarrow$ $\text{intersect}(S \cap T, \mathcal{L}), \text{intersect}(S \cup T, \mathcal{L}) < \text{intersect}(S, \mathcal{L})$.

Thus, $S \cap T$ and $S \cup T$ are in $\text{span}(\mathcal{L})$ [minimality of S]

4.1.4 $\Rightarrow$ $\chi(E(S)) + \chi(E(T)) = \chi(E(S \cup T)) + \chi(E(S \cap T))$

Thus, $\chi(E(S))$ is also in $\text{span}(\mathcal{L})$, contradiction.

- **Lemma 4.1.6**: S intersects $T \in \mathcal{L}$.
  then $\text{intersect}(S \diamond T, \mathcal{L}) < \text{intersect}(S, \mathcal{L})$ for $\diamond = \cup, \cap$.

$\Rightarrow$ If $R (\in \mathcal{L})$ intersects $S \cap T$ or $S \cup T$, R intersects S.
But T intersects S but not $S \cup T$, $S \cap T$ as they are subset of T. (R doesnot intersect T)

- **Prop 4.1.7**: A laminar family over ground set V without singleton has $\leq |V| - 1$ distinct members

- **Cor 4.1.8**: (including singleton) $\leq 2|V| - 1$ distinct members

Proof of 4.1.7: By induction. $|V| = 2$: trivial.

let $n = |V|$ & true for all ground sets of size $< n$.

let $S (\subset V)$ be the maximal set in $\mathcal{L}$.

for any $T \in \mathcal{L}$, $T \subset S$ or $T \cap S = \phi$.

From induction, # sets contained in S : $|S| - 1$.

~~Intersecting sets~~ Other sets form another laminar family

There are, $\leq |V| - |S| - 1$ such sets.

Along with V, # sets $\leq |S| - 1 + |V| - |S| - 1 + 1 = |V| - 1$

⊙ Iterative Leaf-finding MST Algo:

(i) Initialization $F \leftarrow \phi$.

(ii) While $|V(G)| \geq 2$ do

(a) Find opt extreme point x; remove e for $x_e = 0$.

(b) Find v with at most one edge $e = uv$ incident on it $F \leftarrow F \cup e$, $G \leftarrow G \setminus v$

(iii) Return F.

⊙ Correctness :

Lemma 4.1.9 : (Progress) For x with $x_e > 0$ $\forall e \in E$,
$\exists v$ with $d(v) = 1$

$\Rightarrow$ If all v has deg 2 ; $|E| = \frac{1}{2} \sum_{v \in V} d(v) \geq |V|$. — (*)

As, no edge $x_e = 0$, all tight ineq : $x(E(S)) = |S| - 1$.

lemma 4.1.5, $\exists$ $|\mathcal{L}|$ lin indep tight constraints where $\mathcal{L}$ is a laminar family with no singleton set. (we dont need singleton in subtour ineq)

$|E| = \underset{\text{rank lemma}}{|\mathcal{L}|} \underset{\text{Prop 4.1.7}}{\leq} |V| - 1$. — Contradicts (*)

Thm 4.1.10: Returned solution is MST.

$\Rightarrow$ Use induction on # iterations.

For a leaf vertex $x_e = 1$ since $x(\delta(v)) \geq 1$ $\left\{ \begin{array}{l} x(E(V-v)) \leq \frac{|V|}{2} \\ x(E(V)) = |V|-1. \end{array} \right.$

for any spanning tree T' of $G' = G \setminus v$, we can construct a spanning tree $T = T' \cup e$ of G.

Induction on residual problem: MST on $G \setminus v$.

As $x_e = 1$, $x|_{E(G')} = x_{res}$ is a feasible soln for LP (G')

$c(F') \leq c \cdot x_{res}$ from induction

$c(F) = c(F') + c_e \leq c \cdot x \leq$ [cost of MST]

Note: x_{res} is an optimal extreme point solution to subtour LP for G'. So we need to solve the LP only once. $\Rightarrow$ integrality of MST.

© A non-iterative direct proof:

Lemma 4.1.9: $|E| \leq n-1$

$x(E) = n-1$

$x(e) \leq 1$ [for $|S| = 2$]

we must have, $x_e = 1 \ \forall e \in E$ proving integrality.

☑ <u>Iterative 1-edge-finding algorithm</u>:

same as iterative leaf-finding except

(ii) b. Find an edge $e=\{u,v\}$ with $x_e=1$ and update $F \leftarrow F \cup e$, $G \leftarrow G/e$. (contraction)

- Modulo lemma 4.2.1, argument similar to Thm 4.1.10 show that the algorithm returns a MST.

<u>lemma 4.2.1</u>: For any extreme point solution x to the subtour LP with $x_e > 0$ for each edge e, $\exists$ an edge f with $x_f = 1$

- <u>Proof 1</u>. Global counting Argument.
  - Prop 4.1.7 : $|\mathcal{L}| < |V| - 1$.
  - Rank lemma : $|\mathcal{L}| = |E|$.
  - Global degree counting: $|E| \geq |V|$. [each vertex is atleast degree 2]

- <u>Proof 2</u>. Local Token counting Argument.
  - Rank lemma: $|\mathcal{L}| = |E|$.
  - [Assign one token for each $e \in$ support(E)
  - Redistribute tokens so that each set in $\mathcal{L}$ receive one token & still some left $\Rightarrow |E| > |\mathcal{L}|$ ↯.
  - How to redistribute?
    - each edge gives its token to the smallest set containing both of its endpoints.

(contd.)

Let S be any set in $\mathcal{L}$ with children $R_1, \ldots, R_k$ $(k \geq 0)$.

$$x(E(S)) = |S| - 1$$

$$x(E(R_i)) = |R_i| - 1.$$

subtracting,

$$x(E(S)) - \sum_{i=1}^{k} x(E(R_i)) = |S| - \sum_{i=1}^{k} |R_i| + k - 1$$

Let $A = E(S) \setminus [\bigcup_i E(R_i)]$.

S obtains one token for each $e \in A$.

If $A = \phi$, $x(E(S)) = \sum_i x(E(R_i))$ ↯ contradicts linear indep. of these constraints

$|A| \neq 1$ as $x(A)$ is integer, but no edge is integral in it.

Thus, S receives at least two tokens. ↯

<u>Proof</u> 3. [Local Fractional Token Counting Argument]

[Generalization to prev, will be useful in Degree bounded case].

[Assign one token for each $e \in$ Supp(E)

For each edge e, redistribute $\underline{\underline{x_e}}$ fractional token to the smallest set containing both the endpoints.

[Each set in $\mathcal{L}$ contains ≥ 1 token and some extra leftover fractional edge tokens $\Rightarrow$ contradiction]

(contd.)

following Pf 2,

$$x(A) = x(E(S)) - x(E(R_i)) = |S| - \sum_i R_i + K - 1$$

- S obtains exactly x_e fractional token for each edge $e \in A$.
- $A \neq \phi$ from Pf 2. [lin indep]
- $x(A)$ is integer, hence it is at least one, giving S the one token

Since, every edge is not integral, we have extra fractional token of value $(1-x_e)$ for all edge e as unused tokens.

— contradiction.